STOCHASTIC COLLOCATION METHODS for STABILITY ANALYSIS of DYNAMICAL SYSTEMS

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Key reference:

Howard Elman and David Silvester

Collocation methods for exploring perturbations in linear stability analysis

`http://eprints.ma.man.ac.uk/2533/`

HISTORY

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} - \frac{1}{\mathcal{R}} \nabla^2 \vec{u} + \nabla p = \vec{f}$$

$$\nabla \cdot \vec{u} = 0$$

1822

George Stokes (1819–1903)

1883

Osborne Reynolds (1842–1912)

HYDRODYNAMIC STABILITY

steady solution

$$\begin{aligned} \vec{u} \cdot \nabla \vec{u} - \tfrac{1}{\mathcal{R}} \nabla^2 \vec{u} + \nabla p &= 0 \qquad \text{in } D \\ \nabla \cdot \vec{u} &= 0 \qquad \text{in } D \end{aligned}$$

perturbation model

$$\vec{v}(\vec{x}, t) = \vec{u}(\vec{x}) + e^{\lambda t} \delta \vec{u}(\vec{x}), \quad q(\vec{x}, t) = p(\vec{x}) + e^{\lambda t} \delta p(\vec{x})$$

unsteady perturbation evolution

$$\begin{aligned} D(\vec{u}, \delta\vec{u}) - \tfrac{1}{\mathcal{R}} \nabla^2 \delta\vec{u} + \nabla \delta p &= -\lambda\, \delta\vec{u} \qquad \text{in } D \\ -\nabla \cdot \delta\vec{u} &= 0 \qquad \text{in } D \end{aligned}$$

difference term

$$\begin{aligned} D(\vec{u}, \delta\vec{u}) &= \vec{v} \cdot \nabla \vec{v} - \vec{u} \cdot \nabla \vec{u} \\ &= \vec{u} \cdot \nabla \delta\vec{u} + \delta\vec{u} \cdot \nabla \vec{u} + e^{\lambda t}\, \delta\vec{u} \cdot \nabla \delta\vec{u} \end{aligned}$$

LINEARISED STABILITY

steady-state linear eigenvalue problem

$$-\frac{1}{\mathcal{R}}\nabla^2\delta\vec{u} + \vec{u}\cdot\nabla\delta\vec{u} + \delta\vec{u}\cdot\nabla\,\vec{u} + \nabla\delta p = -\lambda\,\delta\vec{u} \qquad \text{in } D$$

$$-\nabla\cdot\delta\vec{u} = 0 \qquad \text{in } D$$

linear algebra

$$\begin{bmatrix} \frac{1}{\mathcal{R}}\boldsymbol{A}+\boldsymbol{N}+\boldsymbol{W} & B^T \\ B & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix} = -\lambda\begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}$$

Important points

- for a (fixed) given value of $\mathcal{R}$, If $\mathbb{R}(\lambda) < 0$ for all eigenvalues, then the flow problem is linearly stable.
- linear instability $\implies$ nonlinear instability.
- a Hopf bifurcation (breakdown to a periodic flow) is a critical value $\mathcal{R}^*$ for which $\mathbb{R}(\lambda^*) \leq 0$ with $\lambda^* = \pm\theta i$.

Abstract of talk

Eigenvalue analysis is a well-established tool for stability analysis of dynamical systems. However, there are situations where eigenvalues miss important features of physical models. For example, in models of incompressible fluid dynamics, there are examples where linear stability analysis predicts stability but transient simulations exhibit significant growth of infinitesimal perturbations.

$$
\begin{aligned}
-\tfrac{1}{\mathcal{R}}\nabla^2\delta\vec{u} + \vec{u}\cdot\nabla\delta\vec{u} + \delta\vec{u}\cdot\nabla\,\vec{u} + \nabla\delta p &= -\lambda\,\delta\vec{u} \qquad &\text{in } D\\
-\nabla\cdot\delta\vec{u} &= 0 \qquad &\text{in } D
\end{aligned}
$$

Pseudospectra

Question: how sensitive are the eigenvalues λ to perturbations in the base flow?

$$
\begin{bmatrix} \tfrac{1}{\mathcal{R}}\boldsymbol{A}+\boldsymbol{N}+\boldsymbol{W} & B^T \\ B & 0 \end{bmatrix}
\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}
= -\lambda
\begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & 0 \end{bmatrix}
\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}
$$

Answer: construct randomly perturbed problem(s)

$$
\begin{bmatrix} \tfrac{1}{\mathcal{R}}\boldsymbol{A}+\boldsymbol{N}+\boldsymbol{W}+\boldsymbol{S} & B^T \\ B & 0 \end{bmatrix}
\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}
= -\lambda
\begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & 0 \end{bmatrix}
\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}
$$

Key points

- ♡ The perturbed base velocity is (discretely) divergence-free: $\nabla_h\cdot\delta\vec{u}_h = 0$
- ♡ The perturbation is required to be nondissipative: $-\boldsymbol{S} = \boldsymbol{S}^T$
- ♣ Computationally expensive!

Eigenvalue problem

$$J(\vec{u}_h^*)\,\boldsymbol{v} = -\lambda M\boldsymbol{v}$$

Let $\vec{u}_h^* + \delta\vec{u}_h$ be a perturbation of the discrete steady solution $\vec{u}_h^*$.

Idea: generate perturbations $\delta\vec{u}_h = \delta\vec{u}_h(\boldsymbol{\xi})$ in a systematic way, depending on some (other) parameters $\boldsymbol{\xi} := (\xi_1, \ldots, \xi_m)$.

Let $g(\boldsymbol{\xi})$ be the rightmost eigenvalue of the perturbed problem

$$\hat{J}(\vec{u}_h^*, \delta\vec{u}_h(\boldsymbol{\xi}))\,\boldsymbol{v} = -\lambda M\boldsymbol{v}$$

and define $g^{(l)}(\boldsymbol{\xi})$ to be a (cheap-to-compute) surrogate approximation

SURROGATE MODEL II

Eigenvalue problem

$$J(\vec{u}_h^*)\,\boldsymbol{v} = -\lambda M\boldsymbol{v}$$

Let $\vec{u}_h^* + \delta\vec{u}_h$ be a perturbation of the discrete steady solution $\vec{u}_h^*$.

Idea: generate perturbations $\delta\vec{u}_h = \delta\vec{u}_h(\boldsymbol{\xi})$ in a systematic way, depending on some (other) parameters $\boldsymbol{\xi} := (\xi_1, \ldots, \xi_m)$.

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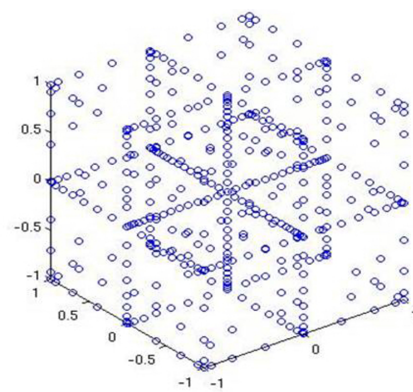
Pseudospectral experiment: study values of $g^{(l)}(\boldsymbol{\xi})$ by *sampling* $\boldsymbol{\xi}$ using sparse grid collocation. In all experiments: we use the spinterp package (Klimke & Wohlmuth).



SURROGATE MODEL III

Eigenvalue problem

$$J(\vec{u}_h^*)\,\boldsymbol{v} = -\lambda M \boldsymbol{v}$$

Let $\vec{u}_h^* + \delta\vec{u}_h$ be a perturbation of the discrete steady solution $\vec{u}_h^*$.

Idea: generate perturbations $\delta\vec{u}_h = \delta\vec{u}_h(\boldsymbol{\xi})$ in a systematic way, depending on some (other) parameters $\boldsymbol{\xi} := (\xi_1, \dots, \xi_m)$.

Idea^2*:* define

$$\vec{u}_h(\vec{x}, \omega) = \vec{u}_h^*(\vec{x}) + \sigma \sum_{j=1}^{m} \sqrt{\mu_j}\, \underbrace{\nabla_h \times \phi_j(\vec{x})}_{\nabla_h \cdot (\ ) = 0}\, \xi_j(\omega)$$

where $\xi_m : \Omega \to [-1, 1]$ are i.i.d. bounded random variables and $\{(\mu_j, \phi_j)\}_{j=1}^m$ are the dominant eigenpairs of the correlation matrix

$$C_{ij} = \exp\left(-\tfrac{1}{4} \| \vec{x}_i - \vec{x}_j \|_{\ell_2}^2\right),$$

where $\vec{x}_i, \vec{x}_j$ are vertices of the rectangular grid.

Computational results

What have we learned?

- ♡ The critical eigenvalues are reasonably stable.
- ♡ New relatively cheap method for pseudospectra is predictive.

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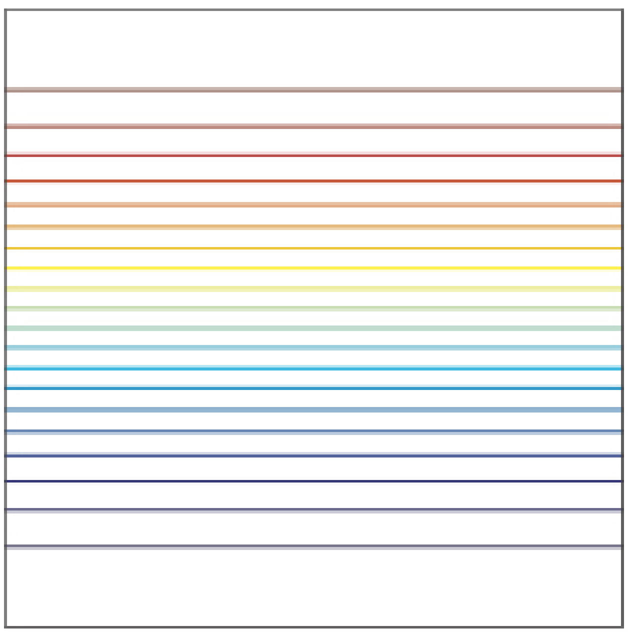
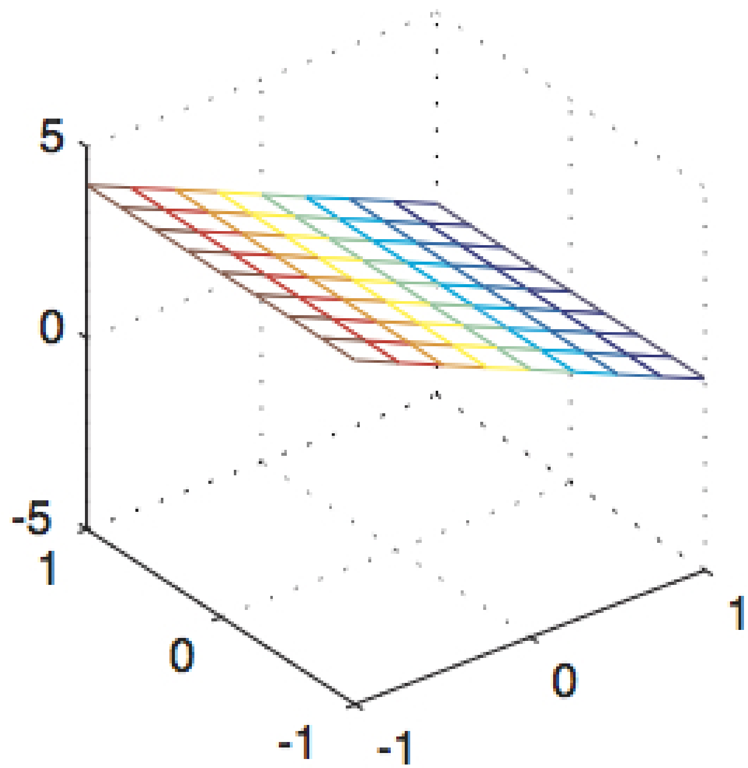
- ♡ New relatively cheap method for pseudospectra is predictive.

What's next?

Poiseuille flow in a finite channel (of length L)





IFISS

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Incompressible Flow & Iterative Solver Software

An open-source software package

Summary

IFISS is a graphical package for the interactive numerical study of incompressible flow problems which can be run under Matlab or Octave. It includes algorithms for discretization by mixed finite element methods and a posteriori error estimation of the computed solutions. The package can also be used as a computational laboratory for experimenting with state-of-the-art preconditioned iterative solvers for the discrete linear equation systems that arise in incompressible flow modelling.

Key Features

Key features include

- implementation of a variety of mixed finite element approximation methods;
- automatic calculation of stabilization parameters where appropriate;
- a posteriori error estimation for steady problems;
- a range of state-of-the-art preconditioned Krylov subspace solvers ;
- built-in geometric and algebraic multigrid solvers and preconditioners;
- fully implicit self-adaptive time stepping algorithms;
- useful visualization tools.

The developers of the IFISS package are David Silvester (School of Mathematics, University of Manchester), Howard Elman (Computer Science Department, University of Maryland), and Alison Ramage (Department of Mathematics and Statistics, University of Strathclyde).

Links

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Overview

Sample output

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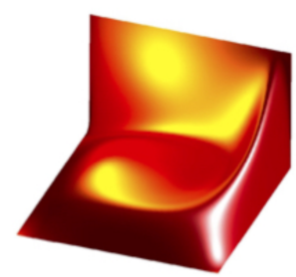

The IFISS logo represents the solution of the *double glazing* convection-diffusion problem. It can be reproduced in IFISS via the function **ifisslogo**.