Robustness of mixing via bottleneck sequences

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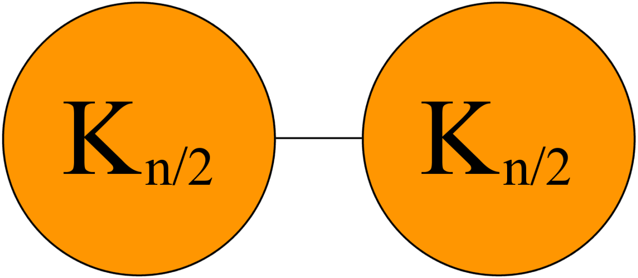
- There are of course many ways to bound the mixing time. We will look at conductance-based bounds.

An example: the dumbbell graph



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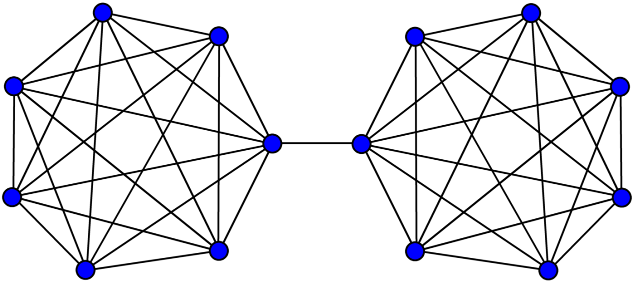

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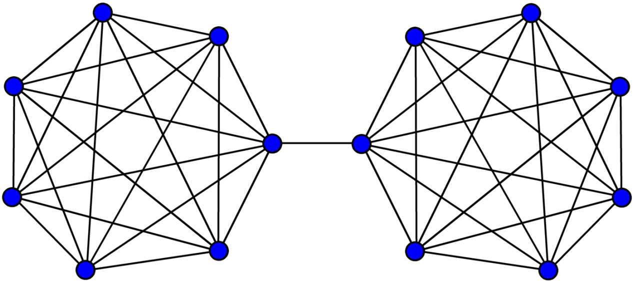

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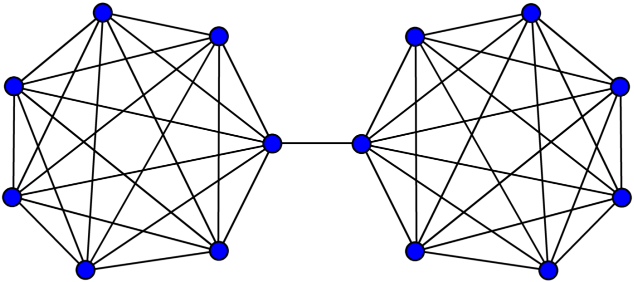

- Starting from the left-hand side, it takes us time $\asymp n^2$ to reach the right-hand side.
- The invariant measure of the right-hand side is $1/2$, so it seems clear (and it is easy to prove) that the mixing time is at least cn^2.

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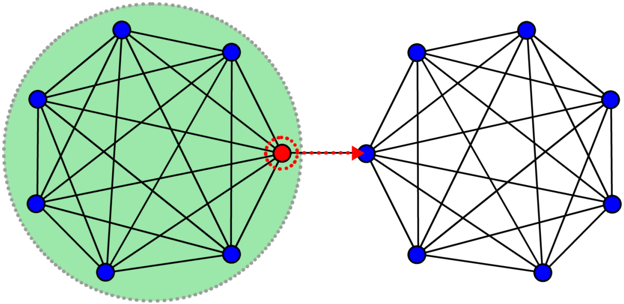

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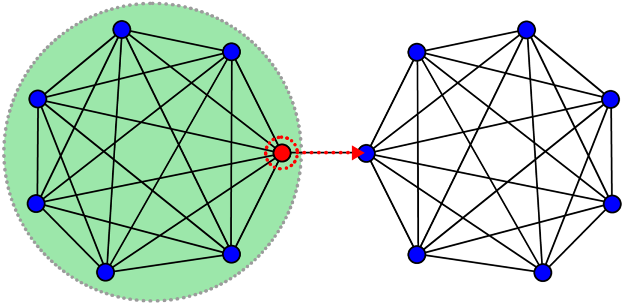
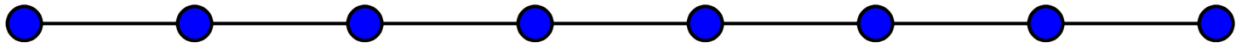


Note that if A is the left-hand side of the dumbbell graph, then $\Phi(A) \asymp 1/n^2$.

First guess: $t_{\text{MIX}} \asymp \max_{A \subset V} 1/\Phi(A)$?

A second example: the path of length n



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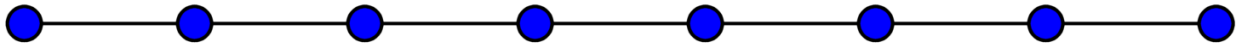

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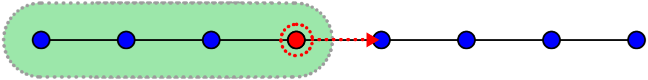

Unfortunately our guess is not correct. Again it takes time $\asymp n^2$ to reach the right-hand side of this graph, so $t_{\text{MIX}} \geq cn^2 \ldots$ but $\max_{A \subset V} 1/\Phi(A) \asymp n$.

The Lovász/Kannan/Fountoulakis/Reed/Morris/Peres bound

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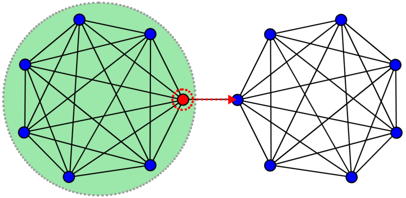
- This built on work of Lovász and Kannan. A similar bound was given by Morris and Peres using evolving sets, which in particular works for non-reversible chains.

The L/K/F/R/M/P bound applied to the dumbbell



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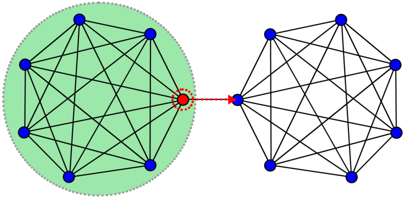

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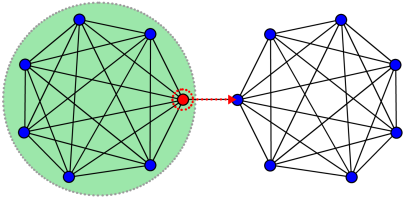

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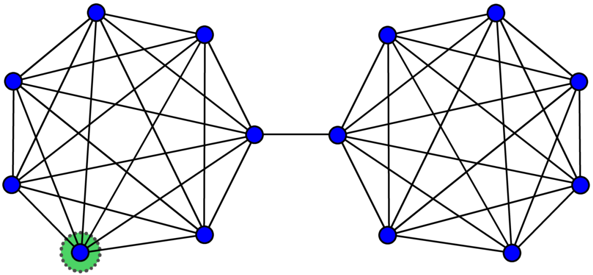

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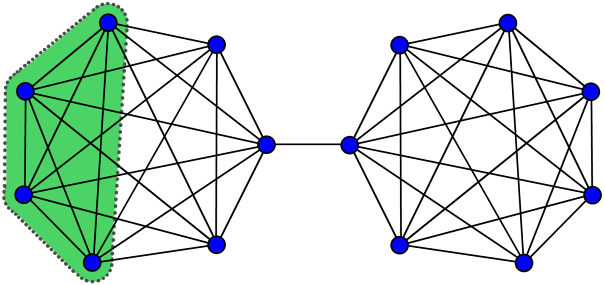

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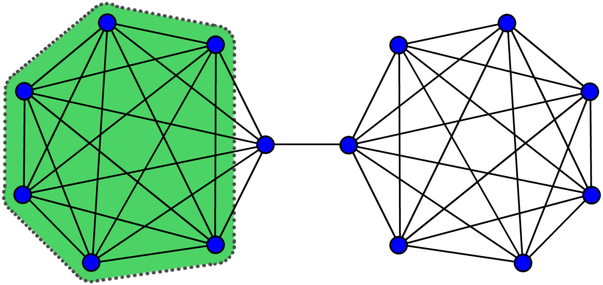

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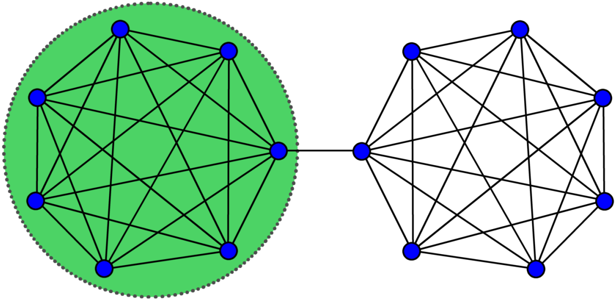

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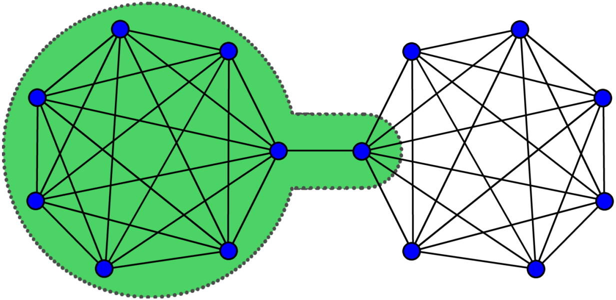

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- For the dumbbell graph, this gives $t_{\mathrm{MIX}} \lesssim n^2$.
- For the path, it also gives $t_{\mathrm{MIX}} \lesssim n^2$.

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If G and H are roughly isometric (with constant r) and have bounded degree, are their mixing times within a constant factor (depending only on r, not the graphs)?

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- Ding and Peres constructed a graph where replacing some edges by two edges end to end decreases the mixing time by an unbounded factor.
- Nonetheless, we may ask: are there large classes of graphs such that the mixing time is robust under rough isometry?
- We start with trees. (Peres and Sousi already proved that the mixing time is robust under rough isometry on trees, but trees give an illuminating application of our bottleneck sequence tools.)

Bounding the mixing time on trees

Recall our first result: for any $\theta \in (0,1)$,

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It is also easy to show (an application of Moon's lemma, or prove directly by induction) that on trees,

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But on trees, if S and S^c are both connected, then the boundary of S is exactly one vertex, so the two bounds agree. And they are robust under rough isometry.

Graphs roughly isometric to trees

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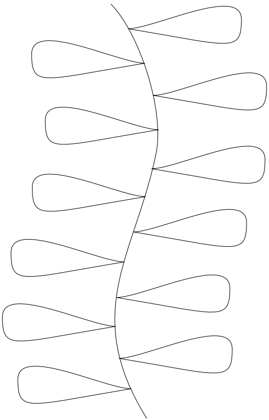
What about an upper bound?

An example when the bottleneck sequence bound is not tight: the beanstalk graph



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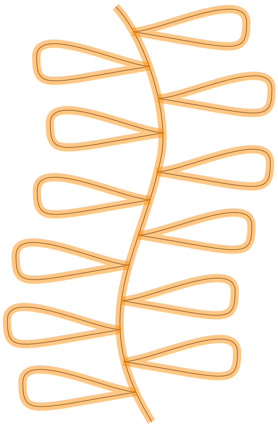

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- $C \subset C'$, $C' \setminus C \subset D^c$, C' connected
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Dasher: From (C, D), D' valid if

- $D \cup C \subset D'$, $(D')^c$ connected
- $\partial D'$ is α-near to C
- D' is a β-adjustment of C
- If $s \in D'$ then s is α-near to C and $D' = V(G)$.

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Theorem (Addario-Berry, R.)

For any $\alpha, \beta, \gamma \in (0,1)$, there exists a strategy for Crawler such that for any valid moves by Dasher,

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Note this bound holds for all graphs, not just tree-like graphs.

Now play the game on a graph G that is roughly isometric (with constant r) to a tree T (both with bounded degree).

The bottleneck sequence game bound

Theorem (Addario-Berry, R.)

For any $\alpha, \beta, \gamma \in (0,1)$, there exists a strategy for Crawler such that for any valid moves by Dasher,

$$t_{\text{MIX}}(G) \leq C \sum_{j=1}^{k} \frac{1}{\Phi(D_j)}.$$

Note this bound holds for all graphs, not just tree-like graphs.

Now play the game on a graph G that is roughly isometric (with constant r) to a tree T (both with bounded degree).

We devise a strategy for Dasher such that whatever moves Crawler makes,

$$\sum_{j=1}^{k} \frac{1}{\Phi(D_j)} \leq C'(r) t_{\text{MIX}}(T)$$

The mixing time is robust on bounded degree graphs that are roughly isometric to trees

Theorem (Addario-Berry, R.)

If G is roughly isometric (with constant r) to a tree T, and both have degree at most Δ, then

$$c(r,\Delta)t_{\mathrm{MIX}}(T) \le t_{\mathrm{MIX}}(G) \le C(r,\Delta)t_{\mathrm{MIX}}(T).$$