Balanced truncation model reduction: algorithms and applications

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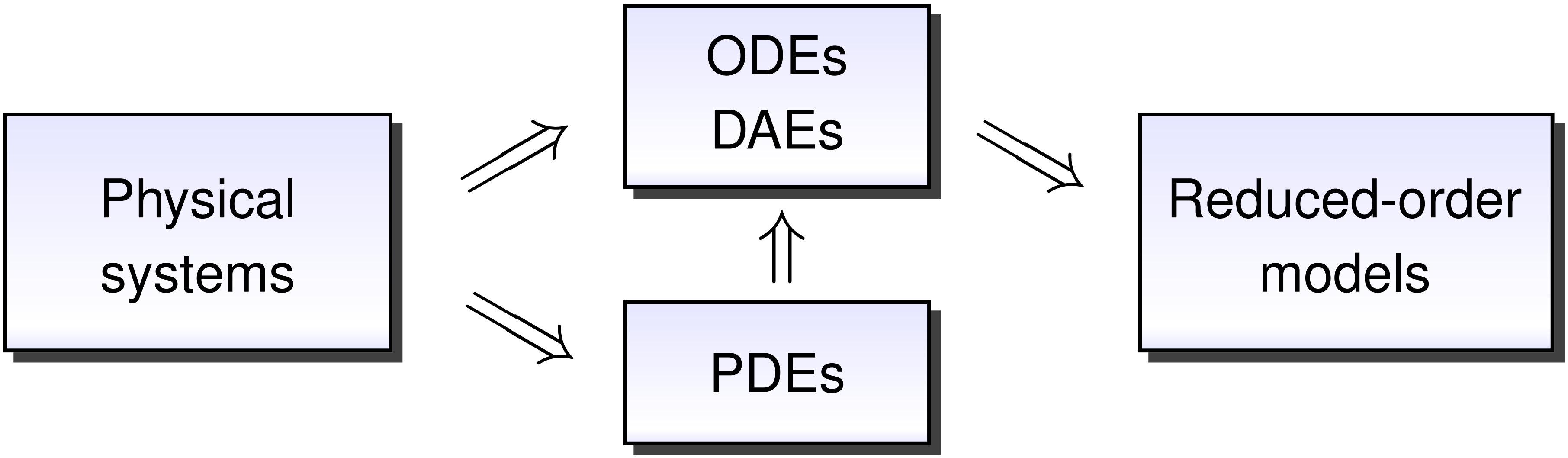
Motivation



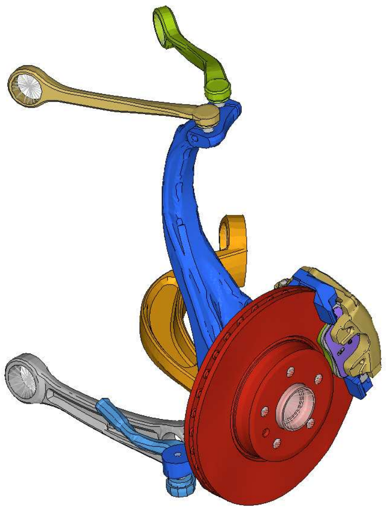
Model reduction ($=$ *dimension reduction*, *order reduction*)

$=$ reduction of the state space dimension

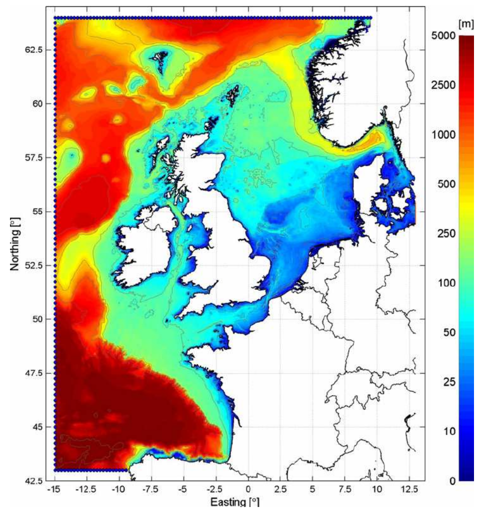
$\Rightarrow$ reduction of computational complexity and storage requirements

Applications

- Circuit simulation and electromagnetics
  (electrical networks, semiconductor devices, power systems, ...)
- Structures, vibrations and acoustics
  (bridges, buildings, machine tools, brake squeal, MEMS, ...)
- Weather prediction and data assimilation
  (North Sea level forecast, Pacific storm tracking, air pollution prediction, ...)
- Biological systems and chemical engineering
  (neural networks, molecular systems, chemical reactions, ...)



[Mehrmann/Schröder'15]



[Altaf/Verlaan/Heemink'12]

[McCaffrey'13]

Outline

Part I

- Model order reduction problem
- Balanced truncation model reduction
- Balancing-related model reduction techniques

Part II

- Balanced truncation for differential-algebraic equations
- Balanced truncation for second-order systems

Part III

- Balanced truncation for parametric systems
- Related topics and open problems

Model reduction problem

Given a large-scale control system

$$u \longrightarrow \boxed{\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) \\ y(t) &= h(x(t), u(t), t) \end{aligned}} \longrightarrow y$$

where $u \in \mathbb{R}^m$ – input, $x \in \mathbb{R}^n$ – state, $y \in \mathbb{R}^p$ – output,

$$f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n, \quad h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p,$$

find a reduced-order model

$$u \longrightarrow \boxed{\begin{aligned} \dot{\widetilde{x}}(t) &= \widetilde{f}(\widetilde{x}(t), u(t), t) \\ \widetilde{y}(t) &= \widetilde{h}(\widetilde{x}(t), u(t), t) \end{aligned}} \longrightarrow \widetilde{y}$$

where $u \in \mathbb{R}^m$, $\widetilde{x} \in \mathbb{R}^\ell$, $\widetilde{y} \in \mathbb{R}^p$, $\ell \ll n$.

Model reduction problem: linear systems

Given a large-scale linear control system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$,

find a reduced-order model

$$\begin{aligned} \dot{\widetilde{x}}(t) &= \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}\,u(t) \\ \widetilde{y}(t) &= \widetilde{C}\,\widetilde{x}(t) + \widetilde{D}\,u(t) \end{aligned}$$

where $\widetilde{A} \in \mathbb{R}^{\ell\times\ell}$, $\widetilde{B} \in \mathbb{R}^{\ell\times m}$, $\widetilde{C} \in \mathbb{R}^{p\times\ell}$, $\widetilde{D} \in \mathbb{R}^{p\times m}$, $\ell \ll n$.

Model reduction problem: linear systems

Laplace transform: $u(t) \mapsto \boldsymbol{u}(s) = \int_0^\infty e^{-st} u(t)\,dt$,

$$x(t) \mapsto \boldsymbol{x}(s), \quad y(t) \mapsto \boldsymbol{y}(s)$$

$$\hookrightarrow \quad \boldsymbol{x}(s) = (sI - A)^{-1} B \boldsymbol{u}(s) + (sI - A)^{-1} x(0)$$

$$\boldsymbol{y}(s) = \big(C(sI - A)^{-1} B + D\big)\, \boldsymbol{u}(s) + C(sI - A)^{-1} x(0)$$

with the transfer function $\boldsymbol{G}(s) = C(sI - A)^{-1} B + D$

Given $\boldsymbol{G}(s) = C(sI - A)^{-1} B + D$ with $A \in \mathbb{R}^{n \times n}$,

find $\widetilde{\boldsymbol{G}}(s) = \widetilde{C}(sI - \widetilde{A})^{-1} \widetilde{B} + \widetilde{D}$ with $\widetilde{A} \in \mathbb{R}^{\ell \times \ell}$, $\ell \ll n$,

such that $\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|$ is small.

Model reduction: goals

- Preserve system properties
  - stability ($\lambda_j(A) \in \mathbb{C}^-$)
  - passivity (= system does not generate energy)
  - contractivity ($\|y\|_{\mathcal{L}_2} \leq \|u\|_{\mathcal{L}_2}$)
  - . . .
- Satisfy desired error tolerance

  $\|\tilde{\boldsymbol{G}} - \boldsymbol{G}\| \leq tol$ or $\|\tilde{y} - y\| \leq tol \cdot \|u\|$ for all $u \in \mathcal{U}$

  $\hookrightarrow$ need for computable error bounds
- Automatic generation of reduced-order models
- Use numerically stable and efficient methods

Approximation error

Fourier transform: $u(t) \mapsto \boldsymbol{u}(i\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} u(t)\, dt, \quad y(t) \mapsto \boldsymbol{y}(i\omega)$

$$\hookrightarrow \quad \boldsymbol{y}(i\omega) = \big(C(i\omega I - A)^{-1}B + D\big)\, \boldsymbol{u}(i\omega) = \boldsymbol{G}(i\omega)\boldsymbol{u}(i\omega)$$

$$\hookrightarrow \quad \|u\|_{\mathcal{L}_2}^2 = \int_{-\infty}^{\infty} \|u(t)\|^2 dt = \|\boldsymbol{u}\|_{\mathcal{L}_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\boldsymbol{u}(i\omega)\|^2 d\omega$$

$$\hookrightarrow \quad \|\boldsymbol{G}\|_{\mathcal{H}_\infty} := \sup_{\boldsymbol{u} \neq 0} \frac{\|\boldsymbol{G}\boldsymbol{u}\|_{\mathcal{L}_2}}{\|\boldsymbol{u}\|_{\mathcal{L}_2}} = \sup_{\omega \in \mathbb{R}} \|\boldsymbol{G}(i\omega)\|_2$$

Approximation error: $\|\widetilde{y} - y\|_{\mathcal{L}_2} = \|\widetilde{\boldsymbol{y}} - \boldsymbol{y}\|_{\mathcal{L}_2} \leq \|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathcal{H}_\infty} \|u\|_{\mathcal{L}_2}$

Approximation by projection

Let $T \in \mathbb{R}^{n\times\ell}$ and $W \in \mathbb{R}^{n\times\ell}$ such that $W^T T = I_\ell$.

- Approximate the state $x(t) \approx T\,\widetilde{x}(t)$ with $\widetilde{x}(t) \in \mathbb{R}^\ell$

$$\hookrightarrow \qquad \begin{aligned} T\,\dot{\widetilde{x}}(t) &= A\,T\,\widetilde{x}(t) + B\,u(t) + \rho(t) \\ \widetilde{y}(t) &= C\,T\,\widetilde{x}(t) + D\,u(t) \end{aligned}$$

- Project the state equation (Petrov-Galerkin projection)

$$\begin{aligned} W^T T\,\dot{\widetilde{x}}(t) &= W^T A\,T\,\widetilde{x}(t) + W^T B\,u(t) \\ \widetilde{y}(t) &= C\,T\,\widetilde{x}(t) + D\,u(t) \end{aligned}$$

- Reduced-order model

$$\begin{aligned} \dot{\widetilde{x}}(t) &= \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}\,u(t) \\ \widetilde{y}(t) &= \widetilde{C}\,\widetilde{x}(t) + \widetilde{D}\,u(t) \end{aligned}$$

with $\widetilde{A} = W^T A\,T$, $\widetilde{B} = W^T B$, $\widetilde{C} = C\,T$, $\widetilde{D} = D$

Outline

- Model order reduction problem
- Balanced truncation model reduction
  - singular value decomposition
  - controllability and observability Gramians
  - Hankel singular values
  - numerical methods for Lyapunov equations
- Balancing-related model reduction techniques
- Balanced truncation for differential-algebraic equations
- Balanced truncation for second-order systems
- Balanced truncation for parametric systems
- Related topics and open problems

SVD-based approximation

Given $X \in \mathbb{R}^{n\times m}$ with $\operatorname{rank} X = r$, find $\widetilde{X} \in \mathbb{R}^{n\times m}$ such that $\operatorname{rank} \widetilde{X} = \ell < r$ and $\|\widetilde{X} - X\|_2 \to \min$.

Singular value decomposition:

$$X = U\Sigma V^T = [\,u_1, \ldots, u_r\,] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} [\,v_1, \ldots, v_r\,]^T$$

$$= \sigma_1 u_1 v_1^T + \ldots + \sigma_\ell u_\ell v_\ell^T + \sigma_{\ell+1} u_{\ell+1} v_{\ell+1}^T + \ldots + \sigma_r u_r v_r^T,$$

where $\sigma_j = \sqrt{\lambda_j(X^T X)} > 0$ are the singular values of X.

$\rightsquigarrow \quad \widetilde{X} = (\sigma_1 u_1) v_1^T + \ldots + (\sigma_\ell u_\ell) v_\ell^T$ with $\|\widetilde{X} - X\|_2 = \sigma_{\ell+1}$

Storage: $X \;\rightsquigarrow\; 4nm$ Bytes , $\widetilde{X} \;\rightsquigarrow\; 4\,(n+m)\ell$ Bytes

Example: image compression with SVD

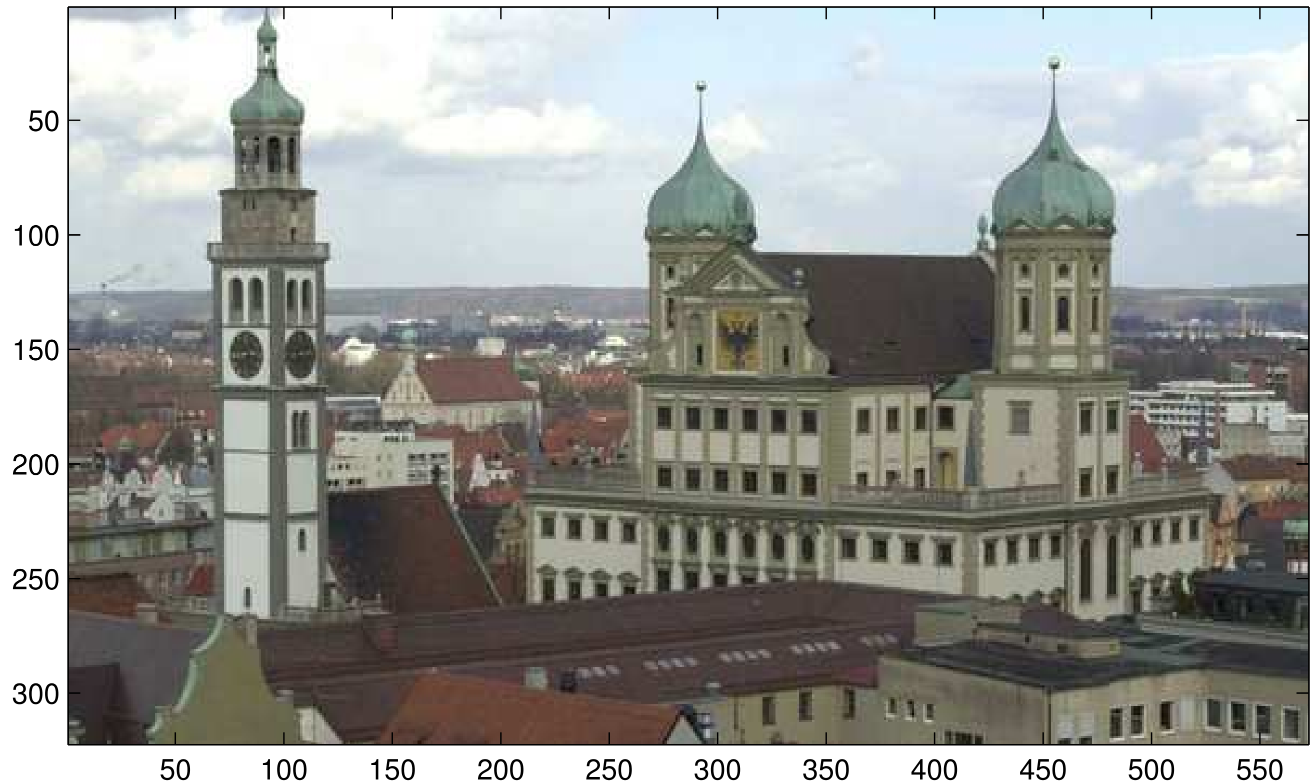


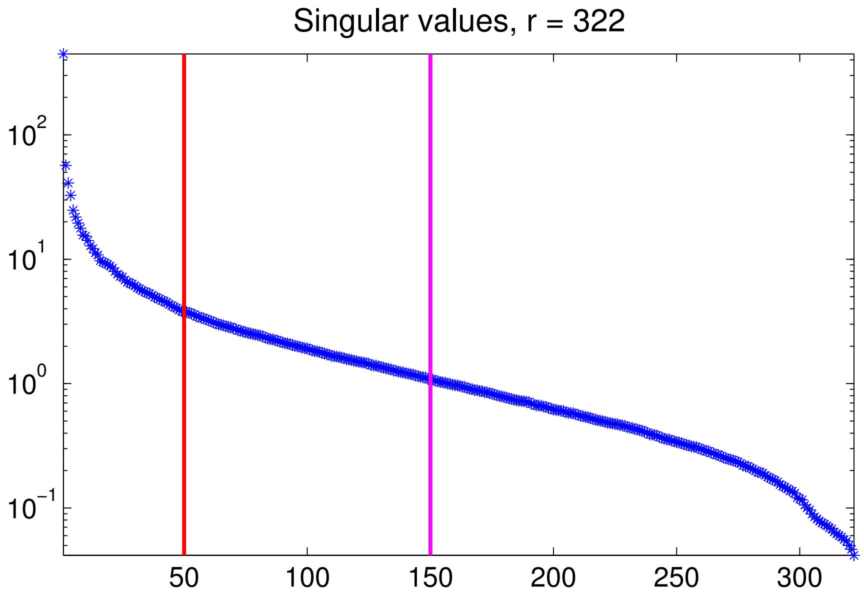
Image $=$ $n \times k$ pixels $=$ k columns with n entries (RGB color values)

$\hookrightarrow$ $n \times k \times 3$ tensor or $n \times 3k$ matrix $X = \begin{pmatrix} * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \end{pmatrix}$

$\hookrightarrow$ storage: $X \rightsquigarrow 12nk$ Bytes (2.11 MB)

Example: image compression with SVD

$322 \times 572 \;\; \rightsquigarrow 2.11$ MB



$\ell = 150 \quad \rightsquigarrow \quad 1.17$ MB

$\ell = 50 \quad \rightsquigarrow \quad 0.39$ MB

Input and output energy

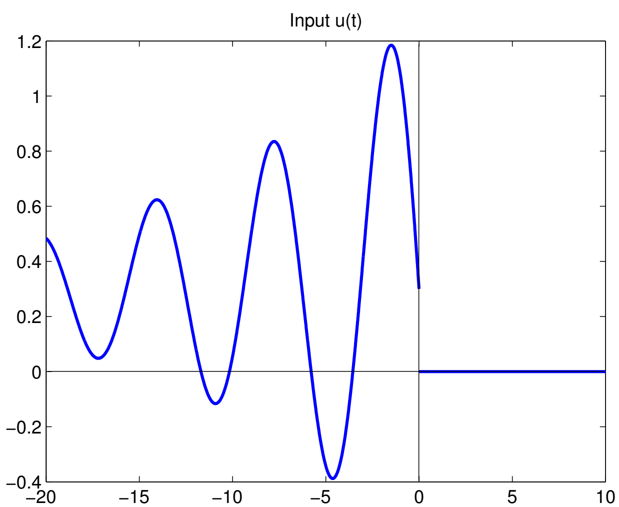
$$\dot{x}(t) = A\,x(t) + B\,u(t), \quad y(t) = C\,x(t)$$

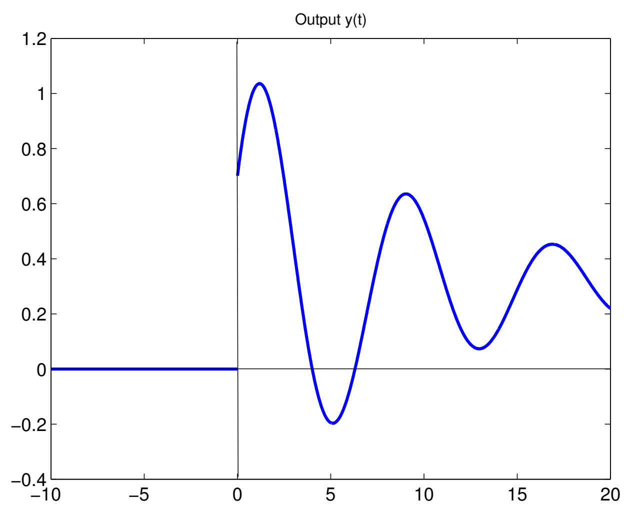
Input energy:

$$E_u(x_0) = \min_{\substack{u \in \mathcal{L}_2(-\infty,0) \\ x(-\infty)=0 \\ x(0)=x_0}} \int_{-\infty}^{0} \|u(t)\|^2\,dt$$

Output energy:

$$E_y(x_0) = \int_{0}^{\infty} \|y(t)\|^2\,dt$$





$$u(t),\ t \in (-\infty,\ 0) \ \Rightarrow\ x(0) = x_0 \ \Rightarrow\ y(t),\ t \in [0,\ \infty)$$

Gramians

Lyapunov equations: ($\lambda_j(A) \in \mathbb{C}^-$)

$A\,X + X\,A^T = -B\,B^T$ $\rightsquigarrow$ X – controllability Gramian

$A^T Y + Y\,A = -C^T C$ $\rightsquigarrow$ Y – observability Gramian

$\hookrightarrow$ $E_u(x_0) = x_0^T\,X^{-1}\,x_0$, $E_y(x_0) = x_0^T\,Y x_0$

- (A, B, C, D) is balanced if $X = Y = \operatorname{diag}(\xi_1, \ldots, \xi_n)$
- $\xi_j = \sqrt{\lambda_j(XY)}$ are Hankel singular values
- $X = RR^T$, $Y = LL^T$ $\hookrightarrow$ $\xi_j = \sigma_j(L^T R)$

Balanced truncation: idea

- Balance the dynamical system

$$(\hat{A},\ \hat{B},\ \hat{C},\ \hat{D}) \;=\; (\hat{T}^{-1}A\hat{T},\ \hat{T}^{-1}B,\ C\hat{T},\ D)$$

$$= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ [\, C_1,\ C_2 \,],\ D \right)$$

$$\hookrightarrow \quad T^{-1}XT^{-T} = T^TYT = \mathrm{diag}(\, \xi_1,\ \ldots,\ \xi_\ell,\ \xi_{\ell+1},\ \ldots,\ \xi_n \,)$$

- Truncate the states corresponding to small Hankel singular values

$$\hookrightarrow \quad (\widetilde{A},\ \widetilde{B},\ \widetilde{C},\ \widetilde{D}) = (\, A_{11},\ B_1,\ C_1,\ D \,)$$

[Mullis/Roberts'76, Moore'81]

Balanced truncation algorithm

1. Compute $X = RR^T$ and $Y = LL^T$.

2. Compute the SVD $L^TR = [\,U_1,\; U_2\,]\begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}[\,V_1,\; V_2\,]^T$,

   with $\Sigma_1 = \mathrm{diag}(\xi_1, \ldots, \xi_\ell)$, $\quad \Sigma_2 = \mathrm{diag}(\xi_{\ell+1}, \ldots, \xi_n)$.

3. Compute the reduced-order model

   $$(\,\widetilde{A},\; \widetilde{B},\; \widetilde{C},\; \widetilde{D}\,) = (\,W^TAT,\; W^TB,\; CT,\; D\,)$$

   with $W = LU_1\Sigma_1^{-1/2} \in \mathbb{R}^{n\times\ell}$, $\quad T = RV_1\Sigma_1^{-1/2} \in \mathbb{R}^{n\times\ell}$.

Properties

- $(\,\widetilde{A},\, \widetilde{B},\, \widetilde{C},\, \widetilde{D}\,)$ is asymptotically stable [Pernebo/Silvermann'82]
- error bound: $\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathcal{H}_\infty} \leq 2(\xi_{\ell+1} + \ldots + \xi_n)$ [Enns'84, Glover'84]
- need to solve large-scale Lyapunov equations

Numerical methods for Lyapunov equations

$$AX + XA^T = -BB^T \quad \rightsquigarrow \quad X = RR^T$$

$$A^TY + YA = -C^TC \quad \rightsquigarrow \quad Y = LL^T$$

- Hammarling method [Hammarling'86, Penzl'98]
  (small, dense)
- Sign function method [Roberts'71, Byers'87, Larin/Aliev'93, Benner/Quintana-Ortí'99]
  (medium, dense)
- $\mathcal{H}$-matrices based methods [Grasedyck/Hackbush/Khoromskij'03, Benner/Baur'04]
  (large, dense+structure / sparse)
- Krylov subspace methods [Saad'90, Jaimoukha/Kasenally'94, Simoncini'06]
  (large, sparse)
- Alternating direction implicit (ADI) method [Wachspress'88, Penzl'99, Li/White'02, Benner/Kürschner/Saak'14]
  (large, sparse)

ADI method

$$(A+\tau_k I)X_{k-1/2} = -BB^T - X_{k-1}(A-\tau_k I)^T$$
$$(A+\overline{\tau}_k I)\ X_k^T = -BB^T - X_{k-1/2}^T(A-\overline{\tau}_k I)^T$$

- $\lim\limits_{k\to\infty} X_k = X$ with $X - X_k = \mathcal{A}_k X \mathcal{A}_k^*$, where

  $\mathcal{A}_k = (A+\tau_1 I)^{-1}(A-\tau_1 I)\cdot\ldots\cdot(A+\tau_k I)^{-1}(A-\tau_k I), \quad \tau_j \in \mathbb{C}^-$

- optimal shift parameters: [Wachspress'88]

  $$\{\tau_1,\ldots,\tau_k\} = \operatorname*{arg\,min}_{\tau_1,\ldots,\tau_k\in\mathbb{C}^-} \max_{t\in\mathrm{Sp(A)}} \frac{|(t-\tau_1)\cdot\ldots\cdot(t-\tau_k)|}{|(t+\tau_1)\cdot\ldots\cdot(t+\tau_k)|}$$

- suboptimal shift parameters [Penzl'99]

  $$\{\tau_1,\ldots,\tau_k\} = \operatorname*{arg\,min}_{\tau_1,\ldots,\tau_k\in\mathbb{C}^-} \max_{t\in\mathcal{R}_+\cup(1/\mathcal{R}_-)} \frac{|(t-\tau_1)\cdot\ldots\cdot(t-\tau_k)|}{|(t+\tau_1)\cdot\ldots\cdot(t+\tau_k)|},$$

  where $\mathcal{R}_+$ and $\mathcal{R}_-$ are the sets of Ritz values of A and A^{-1}

- X_k is symmetric, positive semidefinite $\hookrightarrow$ $X_k = Z_k Z_k^T$

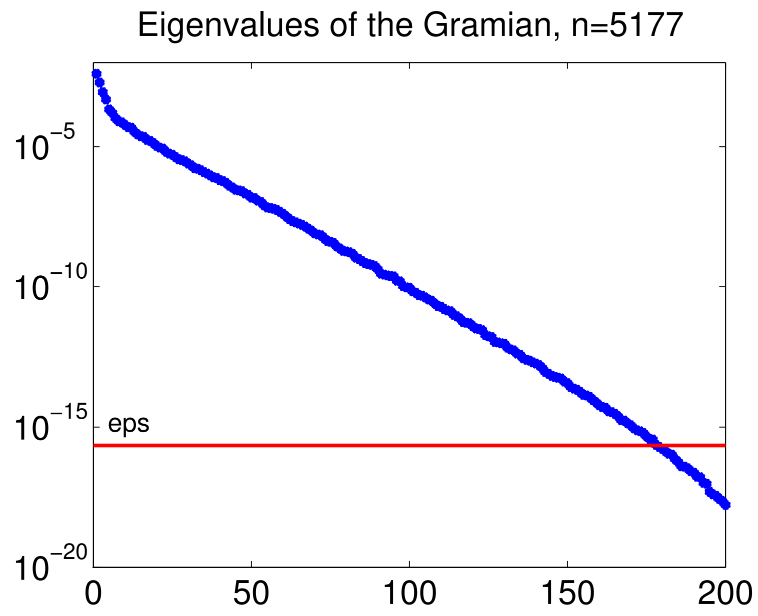
Low-rank approximations

Lyapunov equation: $A\,X + X\,A^T = -BB^T$

$$X = \sum_{j=1}^{n} \lambda_j(X) v_j v_j^T = RR^T, \quad R \in \mathbb{R}^{n\times n}$$

$$\Downarrow \quad \lambda_j(X) \approx 0, \quad j = r+1, \ldots, n$$

$$X \approx \sum_{j=1}^{r} \lambda_j(X) v_j v_j^T = \widetilde{R}\widetilde{R}^T, \quad \widetilde{R} \in \mathbb{R}^{n\times r}$$



$\hookrightarrow$ compute a low-rank approximation to X

Low-rank ADI method

$$V_0 = B, \qquad Z_0 = [\,], \qquad k = 1,$$

while $\|V_{k-1}^T V_{k-1}\|_F \geq tol\, \|B^T B\|_F$

$$F_k = (A + \tau_k I)^{-1} V_{k-1},$$

$$V_k = V_{k-1} - 2\mathrm{Re}(\tau_k) F_k,$$

$$Z_k = [\, Z_{k-1}, \quad \sqrt{-2\mathrm{Re}(\tau_k)}\, F_k \,],$$

$$k \leftarrow k + 1$$

end

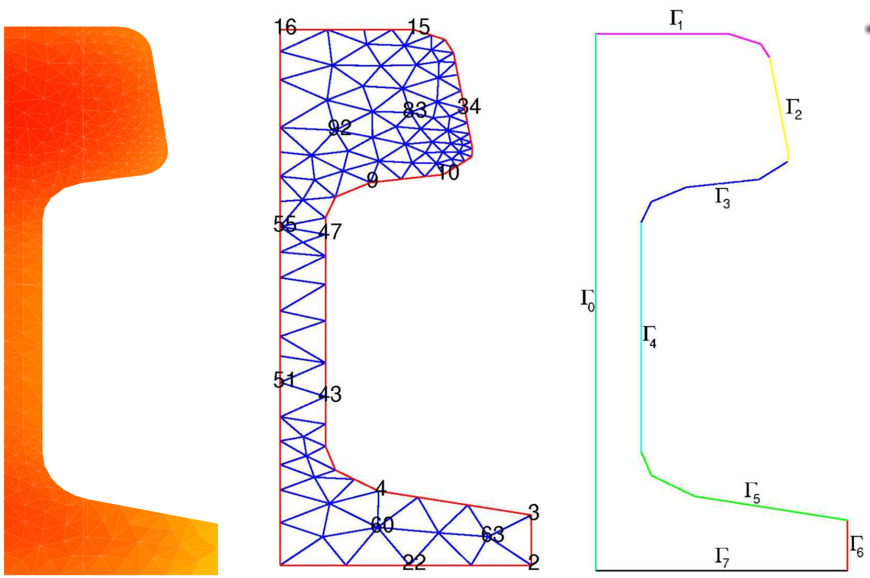
- low-rank approximation $X \approx Z_k Z_k^T$ with $Z_k \in \mathbb{R}^{n \times km}$
- solve linear systems $(A + \tau_k I) z = v$
- low-rank residuals $A Z_k Z_k^T + Z_k Z_k^T A^T + B B^T = V_k V_k^T$ with $V_k \in \mathbb{R}^{n \times k}$ $\hookrightarrow$ fast stopping criterion
- adaptive ADI shift computation

[Benner/Kürschner/Saak'14]

Example: optimal steel cooling

- Mathematical model

$$\begin{aligned}
\partial_t \theta &= \frac{\lambda}{c\,\rho}\Delta\theta && \text{in} \quad \Omega \times (0,T)\\
\partial_\nu \theta &= \frac{q_k}{\lambda}(u_k - \theta) && \text{on} \quad \Gamma_{k,\, k=1,\ldots,7}\\
\partial_\nu \theta &= 0 && \text{on} \quad \Gamma_0
\end{aligned}$$



- FEM model

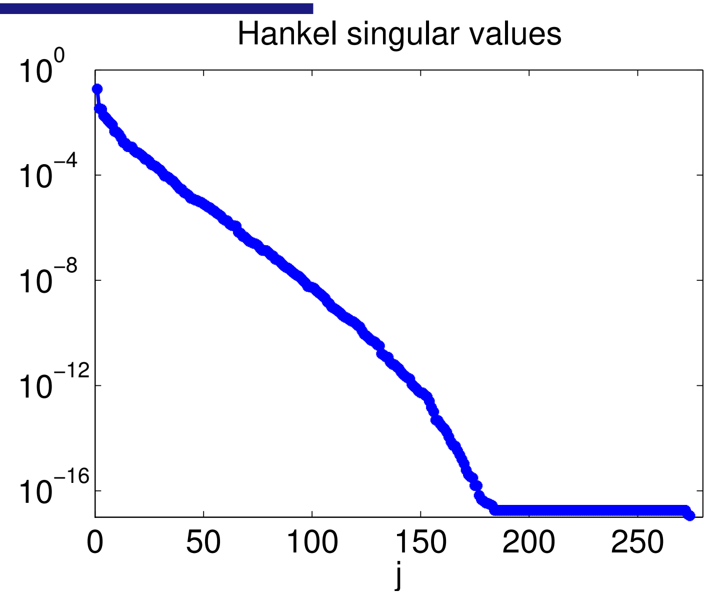
$$\begin{aligned}
E\,\dot{\theta}_h &= A\,\theta_h + B\,u, \qquad \theta_h \in \mathbb{R}^n\\
y &= C\,\theta_h
\end{aligned}$$

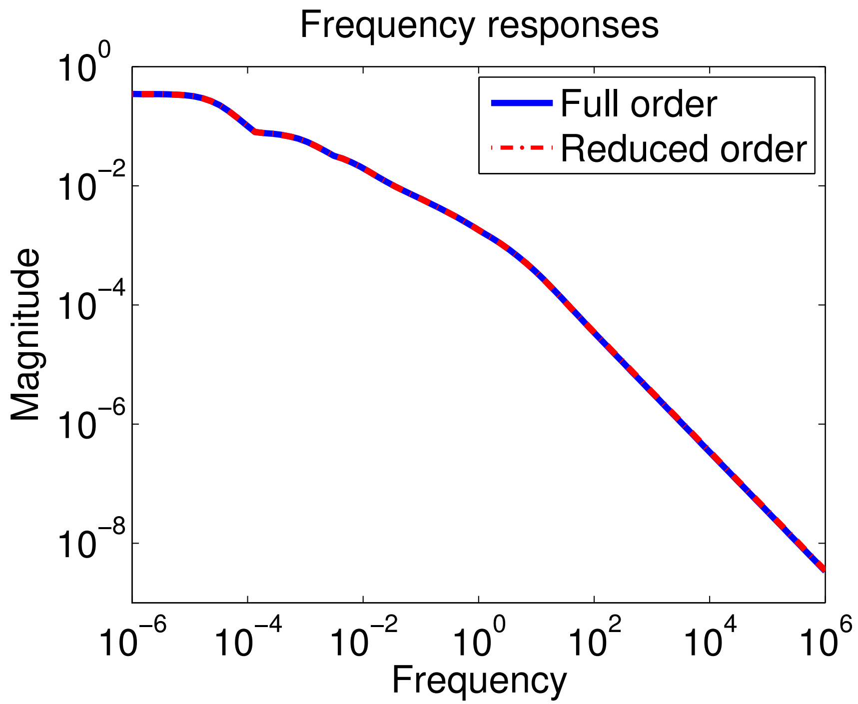
with $n = 1357 \,/\, 20209 \,/\, 79841 \,/\, \ldots$

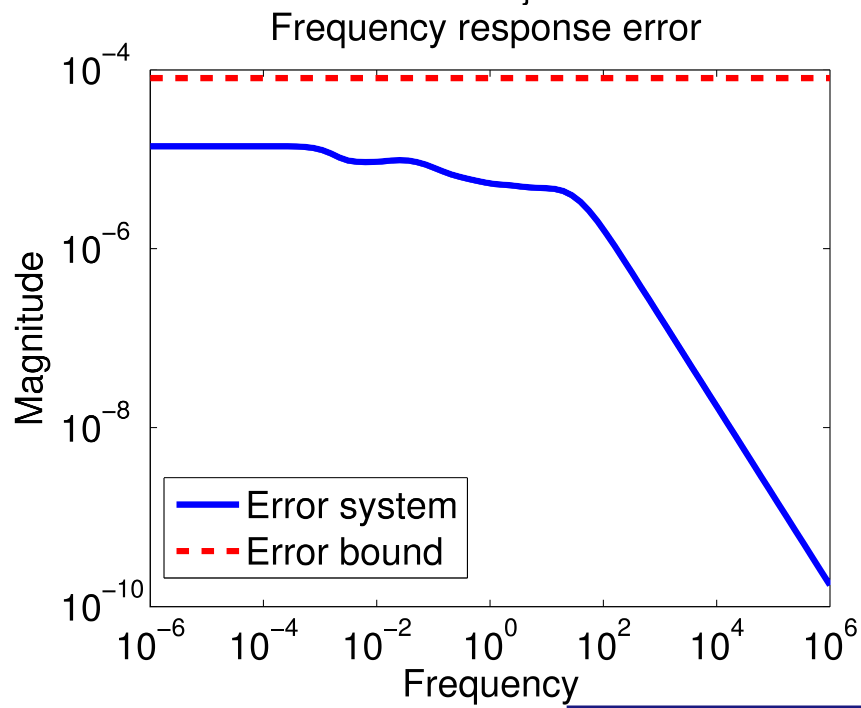
[Oberwolfach Benchmark Collection]

Example: optimal steel cooling

- $n = 20209, \;\; m = 7, \;\; p = 6$
- $X \approx \widetilde{R}\widetilde{R}^T, \quad \widetilde{R} \in \mathbb{R}^{n\times 357}$
- $Y \approx \widetilde{L}\widetilde{L}^T, \quad \widetilde{L} \in \mathbb{R}^{n\times 276}$
- Reduced system: $\ell = 52$







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- Model order reduction problem
- Balanced truncation model reduction
- Balancing-related model reduction techniques
  - positive real balanced truncation
  - bounded real balanced truncation
  - numerical methods for Riccati equations
- Model reduction of differential-algebraic equations
- Balanced truncation for second-order systems
- Balanced truncation for parametric systems
- Related topics and open problems

Positive real balanced truncation

- System is passive $\iff$ $\boldsymbol{G}(s)$ is positive real
  i.e., $\boldsymbol{G}(s) + \boldsymbol{G}^*(s) \geq 0$ for all $s \in \mathbb{C}^+$
- Positive real Gramians $X_{\rm PR}$ and $Y_{\rm PR}$ are stabilizing solutions of the algebraic Riccati equations

$$AX + XA^T + (XC^T - B)(D + D^T)^{-1}(XC^T - B)^T = 0,$$

$$A^TY + YA + (B^TY - C)^T(D + D^T)^{-1}(B^TY - C) = 0.$$

- $\xi_j^{\rm PR} = \sqrt{\lambda_j(X_{\rm PR}Y_{\rm PR})}$ are positive real characteristic values

$\hookrightarrow$ error bound: $\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathcal{H}_\infty} \leq c\left(\xi_{\ell+1}^{\rm PR} + \ldots + \xi_n^{\rm PR}\right)$

with $c = 2\,\|(D+D^T)^{-1}\|_2\|\boldsymbol{G} + D^T\|_{\mathcal{H}_\infty}\|\widetilde{\boldsymbol{G}} + D^T\|_{\mathcal{H}_\infty}$

$\hookrightarrow$ passivity is preserved

[Green'88, Ober'91]

Bounded real balanced truncation

- System is contractive $\iff$ $\boldsymbol{G}(s)$ is bounded real
  i.e., $I - \boldsymbol{G}^*(s)\boldsymbol{G}(s) \geq 0$ for all $s \in \mathbb{C}^+$

- Bounded real Gramians $X_{\rm BR}$ and $Y_{\rm BR}$ are stabilizing solutions of the algebraic Riccati equations

$$AX + XA^T + (XC^T + BD^T)(I - DD^T)^{-1}(XC^T + BD^T)^T = 0,$$

$$A^TY + YA + (B^TY + D^TC)^T(I - D^TD)^{-1}(B^TY + D^TC) = 0.$$

- $\xi_j^{\textsf{BR}} = \sqrt{\lambda_j(X_{\rm BR}Y_{\rm BR})}$ are bounded real characteristic values

  $\hookrightarrow$ error bound: $\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathcal{H}_\infty} \leq 2\left(\xi_{\ell+1}^{\textsf{BR}} + \ldots + \xi_n^{\textsf{BR}}\right)$

  $\hookrightarrow$ contractivity is preserved

[Opdenacker/Jonckheere'88, Ober'91]

Numerical methods for Riccati equations

Riccati equation: $BB^T + AX + XA^T \pm XC^TCX = 0 \quad \rightsquigarrow \quad X \approx \widetilde{R}\widetilde{R}^T$

- Newton's method [Kleinman'68, ..., Benner/Kürschner/Saak'16]
- Sign function method [Roberts'80, Byers'87, Benner/Quintana-Ortí'99]
- $\mathcal{H}$-matrices based methods [Grasedyck/Hackbush/Khoromskij'03]
- Structured doubling algorithm [Li/Chu/Lin/Weng'13]
- Structured invariant subspace methods [Paige/Van Loan'81, Benner/Mehrmann/Xu'98, Kressner'05, ...]
- ADI-type methods [Wong/Balakrishnan'05, Benner/Bujanović/Kürschner/Saak'17]
- Low-rank subspace iteration method [Amodei and Buchot'10, Lin/Simoncini'15, Massoudi/Opmeer/Reis'16]
- Krylov subspace methods [Jaimoukha/Kasenally'94, Heyouni/Jbilou'08, Simoncini'16]

Conclusions

- Balanced truncation for continuous-time systems
  - energy interpretation
  - system-theoretic properties are preserved
  - global computable error bounds
  - using modern numerical linear algebra algorithms for solving large-scale Lyapunov and Riccati equations
- Balanced truncation for discrete-time systems

$$\begin{aligned} E\,x_{k+1} &= A\,x_k + B\,u_k \\ y_k &= C\,x_k + D\,u_k \end{aligned}$$

[Al-Saggaf'86]

  - Gramians satisfy the discrete-time Lyapunov equations

    $$AXA^T - X = -BB^T, \qquad A^T Y A - Y = -C^T C,$$

    which can be solved by the squared Smith method [Smith'68]
  - error bound: $\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathcal{H}_\infty} \le 2\,(\xi_{\ell+1} + \ldots + \xi_n)$ [Hinrichsen/Pritchard'90]

Conclusions

- Other balancing-related model reduction techniques
  - linear-quadratic Gaussian truncation [Jonckeere/Silverman'83]
  - stochastic balanced truncation [Desai/Pal'88, Green'88]
  - frequency weighted balanced truncation [Enns'84, Zhou'95]
  - fractional balanced truncation [Ober/McFarlane'88, Meyer'90]
  - Cross-Gramian balanced truncation [Fernando/Nicholson'84]

- Balanced truncation for systems with many inputs **or** outputs [Benner/Schneider'10]

Outline

Part I

- Model order reduction problem
- Balanced truncation model reduction
- Balancing-related model reduction techniques

Part II

- Balanced truncation for differential-algebraic equations
- Balanced truncation for second-order systems

Part III

- Balanced truncation for parametric systems
- Related topics and open problems

Balanced truncation

Idea: Balance the system (A, B, C, D) and truncate the states corresponding to small Hankel singular values

Algorithm:

1. Solve the Lyapunov equations

$$A\,X + X\,A^T = -BB^T, \qquad A^T Y + Y\,A = -C^T C$$

for $X \approx \widetilde{R}\widetilde{R}^T$ and $Y \approx \widetilde{L}\widetilde{L}^T$.

2. Compute the SVD $\widetilde{L}^T \widetilde{R} = [\,U_1,\; U_2\,]\begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}[\,V_1,\; V_2\,]^T$,

with $\Sigma_1 = \mathrm{diag}(\xi_1, \ldots, \xi_\ell)$, $\Sigma_2 = \mathrm{diag}(\xi_{\ell+1}, \ldots, \xi_n)$.

3. $(\,\widetilde{A},\; \widetilde{B},\; \widetilde{C},\; \widetilde{D}\,) = (\,W^T A T,\; W^T B,\; CT,\; D\,)$ with

$$W = \widetilde{L} U_1 \Sigma_1^{-1/2} \in \mathbb{R}^{n\times\ell}, \qquad T = \widetilde{R} V_1 \Sigma_1^{-1/2} \in \mathbb{R}^{n\times\ell}.$$

Outline

- Model order reduction problem
- Balanced truncation model reduction
- Balancing-related model reduction techniques
- Balanced truncation for differential-algebraic equations
  - properties of DAEs
  - proper and improper Gramians
  - proper and improper Hankel singular values
  - numerical methods for projected Lyapunov equations
- Balanced truncation for second-order systems
- Balanced truncation for parametric systems
- Related topics and open problems

Linear DAE control systems

Time domain representation

u

$$
\begin{aligned}
E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\
y(t) &= C\,x(t) + D u(t)
\end{aligned}
$$

y

where $E,\, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$,

$\lambda E - A$ is regular $(\det(\lambda E - A) \not\equiv 0)$.

Frequency domain representation

Laplace transform: $u(t) \mapsto \boldsymbol{u}(s)$, $y(t) \mapsto \boldsymbol{y}(s)$

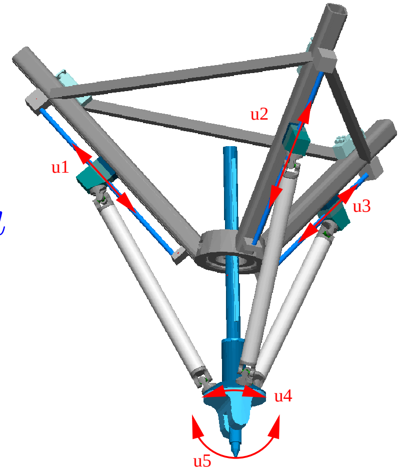
$\hookrightarrow$ $\boldsymbol{y}(s) = \big(C(sE - A)^{-1}B + D\big)\boldsymbol{u}(s) + C(sE - A)^{-1}Ex(0)$

with the transfer function $\boldsymbol{G}(s) = C(sE - A)^{-1}B + D$

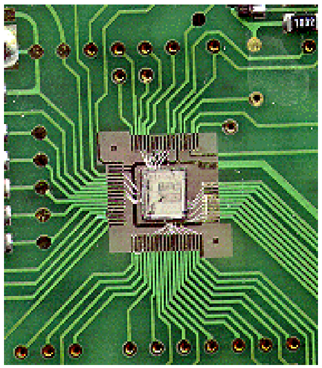
Applications

- Multibody systems with constraints

$$\begin{bmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{v}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ K & D & -G^T \\ G & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{p} \\ \mathbf{v} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \\ B_3 \end{bmatrix} u$$

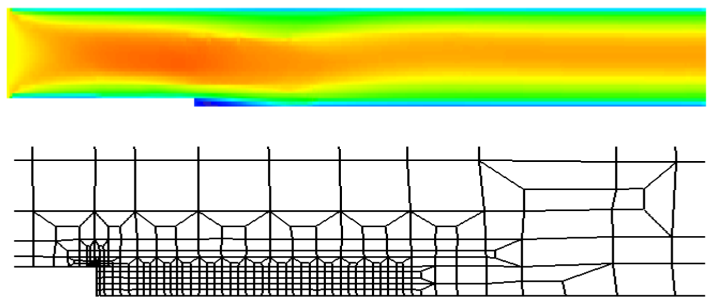


- Electrical circuits



$$\begin{bmatrix} A_C \mathcal{C} A_C^T & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\mathbf{e}} \\ \dot{\mathbf{j}}_L \\ \dot{\mathbf{j}}_V \end{bmatrix} = \begin{bmatrix} -A_R R^{-1} A_R^T & -A_L^T & -A_V^T \\ A_L^T & 0 & 0 \\ A_V^T & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{e} \\ \mathbf{j}_L \\ \mathbf{j}_V \end{bmatrix} - \begin{bmatrix} A_I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} i_V \\ v_I \end{bmatrix}$$

- Semidiscretized Stokes equation



$$\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{v} \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

DAEs are not ODEs! [Petzold'82]

- DAEs may have no solutions or solution may be nonunique
- Initial conditions $x(0) = x_0$ should be consistent
  $\rightsquigarrow$ distributional solutions
- Control $u(t)$ should be sufficiently smooth
  $\rightsquigarrow$ distributional solutions
- Drift off effects may occur in the numerical solution
- Index concepts:
  differentiation index, geometric index, perturbation index, strangeness index, structural index, tractability index, unsolvability index, ...

Model reduction problem

Given a large-scale DAE control system

$$u \longrightarrow \boxed{\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}} \longrightarrow y$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$,

find a reduced-order model

$$u \longrightarrow \boxed{\begin{aligned} \widetilde{E}\,\dot{\widetilde{x}}(t) &= \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}\,u(t) \\ \widetilde{y}(t) &= \widetilde{C}\,\widetilde{x}(t) + \widetilde{D}\,u(t) \end{aligned}} \longrightarrow \widetilde{y}$$

where $\widetilde{E}, \widetilde{A} \in \mathbb{R}^{\ell\times \ell}$, $\widetilde{B} \in \mathbb{R}^{\ell\times m}$, $\widetilde{C} \in \mathbb{R}^{p\times \ell}$, $\widetilde{D} \in \mathbb{R}^{p\times m}$, $\ell \ll n$.

Decoupling of DAEs

Weierstraß canonical form:

$$E = T_l \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} T_r, \qquad A = T_l \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} T_r,$$

where J – Jordan block ($\lambda_j(J)$ are finite eigenvalues of $\lambda E - A$),
N – nilpotent ($N^{\nu-1} \neq 0$, $N^\nu = 0$ $\leadsto$ ν is index of $\lambda E - A$).

Slow subsystem	Fast subsystem
$\dot{x}_1(t) = J\,x_1(t) + B_1 u(t)$	$N\dot{x}_2(t) = x_2(t) + B_2 u(t)$
$y_1(t) = C_1 x_1(t)$	$y_2(t) = C_2 x_2(t) + D\,u(t)$
$\Rightarrow\ x_1(t) = e^{Jt}x_1(0) + \int_0^t e^{J(t-\tau)}B_1 u(\tau)\,d\tau$	$\Rightarrow\ x_2(t) = -\sum_{k=0}^{\nu-1} N^k B_2 u^{(k)}(t)$

Idea: define the controllability and observability Gramains for each subsystem and reduce the subsystems separately.

Proper and improper Gramians

Consider the projectors

$$P_r = T_r^{-1}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_r, \qquad P_l = T_l\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_l^{-1}, \qquad \begin{aligned} Q_r &= I - P_r, \\ Q_l &= I - P_l. \end{aligned}$$

- The proper controllability and observability Gramians solve the projected continuous-time Lyapunov equations

$$E\,\mathcal{G}_{pc}A^T + A\,\mathcal{G}_{pc}E^T = -P_lBB^TP_l^T, \qquad \mathcal{G}_{pc} = P_r\mathcal{G}_{pc}P_r^T,$$
$$E^T\mathcal{G}_{po}\,A + A^T\mathcal{G}_{po}E = -P_r^TC^TCP_r, \qquad \mathcal{G}_{po} = P_l^T\mathcal{G}_{po}P_l.$$

- The improper controllability and observability Gramians solve the projected discrete-time Lyapunov equations

$$A\,\mathcal{G}_{ic}\,A^T - E\,\mathcal{G}_{ic}E^T = Q_lBB^TQ_l^T, \qquad \mathcal{G}_{ic} = Q_r\mathcal{G}_{ic}Q_r^T,$$
$$A^T\mathcal{G}_{io}\,A - E^T\mathcal{G}_{io}\,E = Q_r^TC^TCQ_r, \qquad \mathcal{G}_{io} = Q_l^T\mathcal{G}_{io}Q_l.$$

Balanced truncation for DAEs

- $\boldsymbol{G} = (\,E,\ A,\ B,\ C,\ D\,)$ is balanced, if the Gramians satisfy

$$\mathcal{G}_{pc} = \mathcal{G}_{po} = \begin{bmatrix} \Sigma & \\ & 0 \end{bmatrix}, \qquad \mathcal{G}_{ic} = \mathcal{G}_{io} = \begin{bmatrix} 0 & \\ & \Theta \end{bmatrix}$$

with $\Sigma = \mathrm{diag}(\xi_1, \ldots, \xi_{n_f})$ and $\Theta = \mathrm{diag}(\theta_1, \ldots, \theta_{n_\infty})$.

- $\xi_j = \sqrt{\lambda_j(\mathcal{G}_{pc} E^T \mathcal{G}_{po} E)}$ are the proper Hankel singular values

  $\theta_j = \sqrt{\lambda_j(\mathcal{G}_{ic} A^T \mathcal{G}_{io} A)}$ are the improper Hankel singular values

Idea: balance the system and truncate the states corresponding to small proper and zero improper Hankel singular values.

Example

$$N\dot{x}(t) = x(t) + Bu(t) \qquad \text{with} \quad N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 0.1 \\ 0 \end{bmatrix}, \quad C^T = \begin{bmatrix} 0.04 \\ 30 \\ 1 \end{bmatrix}$$
$$y(t) = Cx(t)$$

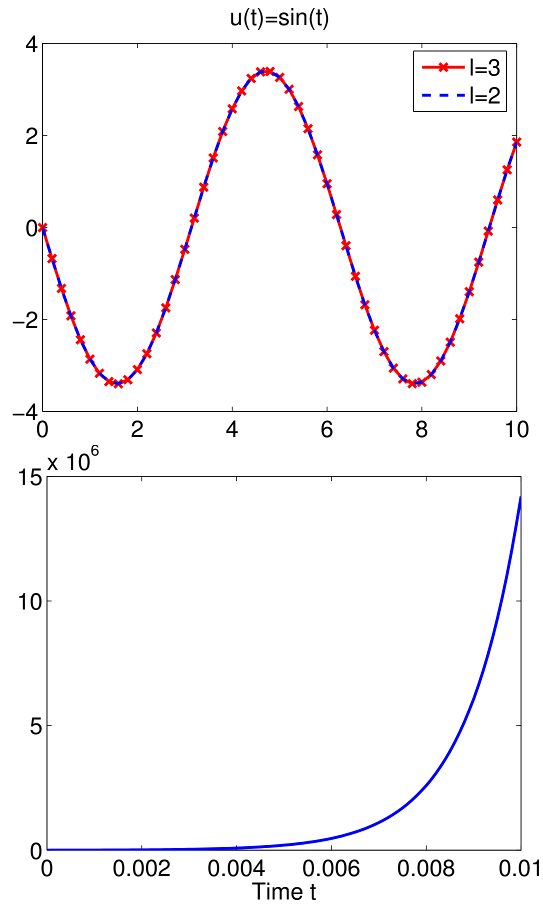
Improper Hankel singular values $\theta_1 = 3.4, \ \theta_2 = 4.7 \cdot 10^{-6}, \ \theta_3 = 0$

- Reduced-order system: $\ell = 2$

$$\begin{bmatrix} 1.2 & 1.2 \\ -1.2 & -1.2 \end{bmatrix} \dot{\tilde{x}}(t) = \begin{bmatrix} 10^3 & 0 \\ 0 & 10^3 \end{bmatrix} \tilde{x}(t) + \tilde{B}u(t)$$
$$\tilde{y}(t) = \tilde{C}\,\tilde{x}(t)$$

- Reduced-order system: $\ell = 1$

$$\dot{\tilde{x}}(t) = 850\,\tilde{x}(t) + 1567u(t)$$
$$\tilde{y}(t) = 1.9\,\tilde{x}(t)$$



Balanced truncation for DAEs

1. Solve the projected Lyapunov equations for

$$\mathcal{G}_{pc} = R_p R_p^T, \quad \mathcal{G}_{po} = L_p L_p^T, \quad \mathcal{G}_{ic} = R_i R_i^T, \quad \mathcal{G}_{io} = L_i L_i^T;$$

2. Compute the SVD

$$L_p^T E R_p = [U_1, \, U_2] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} [V_1, \, V_2]^T;$$

3. Compute the SVD

$$L_i^T A R_i = [U_3, \, U_4] \begin{bmatrix} \Theta & \\ & 0 \end{bmatrix} [V_3, \, V_4]^T;$$

4. $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (W^T E T, \, W^T A T, \, W^T B, \, C T, \, D)$ with

$$W = [\, L_p U_1 \Sigma_1^{-1/2}, \, L_i U_3 \Theta^{-1/2}], \quad T = [\, R_p V_1 \Sigma_1^{-1/2}, \, R_i V_3 \Theta^{-1/2}\,].$$

Balanced truncation: properties

- Asymptotic stability is preserved
- Error bound:
  - $\boldsymbol{G}(s) = C(sE - A)^{-1}B + D = \boldsymbol{G}_{\rm sp}(s) + \boldsymbol{P}(s)$,

    where $\boldsymbol{G}_{\rm sp}(s) = C_1(sI - J)^{-1}B_1$ is strictly proper,

    $$\boldsymbol{P}(s) = C_2(sN - I)^{-1}B_2 + D = -\sum_{k=0}^{\nu-1} C_2 N^k B_2 s^k + D$$
  - $\widetilde{\boldsymbol{G}}(s) = \widetilde{C}(s\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D} = \widetilde{\boldsymbol{G}}_{\rm sp}(s) + \boldsymbol{P}(s)$

    $\hookrightarrow \quad \|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathcal{H}_\infty} \leq 2(\xi_{\ell_f} + \ldots + \xi_{n_f})$
- $\mathrm{Index}(\widetilde{E}, \widetilde{A}) \leq \mathrm{Index}(E, A)$

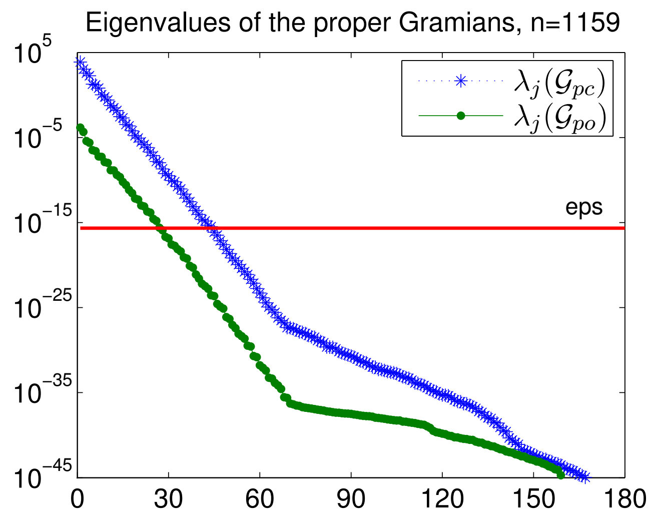
Computing the Gramians

- Instead of the proper Gramians compute their low-rank approximations

  $$\mathcal{G}_{pc} \approx \tilde{R}_p \tilde{R}_p^T \quad \text{and} \quad \mathcal{G}_{po} \approx \tilde{L}_p \tilde{L}_p^T$$

  with $\tilde{R}_p \in \mathbb{R}^{n \times r_{pc}}$, $\tilde{L}_p \in \mathbb{R}^{n \times r_{po}}$, $r_{pc}, r_{po} \ll n$

  $\hookrightarrow$ use the generalized ADI method [St.'08]



- Since $r_{ic} = \text{rank}\,(\mathcal{G}_{ic}) \leq \nu m$ and $r_{io} = \text{rank}\,(\mathcal{G}_{io}) \leq \nu q$, compute the full-rank factors of the improper Gramians

  $$\mathcal{G}_{ic} = R_i R_i^T, \; R_i \in \mathbb{R}^{n \times r_{ic}} \quad \text{and} \quad \mathcal{G}_{io} = L_i L_i^T, \; L_i \in \mathbb{R}^{n \times r_{io}}$$

  $\hookrightarrow$ use the generalized Smith method [St.'08]

- Projectors P_r and P_l are required

  $\hookrightarrow$ exploit the structure of the matrices E and A

Computing the projectors

[✓] semi-explicit systems (index 1) [St.'08]

$$E = \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

[✓] Stokes-like systems (index 2)

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

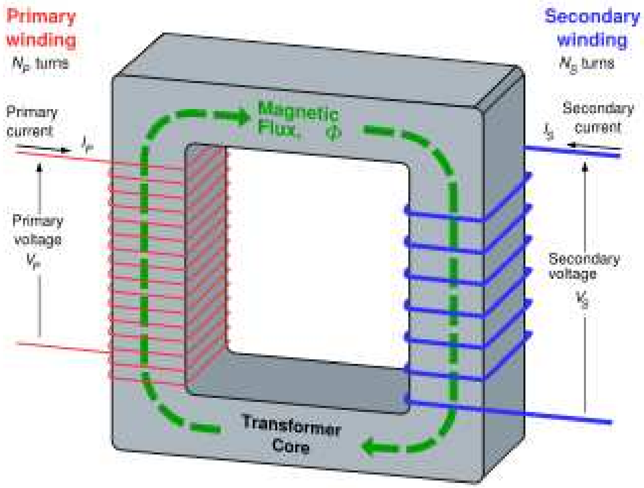
[✓] mechanical systems (index 3)

$$E = \begin{bmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I & 0 \\ D & K & -G^T \\ G & 0 & 0 \end{bmatrix}$$

[✓] electrical circuits (index 1 and 2) [Reis/St.'10,'11]

Remark: For some problems, the explicit computation of the projectors can be avoided [Heinkenschloss/Sorensen/Sun'08, Freitas/Rommes/Martins'08]

Example: one-phase transformer



- Mathematical model

$$\begin{aligned}
\sigma\frac{\partial A}{\partial t} + \nabla\times(\nu_{\rm ir}\nabla\times A) &= 0 && \text{in } \Omega_{\rm ir}\times(0,T)\\
\nabla\times(\nu_{ca}\nabla\times A) &= \omega\, i && \text{in } \Omega_{\rm c}\cup\Omega_{\rm a}\times(0,T)\\
\int_\Omega \omega^T\frac{\partial}{\partial t}A\,dz + R\,i &= u && \text{in } (0,T)\\
A\times n &= 0 && \text{on } \partial\Omega\times(0,T)\\
A &= A_0 && \text{in } \Omega_{\rm ir}
\end{aligned}$$

- FEM model

$$\begin{bmatrix} M_{11} & 0 & 0\\ 0 & 0 & 0\\ X_1^T & X_2^T & 0\end{bmatrix}\frac{d}{dt}\begin{bmatrix}a_1\\a_2\\i\end{bmatrix} = \begin{bmatrix} -K_{11} & -K_{12} & X_1\\ -K_{12}^T & -K_{22} & X_2\\ 0 & 0 & -R\end{bmatrix}\begin{bmatrix}a_1\\a_2\\i\end{bmatrix} + \begin{bmatrix}0\\0\\I\end{bmatrix}u$$

$$y = i$$

Example: one-phase transformer

Transform the DAE into the ODE form [Kerler-Back/St.'17]

$$\hat{E}\,\dot{\hat{x}} = \hat{A}\,\hat{x} + \hat{B}\,u$$
$$y = \hat{C}\,\hat{x}$$

with

$$\hat{E} = \begin{bmatrix} M_{11} + X_1 R^{-1} X_1^T & X_1 R^{-1} X_2^T Z \\ Z^T X_2 R^{-1} X_1^T & Z^T X_2 R^{-1} X_2^T Z \end{bmatrix} > 0, \qquad \hat{x} = \begin{bmatrix} a_1 \\ Z^T a_2 \end{bmatrix} \in \mathbb{R}^{n_d},$$

$$\hat{A} = -\begin{bmatrix} K_{11} & K_{12} Z \\ Z^T K_{12}^T & Z^T K_{22} Z \end{bmatrix} + \begin{bmatrix} K_{12} \\ Z^T K_{22} \end{bmatrix} Y \left(Y^T K_{22} Y \right)^{-1} Y^T \left[\, K_{12}^T,\; K_{22} Z \,\right] < 0,$$

$$\hat{B} = \begin{bmatrix} X_1 \\ Z^T X_2 \end{bmatrix} R^{-1}, \qquad \operatorname{im} Y = \ker X_2^T, \qquad Z = X_2 (X_2^T X_2)^{-1/2},$$

$$\hat{C} = (X_2^T X_2)^{-1} X_2^T \left(I - K_{22} Y (Y^T K_{22} Y)^{-1} Y^T \right) \left[\, K_{12}^T,\; K_{22} Z \,\right] = -\hat{B}^T \hat{E}^{-1} \hat{A}.$$

Example: one-phase transformer

Goal: solve $(\hat{A} + \tau\hat{E})z = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ with

$$\hat{E} = \begin{bmatrix} M_{11} + X_1 R^{-1} X_1^T & X_1 R^{-1} X_2^T Z \\ Z^T X_2 R^{-1} X_1^T & Z^T X_2 R^{-1} X_2^T Z \end{bmatrix}, \qquad Z = X_2 (X_2^T X_2)^{-1/2}$$

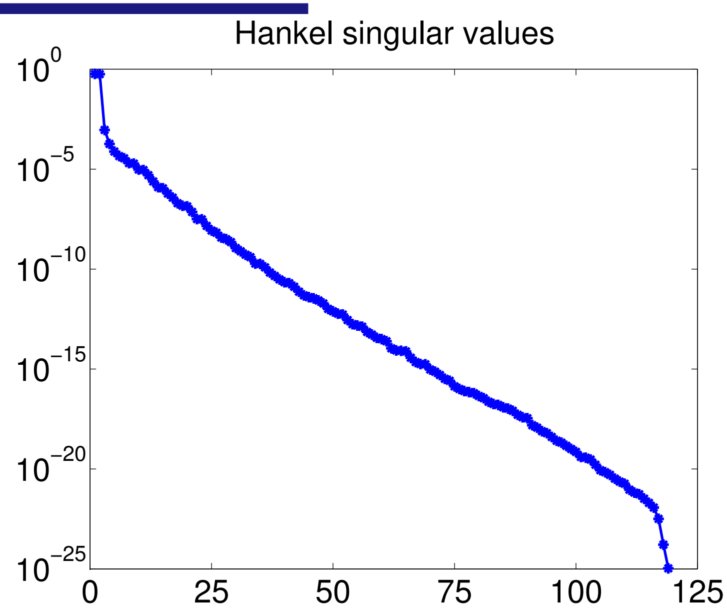
$$\hat{A} = -\begin{bmatrix} K_{11} & K_{12} Z \\ Z^T K_{21} & Z^T K_{22} Z \end{bmatrix} + \begin{bmatrix} K_{12} \\ Z^T K_{22} \end{bmatrix} Y \left(Y^T K_{22} Y\right)^{-1} Y^T [\, K_{21},\ K_{22} Z\,]$$

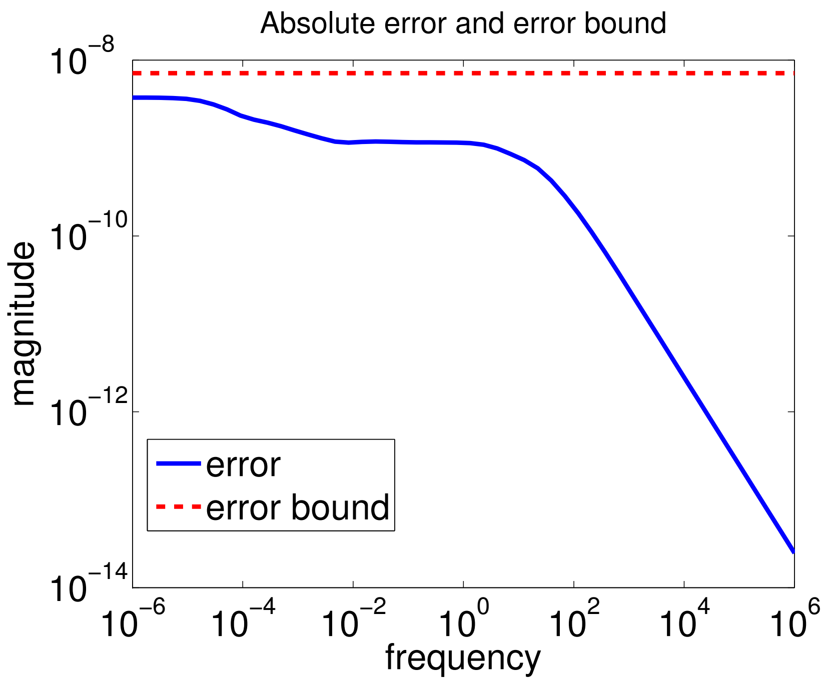
- Solve $\begin{bmatrix} \tau M_{11} - K_{11} & -K_{12} & X_1 \\ -K_{21} & -K_{22} & X_2 \\ \tau X_1^T & \tau X_2^T & -R \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ Z v_2 \\ 0 \end{bmatrix}$.
- Compute $z = \begin{bmatrix} z_1 \\ Z^T z_2 \end{bmatrix}$.

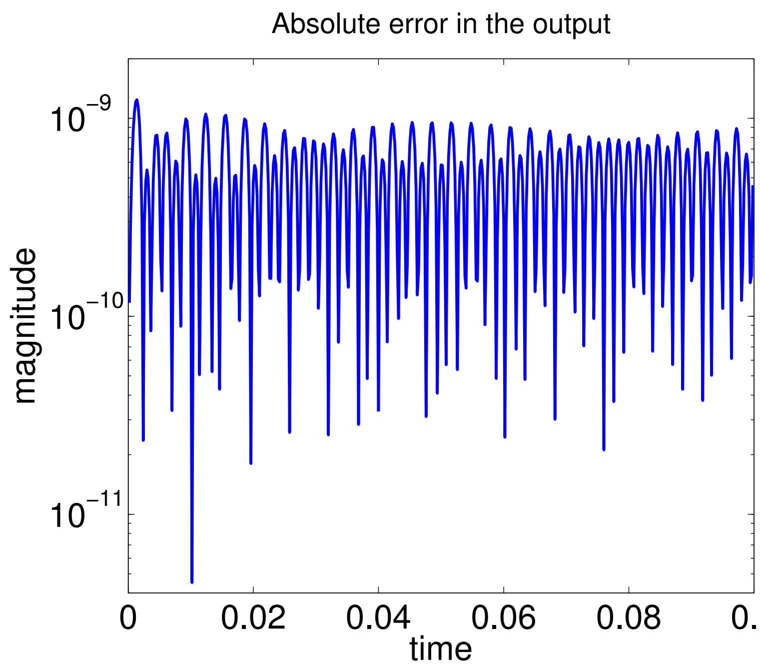
Note: Y is not required!

Example: one-phase transformer

- $n = 17733$, $n_d = 7202$, $n_a = 12531$, $m = 2$
- $X \approx \widetilde{R}\widetilde{R}^T, \quad \widetilde{R} \in \mathbb{R}^{n_d \times 126}$
- Reduced system: $r = 29$
- $t_{orig} = 180.74$ sec, $t_{red} = 0.06$ sec







Outline

- Model order reduction problem
- Balanced truncation model reduction
- Balancing-related model reduction techniques
- Balanced truncation for differential-algebraic equations
- Balanced truncation for second-order systems
  - structure-preserving model reduction
  - position and velocity Gramians
  - position and velocity Hankel singular values
  - second-order balanced truncation
- Balanced truncation for parametric systems
- Related topics and open problems

Second-order control systems

Time domain representation

$u \longrightarrow$

$$\begin{aligned} M\ddot{q}(t) + D\dot{q}(t) + Kq(t) &= B_2u(t) \\ C_2\dot{q}(t) + C_1q(t) &= y(t) \end{aligned}$$

$\longrightarrow y$

where $M, D, K \in \mathbb{R}^{n\times n}$, $B_2 \in \mathbb{R}^{n\times m}$, $C_1, C_2 \in \mathbb{R}^{p\times n}$,

$u \in \mathbb{R}^m$ – input, $q \in \mathbb{R}^n$ – state, $y \in \mathbb{R}^p$ – output.

Frequency domain representation

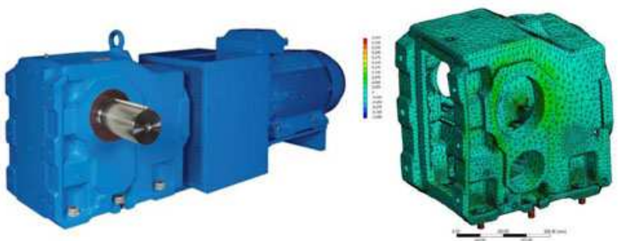
Laplace transform: $u(t) \mapsto \boldsymbol{u}(s)$, $y(t) \mapsto \boldsymbol{y}(s)$ $(q(0) = 0,\ \dot{q}(0) = 0)$

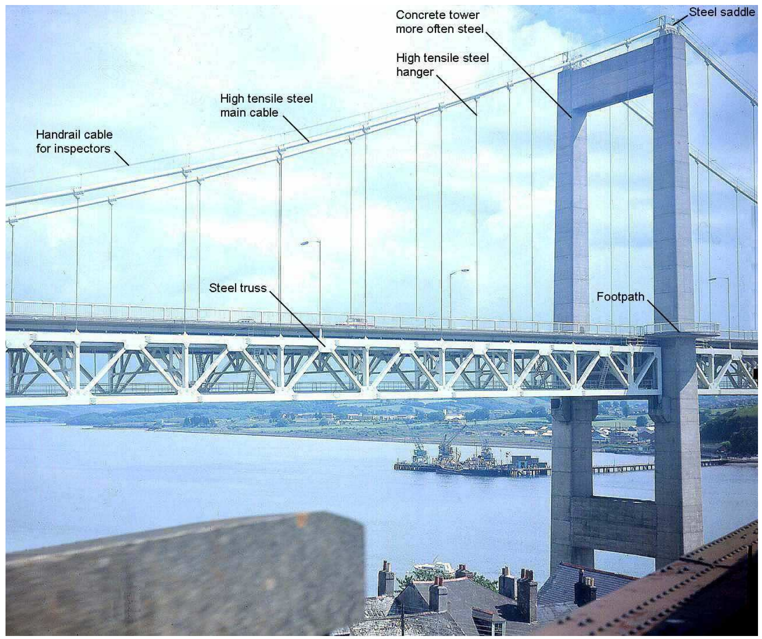
$$\hookrightarrow \quad \boldsymbol{y}(s) = (C_1 + sC_2)(s^2M + sD + K)^{-1}B_2\,\boldsymbol{u}(s) = \boldsymbol{G}(s)\boldsymbol{u}(s)$$

with $\boldsymbol{G}(s) = (C_1 + sC_2)(s^2M + sD + K)^{-1}B_2$

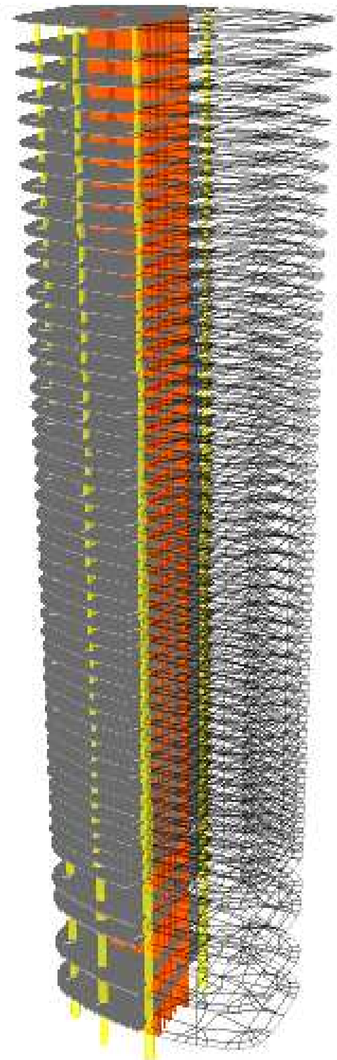
Applications

- Vibration and acoustic systems
  (automotive industry, rotor dynamics, machine tools, civil and earthquake engineering, ...)
- Control of large flexible structures
- MEMS devices design





The Tamar Bridge in England



50-Storey Tower in Kuala Lumpur, Malaysia

Model reduction problem

Given a second-order system

$u \rightarrow$

$$\begin{aligned} M\ddot{q}(t) + D\dot{q}(t) + Kq(t) &= B_2\, u(t) \\ C_2\dot{q}(t) + C_1 q(t) &= y(t) \end{aligned}$$

$\rightarrow y$

with $M, D, K \in \mathbb{R}^{n\times n}$, $B_2 \in \mathbb{R}^{n\times m}$, $C_1, C_2 \in \mathbb{R}^{p\times n}$,

find a reduced-order model

$u \rightarrow$

$$\begin{aligned} \tilde{M}\ddot{\tilde{q}}(t) + \tilde{D}\dot{\tilde{q}}(t) + \tilde{K}\tilde{q}(t) &= \tilde{B}_2\, u(t) \\ \tilde{C}_2\dot{\tilde{q}}(t) + \tilde{C}_1 \tilde{q}(t) &= \tilde{y}(t) \end{aligned}$$

$\rightarrow \tilde{y}$

with $\tilde{M}, \tilde{D}, \tilde{K} \in \mathbb{R}^{\ell\times \ell}$, $\tilde{B}_2 \in \mathbb{R}^{\ell\times m}$, $\tilde{C}_1, \tilde{C}_2 \in \mathbb{R}^{p\times \ell}$ and $\ell \ll n$.

Structure-preserving model reduction

$$\begin{array}{c} M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = B_2u(t) \\ C_2\dot{q}(t) + C_1q(t) = y(t) \end{array} \quad \Longrightarrow \quad \begin{array}{c} \tilde{M}\,\ddot{\tilde{q}}(t) + \tilde{D}\,\dot{\tilde{q}}(t) + \tilde{K}\,\tilde{q}(t) = \tilde{B}_2u(t) \\ \tilde{C}_2\dot{\tilde{q}}(t) + \tilde{C}_1\tilde{q}(t) = \tilde{y}(t) \end{array}$$

Second-order $\Rightarrow$ first-order

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = B_2u(t) \qquad \tilde{M}\,\ddot{\tilde{q}}(t) + \tilde{D}\,\dot{\tilde{q}}(t) + \tilde{K}\,\tilde{q}(t) = \tilde{B}_2u(t)$$

$$C_2\dot{q}(t) + C_1q(t) = y(t) \qquad \tilde{C}_2\dot{\tilde{q}}(t) + \tilde{C}_1\tilde{q}(t) = \tilde{y}(t)$$

$$\Downarrow$$

$$\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned}$$

$$E = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \qquad C = [\,C_1,\ C_2\,]$$

or

$$E = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \qquad B = \begin{bmatrix} B_2 \\ 0 \end{bmatrix}, \qquad C = [\,C_1,\ C_2\,]$$

$$\hookrightarrow \qquad \boldsymbol{G}(s) = (C_1 + sC_2)(s^2M + sD + K)^{-1}B_2 = C(sE - A)^{-1}B$$

Model reduction of the first-order system

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = B_2u(t)$$
$$C_2\dot{q}(t) + C_1q(t) = y(t)$$

$$\tilde{M}\,\ddot{\tilde{q}}(t) + \tilde{D}\,\dot{\tilde{q}}(t) + \tilde{K}\,\tilde{q}(t) = \tilde{B}_2u(t)$$
$$\tilde{C}_2\dot{\tilde{q}}(t) + \tilde{C}_1\tilde{q}(t) = \tilde{y}(t)$$

$$\Downarrow$$

$$\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \tilde{E}\,\dot{\tilde{x}}(t) &= \tilde{A}\,\tilde{x}(t) + \tilde{B}\,u(t) \\ \tilde{y}(t) &= \tilde{C}\,\tilde{x}(t) \end{aligned}$$

$$E = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \qquad C = [\,C_1,\; C_2\,]$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\tilde{E} = W^TET \qquad \tilde{A} = W^TAT \qquad \tilde{B} = W^TB \qquad \tilde{C} = CT$$

First-order $\Rightarrow$ second-order

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = B_2u(t)$$
$$C_2\dot{q}(t) + C_1q(t) = y(t)$$

$$\tilde{M}\,\ddot{\tilde{q}}(t) + \tilde{D}\,\dot{\tilde{q}}(t) + \tilde{K}\,\tilde{q}(t) = \tilde{B}_2u(t)$$
$$\tilde{C}_2\dot{\tilde{q}}(t) + \tilde{C}_1\tilde{q}(t) = \tilde{y}(t)$$

$$\Downarrow \qquad\qquad\qquad \Uparrow\ ?$$

$$\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned} \quad\Longrightarrow\quad \begin{aligned} \tilde{E}\,\dot{\tilde{x}}(t) &= \tilde{A}\,\tilde{x}(t) + \tilde{B}\,u(t) \\ \tilde{y}(t) &= \tilde{C}\,\tilde{x}(t) \end{aligned}$$

$$E = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \qquad C = [\, C_1,\ C_2\,]$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\tilde{E} = W^TET, \qquad \tilde{A} = W^TAT, \qquad \tilde{B} = W^TB, \qquad \tilde{C} = CT$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\tilde{M} = ?, \qquad \tilde{D} = ?, \qquad \tilde{K} = ?, \qquad \tilde{B}_2 = ?, \qquad \tilde{C}_1 = ?, \quad \tilde{C}_2 = ?$$

First-order $\Rightarrow$ second-order

Is it always possible to rewrite a first-order control system as a second-order control system ?

Answer: NO!

But ...

for $W = \begin{bmatrix} W_1 & \\ & W_2 \end{bmatrix}$ and $T = \begin{bmatrix} T_1 & \\ & T_1 \end{bmatrix}$, we have

$$\tilde{E} = W^T E T = \begin{bmatrix} W_1^T T_1 & 0 \\ 0 & W_2^T M T_1 \end{bmatrix}, \quad \tilde{A} = W^T A T = \begin{bmatrix} 0 & W_1^T T_1 \\ -W_2^T K T_1 & -W_2^T D\, T_1 \end{bmatrix},$$

$$\tilde{B} = W^T B = \big[\, 0, \ \ (W_2^T B_2)^T \,\big]^T, \quad \tilde{C} = C T = [\, C_1 T_1, \ \ C_2 T_1 \,]$$

$$\hookrightarrow \quad \tilde{\boldsymbol{G}} = (\, W_2^T M T_1,\ W_2^T D T_1,\ W_2^T K T_1,\ W_2^T B_2,\ C_1 T_1,\ C_2 T_1 \,)$$

Position and velocity Gramians

$$AXE^T + EXA^T = -BB^T \qquad A^TYE + E^TYA = -C^TC$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$X = \begin{bmatrix} X_p & X_{12} \\ X_{12}^T & X_v \end{bmatrix}, \qquad Y = \begin{bmatrix} Y_p & Y_{12} \\ Y_{12}^T & Y_v \end{bmatrix}$$

X_p – position controllability Gramian

X_v – velocity controllability Gramian

Y_p – position observability Gramian

Y_v – velocity observability Gramian

[Meyer/Srinivasan'96]

Hankel singular values

First-order system:

$$\xi_j = \sqrt{\lambda_j(XE^TYE)} \quad – \quad \text{Hankel singular values}$$

Second-order system:

$$\xi_j^p = \sqrt{\lambda_j(X_pY_p)} \quad – \quad \text{position singular values}$$

$$\xi_j^v = \sqrt{\lambda_j(X_vM^TY_vM)} \quad – \quad \text{velocity singular values}$$

$$\xi_j^{pv} = \sqrt{\lambda_j(X_pM^TY_vM)} \quad – \quad \text{position-velocity singular values}$$

$$\xi_j^{vp} = \sqrt{\lambda_j(X_vY_p)} \quad – \quad \text{velocity-position singular values}$$

[Reis/St.'08]

Balancing

First-order system:

$(E,\ A,\ B,\ C)$ is balanced, if $X = Y = \mathrm{diag}(\xi_1, \ldots, \xi_{2n})$.

Second-order system:

$(M,\ K,\ D,\ B_2,\ C_1,\ C_2)$ is position balanced, if

$$X_p = Y_p = \mathrm{diag}(\xi_1^p, \ldots, \xi_n^p).$$

$(M,\ K,\ D,\ B_2,\ C_1,\ C_2)$ is velocity balanced, if

$$X_v = Y_v = \mathrm{diag}(\xi_1^v, \ldots, \xi_n^v).$$

$(M,\ K,\ D,\ B_2,\ C_1,\ C_2)$ is position-velocity balanced, if

$$X_p = Y_v = \mathrm{diag}(\xi_1^{pv}, \ldots, \xi_n^{pv}).$$

$(M,\ K,\ D,\ B_2,\ C_1,\ C_2)$ is velocity-position balanced, if

$$X_v = Y_p = \mathrm{diag}(\xi_1^{vp}, \ldots, \xi_n^{vp}).$$

Second-order balanced truncation (SOBTp)

1. Compute $X = \begin{bmatrix} X_p & X_{12} \\ X_{12}^T & X_v \end{bmatrix}$, $Y = \begin{bmatrix} Y_p & Y_{12} \\ Y_{12}^T & Y_v \end{bmatrix}$ $\Bigg|$ $X_p = R_p R_p^T, \quad X_v = R_v R_v^T$ $Y_p = L_p L_p^T, \quad Y_v = L_v L_v^T$

2. Compute the SVD $R_p^T L_p = [\, U_{p1},\ U_{p2}\,] \begin{bmatrix} \Sigma_{p1} & \\ & \Sigma_{p2} \end{bmatrix} [\, V_{p1},\ V_{p2}\,]^T$,

   where $\Sigma_{p1} = \text{diag}(\xi_1^p, \ldots, \xi_\ell^p)$ and $\Sigma_{p2} = \text{diag}(\xi_{\ell+1}^p, \ldots, \xi_n^p)$;

3. Compute the SVD $R_v^T M^T L_v = [\, U_{v1},\ U_{v2}\,] \begin{bmatrix} \Sigma_{v1} & \\ & \Sigma_{v2} \end{bmatrix} [\, V_{v1},\ V_{v2}\,]^T$,

   where $\Sigma_{v1} = \text{diag}(\xi_1^v, \ldots, \xi_\ell^v)$ and $\Sigma_{v2} = \text{diag}(\xi_{\ell+1}^v, \ldots, \xi_n^v)$;

3. Compute $\tilde{M} = \tilde{W}^T M \tilde{T}$, $\tilde{D} = \tilde{W}^T D \tilde{T}$, $\tilde{K} = \tilde{W}^T K \tilde{T}$, $\tilde{B}_2 = \tilde{W}^T B_2$, $\tilde{C}_1 = C_1 \tilde{T}$, $\tilde{C}_2 = C_2 \tilde{T}$ with $\tilde{W} = L_v V_{v1} \Sigma_{p1}^{-1/2}$, $\tilde{T} = R_p U_{p1} \Sigma_{p1}^{-1/2}$.

Properties of the SOBT

- Stability is not necessarily preserved in the reduced model and, in general, no error bounds
- For symmetric second-order systems with

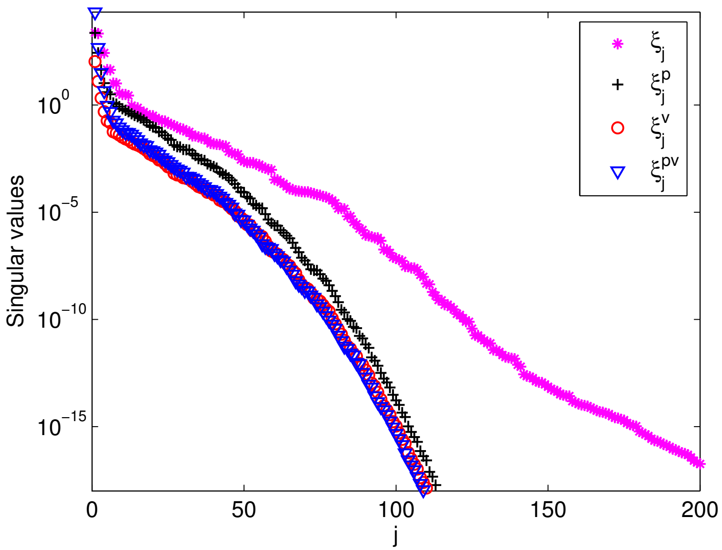
  $M = M^T > 0, \quad D = D^T > 0, \quad K = K^T > 0, \quad B_2 = C_2^T, \quad C_1 = 0,$

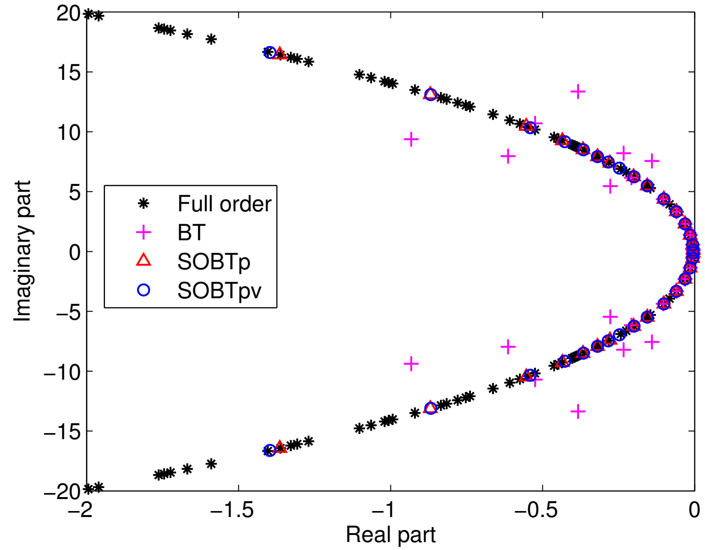
  we have
  - $\boldsymbol{G}(s) = \boldsymbol{G}^T(s)$
  - $\lambda^2 M + \lambda D + K$ is stable
  - $X_p = Y_v$
  - symmetry and stability are preserved
  - no error bounds
- Position and velocity Graminas can be computed using the ADI method without explicit forming the double sized matrices

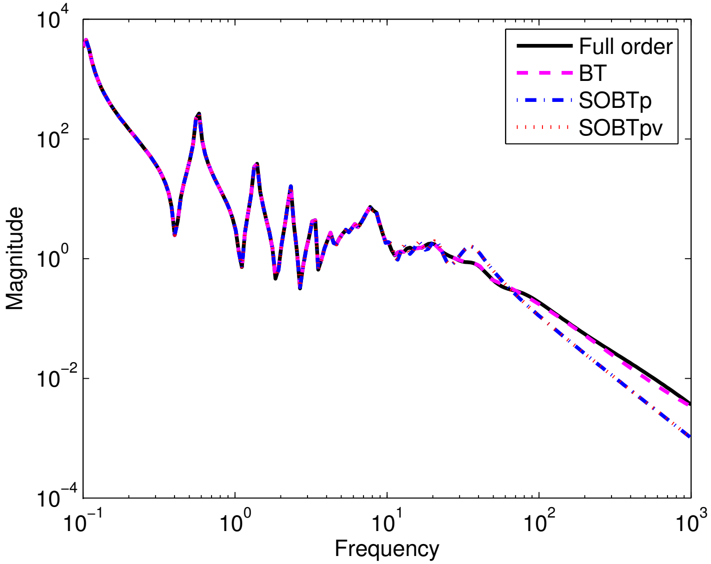
[Benner/Saak'11]

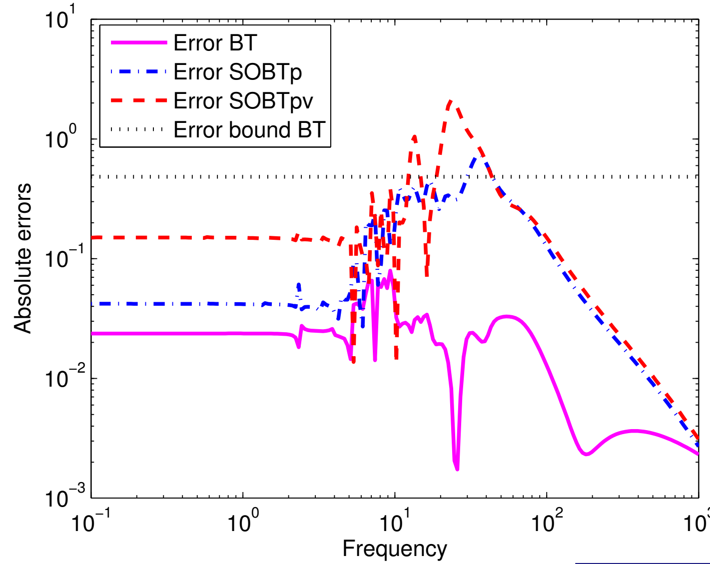
Clamped beam model

$n = 174, \quad m = p = 1 \quad \Longrightarrow \quad \ell = 17$ [Oberwolfach Benchmark Collection]









Conclusions

- Balanced truncation for DAEs
  - proper and improper Gramians
  - algebraic constraints are preserved
  - exploiting the structure of system matrices for computing P_l and P_r and solving the Lyapunov equations
  - other balancing techniques can also be extended to DAEs

    [Reis/St.'10,11, Möckel/Reis/St.'11, Benner/St.'17]
- Balanced truncation for second-order systems
  - position and velocity Gramians
  - second-order structure is preserved
  - stability is not always guaranteed
  - no error bounds

Outline

Part I

- Model order reduction problem
- Balanced truncation model reduction
- Balancing-related model reduction techniques

Part II

- Balanced truncation for differential-algebraic equations
- Balanced truncation for second-order systems

Part III

- Balanced truncation for parametric systems
- Related topics and open problems

Outline

- Model order reduction problem
- Balanced truncation model reduction
- Balancing-related model reduction techniques
- Balanced truncation for differential-algebraic equations
- Balanced truncation for second-order systems
- Balanced truncation for parametric systems
  - reduced basis method for parametric Lyapunov equations
  - parametric balanced truncation
- Related topics and open problems

Model reduction problem

Given a large-scale parametric control system

$u \rightarrow$

$$\begin{aligned} E(p)\dot{x}(t,p) &= A(p)x(t,p) + B(p)u(t) \\ y(t,p) &= C(p)x(t,p) + D(p)u(t) \end{aligned}$$

$\rightarrow y$

where $E(p), A(p) \in \mathbb{R}^{n\times n},\ B(p) \in \mathbb{R}^{n\times m},\ C(p) \in \mathbb{R}^{q\times n},\ D(p) \in \mathbb{R}^{q\times m},$
$p \in \mathbb{P} \subset \mathbb{R}^d$, find a reduced-order model

$u \rightarrow$

$$\begin{aligned} \widetilde{E}(p)\dot{\widetilde{x}}(t,p) &= \widetilde{A}(p)\widetilde{x}(t,p) + \widetilde{B}(p)u(t) \\ \widetilde{y}(t,p) &= \widetilde{C}(p)\widetilde{x}(t,p) + \widetilde{D}(p)u(t) \end{aligned}$$

$\rightarrow \widetilde{y}$

where $\widetilde{E}(p), \widetilde{A}(p) \in \mathbb{R}^{\ell\times \ell},\ \widetilde{B}(p) \in \mathbb{R}^{\ell\times m},\ \widetilde{C}(p) \in \mathbb{R}^{q\times \ell},\ \widetilde{D}(p) \in \mathbb{R}^{q\times m}.$

Balanced truncation algorithm

1. Solve the parametric Lyapunov equations

$$A(p)X(p)E^T(p) + E(p)X(p)A^T(p) = -B(p)B^T(p),$$
$$A^T(p)Y(p)E(p) + E^T(p)Y(p)A(p) = -C^T(p)C(p)$$

for $X(p) \approx \widetilde{R}(p)\widetilde{R}^T(p)$ and $Y(p) \approx \widetilde{L}(p)\widetilde{L}^T(p)$.

2. Compute the SVD

$$\widetilde{L}^T(p)E(p)\widetilde{R}(p) = [\,U_1(p),\ U_2(p)\,]\begin{bmatrix} \Sigma_1(p) & \\ & \Sigma_2(p) \end{bmatrix}\begin{bmatrix} V_1^T(p) \\ V_2^T(p) \end{bmatrix}.$$

3. Compute $\big(\widetilde{E}(p),\ \widetilde{A}(p),\ \widetilde{B}(p),\ \widetilde{C}(p),\ \widetilde{D}(p)\big)$ with

$$\widetilde{E}(p) = W^T(p)E(p)T(p), \qquad \widetilde{A}(p) = W^T(p)A(p)T(p),$$
$$\widetilde{B}(p) = W^T(p)B(p), \qquad \widetilde{C}(p) = C(p)T(p), \qquad \widetilde{D}(p) = D(p),$$
$$W(p) = \widetilde{L}(p)U_1(p)\Sigma_1^{-1/2}(p), \quad T(p) = \widetilde{R}(p)V_1(p)\Sigma_1^{-1/2}(p).$$

Parametric Lyapunov equations

- Lyapunov equation:

$$-A(p)X(p)E^T(p) - E(p)X(p)A^T(p) = B(p)B^T(p),$$

where $E(p), A(p), X(p) \in \mathbb{R}^{n\times n}$, $B(p) \in \mathbb{R}^{n\times m}$

- Operator equation:

$$\mathcal{L}_p(X(p)) = B(p)B^T(p),$$

where $\mathcal{L}_p : \mathbb{S}_+ \longrightarrow \mathbb{S}_+$ is a *Lyapunov operator*

- Linear system:

$$\boldsymbol{L}(p)\,\boldsymbol{x}(p) = \boldsymbol{b}(p),$$

where $\boldsymbol{L}(p) = -E(p) \otimes A(p) - A(p) \otimes E(p) \in \mathbb{R}^{n^2\times n^2}$,

$\boldsymbol{x}(p) = \mathrm{vec}(X(p))$, $\boldsymbol{b}(p) = \mathrm{vec}(B(p)B^T(p)) \in \mathbb{R}^{n^2}$

Reduced basis method: idea

Reduced basis method for $\mathcal{L}_p(X(p)) = B(p)B^T(p)$

- Snapshots collection:
  construct the reduced basis matrix $V_k = [\,Z_1, \,\ldots\,, \,Z_k\,]$, where
  $X(p_j) \approx Z_j Z_j^T$ solves $\mathcal{L}_{p_j}(X(p_j)) = B(p_j)B(p_j)^T$
- Galerkin projection:
  approximate the solution $X(p) \approx V_k \widetilde{X}(p) V_k^T$, where $\widetilde{X}(p)$
  solves $-\widetilde{A}(p)\widetilde{X}(p)\widetilde{E}^T(p) - \widetilde{E}(p)\widetilde{X}(p)\widetilde{A}^T(p) = \widetilde{B}(p)\widetilde{B}^T(p)$
  with $\widetilde{E}(p) = V_k^T E(p) V_k$, $\widetilde{A}(p) = V_k^T A(p) V_k$, $\widetilde{B}(p) = V_k^T B(p)$

Questions

- How to choose the parameters $p_1, \ldots, p_k$?
- How to estimate the error $\mathcal{E}_k(p) = X(p) - V_k \widetilde{X}(p) V_k^T$?
- How to make the computations efficient?

Error estimation

Goal: estimate the error $\mathcal{E}_k(p) = X(p) - V_k \widetilde{X}(p) V_k^T$

Residual $\mathcal{R}_k(p) := B(p)B^T(p) - \mathcal{L}_p\big(V_k \widetilde{X}(p) V_k^T\big) = \mathcal{L}_p\big(\mathcal{E}_k(p)\big)$

- Error estimate

$$\|\mathcal{E}_k(p)\|_F \le \|\mathcal{L}_p^{-1}\|_F \|\mathcal{R}_k(p)\|_F = \frac{\|\mathcal{R}_k(p)\|_F}{\alpha(p)}$$

with $\alpha(p) := \|\mathcal{L}_p^{-1}\|_F^{-1} = \inf\limits_{\|X\|_F=1} \|\mathcal{L}_p(X)\|_F = \sigma_{\min}\big(\mathbf{L}(p)\big)$

- Effectivity of the error estimator

$$1 \le \frac{\|\mathcal{R}_k(p)\|_F}{\alpha(p)\|\mathcal{E}_k(p)\|_F} = \frac{\|\mathcal{L}_p(\mathcal{E}_k(p)\|_F}{\alpha(p)\|\mathcal{E}_k(p)\|_F} \le \frac{\|\mathcal{L}_p\|_F}{\alpha(p)} = \frac{\gamma(p)}{\alpha(p)}$$

with $\gamma(p) := \|\mathcal{L}_p\|_F = \sup\limits_{\|X\|_F=1} \|\mathcal{L}_p(X)\|_F = \sigma_{\max}\big(\mathbf{L}(p)\big)$

Error estimation

Goal: estimate the error $\mathcal{E}_k(p) = X(p) - V_k \widetilde{X}(p) V_k^T$

Residual $\mathcal{R}_k(p) := B(p)B^T(p) - \mathcal{L}_p\big(V_k \widetilde{X}(p) V_k^T\big) = \mathcal{L}_p\big(\mathcal{E}_k(p)\big)$

- Error estimate

$$\|\mathcal{E}_k(p)\|_F \le \|\mathcal{L}_p^{-1}\|_F \|\mathcal{R}_k(p)\|_F = \frac{\|\mathcal{R}_k(p)\|_F}{\alpha(p)} \le \frac{\|\mathcal{R}_k(p)\|_F}{\alpha_{LB}(p)} =: \Delta_k(p)$$

with $\alpha(p) := \|\mathcal{L}_p^{-1}\|_F^{-1} = \inf\limits_{\|X\|_F=1} \|\mathcal{L}_p(X)\|_F = \sigma_{\min}\big(\mathbf{L}(p)\big) \ge \alpha_{LB}(p)$

- Effectivity of the error estimator

$$1 \le \frac{\Delta_k(p)}{\|\mathcal{E}_k(p)\|_F} = \frac{\|\mathcal{R}_k(p)\|_F}{\alpha_{\mathrm{LB}}(p)\|\mathcal{E}_k(p)\|_F} \le \frac{\gamma(p)}{\alpha_{LB}(p)} \le \frac{\gamma_{UB}(p)}{\alpha_{LB}(p)}$$

with $\gamma(p) := \|\mathcal{L}_p\|_F = \sup\limits_{\|X\|_F=1} \|\mathcal{L}_p(X)\|_F = \sigma_{\max}\big(\mathbf{L}(p)\big) \le \gamma_{UB}(p)$

Construction of the reduced basis

Greedy algorithm

Input: tolerance tol, training set $\mathbb{P}_{\rm train} \subset \mathbb{P}$, initial parameter $p_1 \in \mathbb{P}$

- Solve $\mathcal{L}_{p_1}\big(X(p_1)\big) = B(p_1)B^T(p_1)$ for $X(p_1) \approx Z_1 Z_1^T$, $Z_1 \in \mathbb{R}^{n\times r_1}$
- Set $k=2$, $\Delta_1^{\max}=1$ and $V_1 = Z_1$
- while $\Delta_{k-1}^{\max} \geq tol$

  $$p_k = \arg \max_{p\in\mathbb{P}_{\rm train}} \Delta_{k-1}(p) \qquad \% \ \Delta_{k-1}(p) = \frac{\|\mathcal{R}_{k-1}(p)\|_F}{\alpha_{LB}(p)}$$

  $$\Delta_k^{\max} = \Delta_{k-1}(p_k)$$

  solve $\mathcal{L}_{p_k}\big(X(p_k)\big) = B(p_k)B^T(p_k)$ for $X(p_k) \approx Z_k Z_k^T$, $Z_k \in \mathbb{R}^{n\times r_k}$

  $$V_k = [\, V_{k-1}, \ Z_k \,]$$

  $$k \leftarrow k+1$$

  end

Offline-online decomposition

Assumption: affine parameter dependence

$$E(p) = \sum_{i=1}^{n_E} \theta_i^E(p) E_i, \quad A(p) = \sum_{i=1}^{n_A} \theta_i^A(p) A_i, \quad B(p) = \sum_{i=1}^{n_B} \theta_i^B(p) B_i$$

$$\hookrightarrow \; \mathcal{L}_p(X) = \sum_{i=1}^{n_E} \sum_{j=1}^{n_A} \theta_i^E(p)\, \theta_j^A(p)\, \mathcal{L}_{ij}(X), \quad \mathcal{L}_{ij}(X) = -A_j X E_i^T - E_i X A_j^T,$$

$$B(p)B^T(p) = \sum_{i=1}^{n_B} \sum_{j=1}^{n_B} \theta_i^B(p)\, \theta_j^B(p)\, B_i B_j^T$$

Offline: compute the reduced basis matrix $V_k = [Z_1, \ldots, Z_k] \in \mathbb{R}^{n \times r}$.

Online: for $p \in \mathbb{P}$, compute $X(p) \approx V_k \widetilde{X}(p) V_k^T$, where $\widetilde{X}(p)$ solves

$$-\widetilde{A}(p)\widetilde{X}(p)\widetilde{E}^T(p) - \widetilde{E}(p)\widetilde{X}(p)\widetilde{A}^T(p) = \widetilde{B}(p)\widetilde{B}^T(p)$$

with

$$\widetilde{E}(p) = \sum_{j=1}^{n_E} \theta_j^E(p) V_k^T E_j V_k, \quad \widetilde{A}(p) = \sum_{j=1}^{n_A} \theta_j^A(p) V_k^T A_j V_k, \quad \widetilde{B}(p) = \sum_{j=1}^{n_B} \theta_j^B(p) V_k^T B_j.$$

Computation of the residual norm

$$
\begin{aligned}
\|\mathcal{R}_k(p)\|_F^2 &= \|B(p)B^T(p) - \mathcal{L}_p\big(V_k\widetilde{X}(p)V_k^T\big)\|_F^2 \\
&= \sum_{i,j=1}^{n_B}\sum_{f,g=1}^{n_B}\theta^B_{ijfg}(p)\,\mathrm{trace}\big((B_i^T B_f)(B_g^T B_j)\big) \\
&+ 4\sum_{i,j=1}^{n_B}\sum_{f=1}^{n_E}\sum_{g=1}^{n_A}\theta^{AEB}_{ijfg}(p)\,\mathrm{trace}\big(B_i^T(E_fV_k)\widetilde{X}(p)(A_gV_k)^T B_j\big) \\
&+ 2\sum_{i,f=1}^{n_E}\sum_{j,g=1}^{n_A}\theta^{AE}_{ijfg}(p)\,\mathrm{trace}\big((E_fV_k)^T(E_iV_k)\widetilde{X}(p)(A_jV_k)^T(A_gV_k)\widetilde{X}(p)\big) \\
&+ 2\sum_{i,f=1}^{n_E}\sum_{j,g=1}^{n_A}\theta^{AE}_{ijfg}(p)\,\mathrm{trace}\big((E_fV_k)^T(A_jV_k)\widetilde{X}(p)(E_iV_k)^T(A_gV_k)\widetilde{X}(p)\big)
\end{aligned}
$$

with $\theta^B_{ijfg}(p) = \theta^B_i(p)\theta^B_j(p)\theta^B_f(p)\theta^B_g(p)$, $\quad \theta^{AEB}_{ijfg}(p) = \theta^B_i(p)\theta^B_j(p)\theta^E_f(p)\theta^A_g(p)$,

$\theta^{AE}_{ijfg}(p) = \theta^E_i(p)\theta^A_j(p)\theta^E_f(p)\theta^A_g(p)$.

Error estimation: min-θ approach

Assumption: $E(p) = E^T(p) > 0, \quad A(p) + A^T(p) < 0$ for all $p \in \mathbb{P}$

(e.g., $\theta_i^E(p) > 0, \ \ E_i = E_i^T \geq 0, \ \ \bigcap \ker(E_i) = \{0\}$ and

$\theta_i^A(p) > 0, \ \ A_i + A_i^T \leq 0, \ \ \bigcap \ker(A_i + A_i^T) = \{0\}$)

Let $\hat{p} \in \mathbb{P}$ and

$$\theta_{\min}^{\hat{p}}(p) = \min_{\substack{i=1,\ldots,n_E \\ j=1,\ldots,n_A}} \frac{\theta_i^E(p)\theta_j^A(p)}{\theta_i^E(\hat{p})\theta_j^A(\hat{p})}, \qquad \theta_{\max}^{\hat{p}}(p) = \max_{\substack{i=1,\ldots,n_E \\ j=1,\ldots,n_A}} \frac{\theta_i^E(p)\theta_j^A(p)}{\theta_i^E(\hat{p})\theta_j^A(\hat{p})}.$$

Then $\alpha(p) \geq \theta_{\min}^{\hat{p}}(p)\, \lambda_{\min}\big(-A(\hat{p}) - A^T(\hat{p})\big)\, \lambda_{\min}\big(E(\hat{p})\big) =: \alpha_{LB}(p)$,

$$\gamma(p) \leq \theta_{\max}^{\hat{p}}(p)\, \lambda_{\max}\big(-A(\hat{p}) - A^T(\hat{p})\big)\, \lambda_{\max}\big(E(\hat{p})\big) =: \gamma_{UB}(p)$$

for all $p \in \mathbb{P}$. [Son/St.'17]

Parametric balanced truncation

Offline phase: compute the reduced basis matrices V_X and V_Y for the controllability and observability Lyapunov equations; compute all parameter-independent matrices.

Online phase: for given $p \in \mathbb{P}$,

- solve the reduced Lyapunov equations

$$-\widetilde{A}_X(p)\widetilde{X}(p)\widetilde{E}_X^T(p) - \widetilde{E}_X(p)\widetilde{X}(p)\widetilde{A}_X^T(p) = \widetilde{B}(p)\widetilde{B}^T(p),$$

$$-\widetilde{A}_Y^T(p)\widetilde{Y}(p)\widetilde{E}_Y(p) - \widetilde{E}_Y^T(p)\widetilde{Y}(p)\widetilde{A}_Y(p) = \widetilde{C}^T(p)\widetilde{C}(p)$$

with $\widetilde{E}_X(p) = \sum_{j=1}^{n_E} \theta_j^E(p) V_X^T E_j V_X, \quad \widetilde{A}_X(p) = \sum_{j=1}^{n_A} \theta_j^A(p) V_X^T A_j V_X,$

$$\widetilde{E}_Y(p) = \sum_{j=1}^{n_E} \theta_j^E(p) V_Y^T E_j V_Y, \quad \widetilde{A}_Y(p) = \sum_{j=1}^{n_A} \theta_j^A(p) V_Y^T A_j V_Y,$$

$$\widetilde{B}(p) = \sum_{j=1}^{n_B} \theta_j^B(p) V_X^T B_j, \qquad \widetilde{C}(p) = \sum_{j=1}^{n_C} \theta_j^C(p) C_j V_Y.$$

Parametric balanced truncation

$\hookrightarrow$ Gramians $X(p) \approx V_X \widetilde{X}(p) V_X^T = V_X Z_X(p) Z_X^T(p) V_X^T$

$$Y(p) \approx V_Y \widetilde{Y}(p) V_Y^T = V_Y Z_Y(p) Z_Y^T(p) V_Y^T$$

- Compute the SVD

$$\begin{aligned} Z_Y^T(p) V_Y^T E(p) V_X Z_X(p) &= \sum_{j=1}^{n_E} \theta_j^E(p) Z_Y^T(p) V_Y^T E_j V_X Z_X(p) \\ &= [U_1(p),\, U_2(p)] \begin{bmatrix} \Sigma_1(p) & 0 \\ 0 & \Sigma_2(p) \end{bmatrix} \begin{bmatrix} V_1^T(p) \\ V_2^T(p) \end{bmatrix}. \end{aligned}$$

- Compute the reduced model $\big(\widetilde{E}(p), \widetilde{A}(p), \widetilde{B}(p), \widetilde{C}(p), D(p)\big)$ with

$$\widetilde{E}(p) = \sum_{j=1}^{n_E} \theta_j^E(p) W^T(p) V_Y^T E_j V_X T(p), \quad \widetilde{B}(p) = \sum_{j=1}^{n_B} \theta_j^B(p) W^T(p) V_Y^T B_j,$$

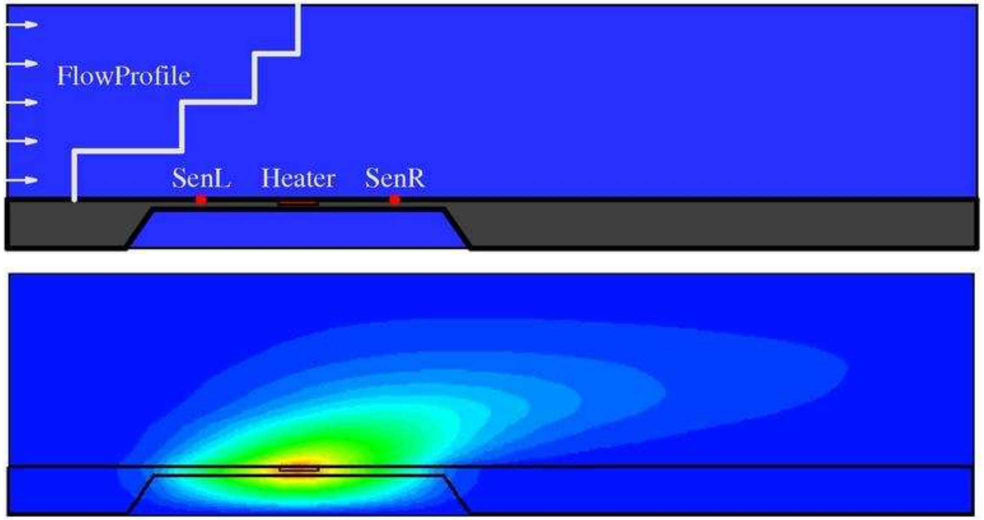
$$\widetilde{A}(p) = \sum_{j=1}^{n_A} \theta_j^A(p) W^T(p) V_Y^T A_j V_X T(p), \quad \widetilde{C}(p) = \sum_{j=1}^{n_C} \theta_j^C(p) C_j V_X T(p),$$

$$T(p) = Z_X(p) V_1(p) \Sigma_1(p)^{-1/2}, \quad W(p) = Z_Y(p) U_1(p) \Sigma_1(p)^{-1/2}.$$

Properties

- Preservation of stability
- Computable error bounds
- Approximation does not rely on solution snapshots and is independent of the training input
- Other error estimation techniques can be used (e.g., successive constraints method)
- Reduced basis method for parametric Riccati equations
  [Haasdonk/Schmidt'15]

Example: anemometer



Mathematical model:

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot \kappa \nabla T - \rho c v \cdot \nabla T + \dot{q}$$

boundary / initial conditions

FEM model:

$$E(p)\,\dot{x} = A(p)\,x + B\,u$$
$$y = C\,x$$

with $E(p) = E_1 + p_1 E_2, \quad A(p) = A_1 + p_2 A_2 + p_3 A_3 \in \mathbb{R}^{n\times n}, \quad p = \begin{bmatrix} c_f \\ \kappa_f \\ c_f v \end{bmatrix},$

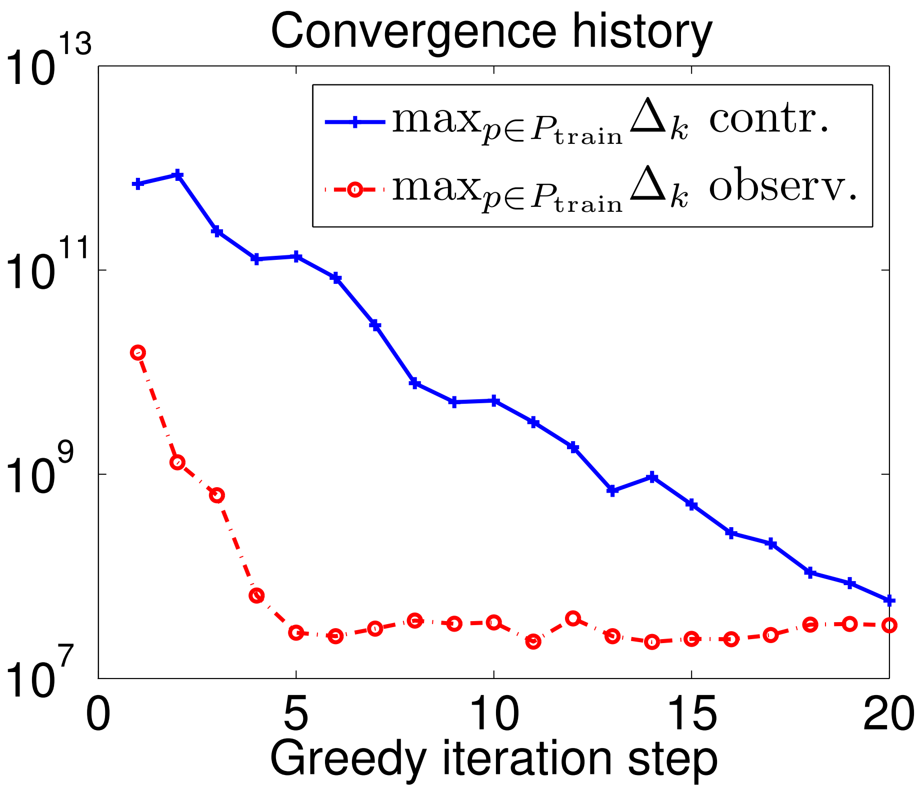
$B, C^T \in \mathbb{R}^n, \quad n = 29008$

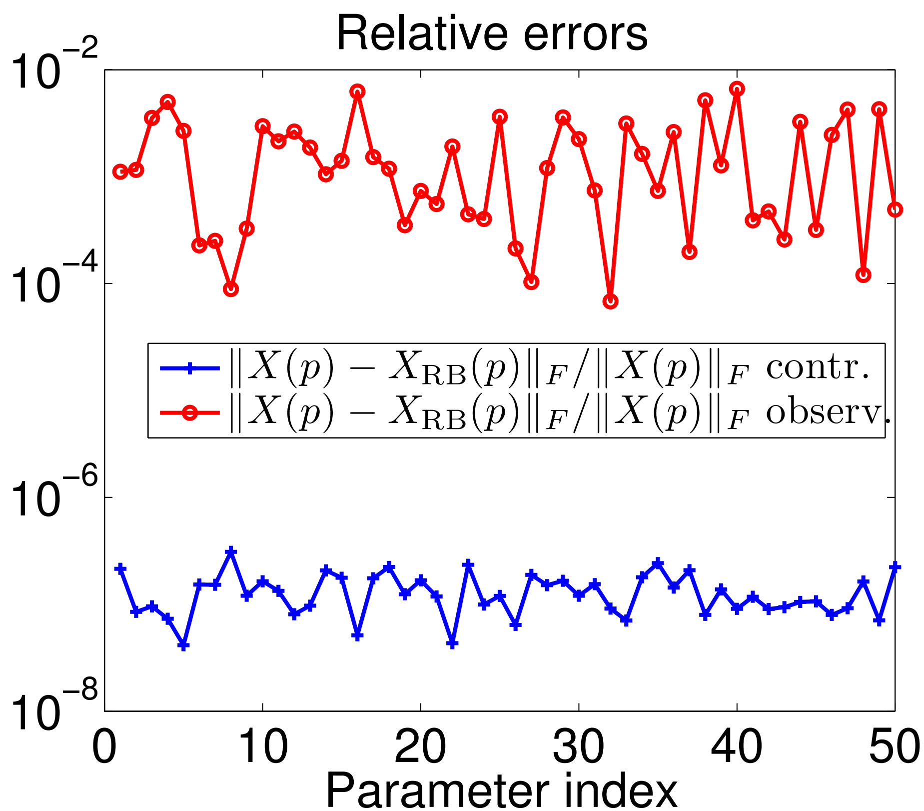
[Moosmann'07, MOR Wiki]

Example: anemometer

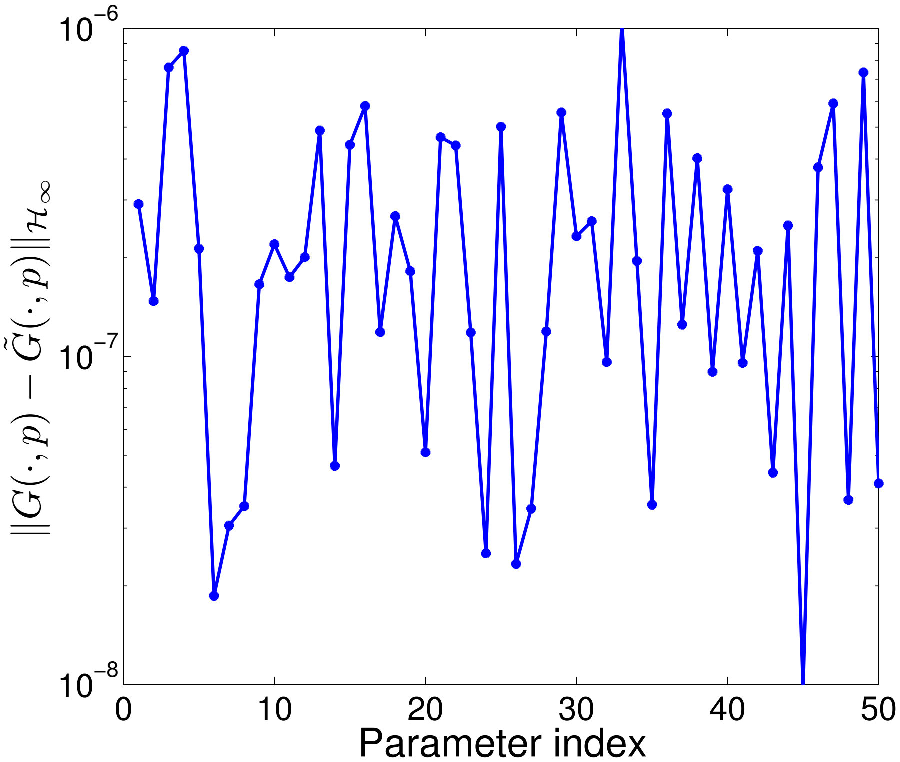
$\mathbb{P}_{\text{train}} = \{10000$ random points$\}$, 20 Greedy iterations

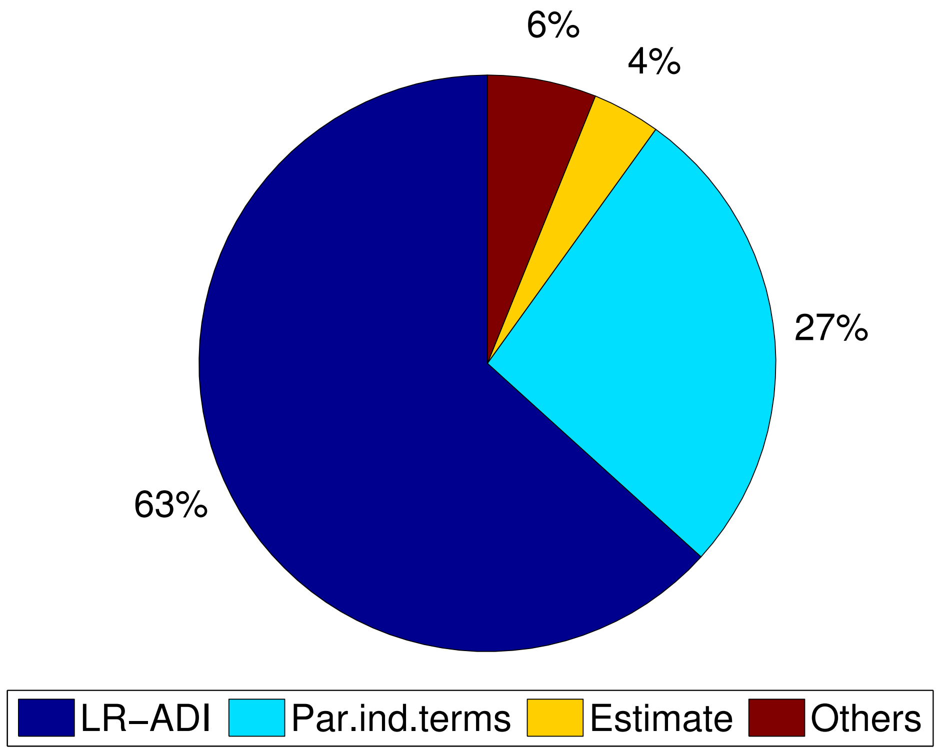
$\mathbb{P}_{\text{test}} = \{50$ random points$\}$





Example: anemometer





Outline

- Model order reduction problem
- Balanced truncation model reduction
- Balancing-related model reduction techniques
- Balanced truncation for differential-algebraic equations
- Balanced truncation for second-order systems
- Balanced truncation for parametric systems
- Related topics and open problems
  - Balanced truncation for linear time-varying systems
  - Balanced truncation for bilinear systems
  - Balanced truncation for quadratic-bilinear systems
  - Balanced truncation for nonlinear systems
  - Balanced truncation for infinite-dimensional systems

BT for linear time-varying systems

- For linear time-varying systems

$$\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\,u(t), \qquad t \in [0,T], \\ y(t) &= C(t)\,x(t) + D(t)\,u(t), \end{aligned}$$

the Gramians satisfy the Lyapunov differential equations

$$\begin{aligned} \dot{X}(t) &= A(t)\,X(t) + X(t)A^T(t) + B(t)B^T(t),\; X(0) = 0, \\ -\dot{Y}(t) &= A^T(t)Y(t) + Y(t)\,A(t) + C^T(t)C(t),\; Y(T) = 0 \end{aligned}$$

[Shokoohi/Silverman/Van Dooren'83, Sandberg'02]

$\hookrightarrow$ use the BDF or Rosenbrock method combined with the LDL^T-type ADI or Krylov subspace methods [Lang/Saak/St.'16]

- projection matrices are time-dependent
- zero initial and final conditions for the Gramians lead to zero initial and final reduced state

BT for bilinear systems

- For bilinear systems [Benner/Damm'11, Benner/Goyal/Redmann'16]

$$\dot{x}(t) = A\,x(t) + \sum_{k=1}^{m} N_k\,x(t)u_k(t) + B\,u(t),$$
$$y(t) = C\,x(t) + D\,u(t),$$

the Gramians satisfy the generalized Lyapunov equations

$$AX + XA^T + \sum_{k=1}^{m} N_k X N_k^T = -BB^T,$$
$$A^TY + YA + \sum_{k=1}^{m} N_k^T Y N_k = -C^TC.$$

$\hookrightarrow$ use the ADI or Krylov subspace methods [Benner/Breiten'12]

$\hookrightarrow$ $(W^TAT,\, W^TN_1T, \ldots, W^TN_mT,\, W^TB,\, CT,\, D)$

- energy functionals: $E_u(x_0) \geq x_0^T X^{-1} x_0,\;\; E_y(x_0) \leq x_0^T Y x_0,\;\; x_0 \in \mathcal{B}(0)$
- computationally expensive $\hookrightarrow$ use truncated Gramians
- no error bounds

BT for quadratic-bilinear systems

- For quadratic-bilinear systems [Benner/Goyal'17]

$$\dot{x}(t) = A\,x(t) + H\left(x(t)\otimes x(t)\right) + \sum_{k=1}^{m} N_k x(t) u_k(t) + B\,u(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$

the Gramians satisfy the generalized Lyapunov equations

$$AX + XA^T + H\,(X\otimes X)\,H^T + \sum_{k=1}^{m} N_k X N_k^T = -BB^T,$$
$$A^TY + YA + (H^{(2)})^T(X\otimes Y)H^{(2)} + \sum_{k=1}^{m} N_k^T Y N_k = -C^TC.$$

$\hookrightarrow$ use the fix point iteration combined with the ADI method

$\hookrightarrow$ $(W^TAT,\, W^TH(T\otimes T),\, W^TN_1T, \ldots, W^TN_mT,\, W^TB,\, CT,\, D)$

- energy functionals: $E_u(x_0) \ge x_0^T X^{-1} x_0,\;\; E_y(x_0) \le x_0^T Y x_0,\;\; x_0 \in \mathcal{B}(0)$
- computationally expensive $\hookrightarrow$ use truncated Gramians
- no error bounds

BT for nonlinear systems

- For nonlinear systems [Scherpen'94, Fujimoto/Scherpen'10]

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + g(x(t))u(t),\\ y(t) &= h(x(t)),\end{aligned}$$

the input and output energy functionals $E_u(x_0)$ and $E_y(x_0)$ satisfy the partial differential equations

$$\frac{\partial E_c}{\partial x} f(x) + \frac{1}{4}\frac{\partial E_c}{\partial x} g(x) g^T(x) \frac{\partial^T E_c}{\partial x} = 0, \quad E_c(0) = 0,$$

$$\frac{\partial E_o}{\partial x} f(x) + h(x)h^T(x) = 0, \quad E_o(0) = 0.$$

  - computationally very expensive

BT for infinite-dimensional systems

- For infinite-dimensional systems

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t), \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}$$

with $A : \mathcal{D}(A) \subset \mathcal{X} \to \mathcal{X}$, $B : \mathcal{U} \to \mathcal{D}(A^*)'$, $C : \mathcal{X} \to \mathcal{Y}$, $D : \mathcal{U} \to \mathcal{Y}$, where $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, the Gramians satisfy the operator Lyapunov equations

$$\begin{aligned} 2\,\mathrm{Re}\langle Xv, A^*v\rangle_{\mathcal{X}} + \|B'v\|^2_{\mathcal{U}} &= 0 \quad \text{for all } v \in \mathcal{D}(A^*), \\ 2\,\mathrm{Re}\langle A\,v,\, Y\,v\rangle_{\mathcal{X}} + \|Cv\|^2_{\mathcal{Y}} &= 0 \quad \text{for all } v \in \mathcal{D}(A). \end{aligned}$$

[Glover/Curtain/Partingto'88, Guiver/Opmeer'13, Reis/Selig'14]

$\hookrightarrow$ use the finite-rank ADI iteration [Reis/Opmeer/Wollner'13]

- error bound $\|\boldsymbol{G} - \widetilde{\boldsymbol{G}}\|_{\mathcal{H}_\infty} \leq 2 \sum\limits_{j=\ell+1}^{\infty} \xi_j$

Conclusion

- General framework for balanced truncation model reduction
  - input and output energy functionals
  - controllability and observability Gramians
  - (Hankel) singular values
  - balanced realization
- Properties
  - preservation of physical properties
  - computable error bounds
  - independence of the control
- Numerical solution of Lyapunov, Riccati, Lur’e equations

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