ASYMPTOTIC APERIODICITY AND THE STRONG RATIO LIMIT PROPERTY

Erik A. van Doorn
Department of Applied Mathematics
University of Twente
Enschede, The Netherlands

London Mathematical Society – EPSRC Durham Symposium

Markov Processes, Mixing Times and Cutoff

27 July - 4 August 2017

outline

1. general setting: discrete-time Markov chains

(i) strong ratio limit property (SRLP)

(ii) conditions for SRLP

(iii) problem

2. specific setting: birth-death processes

(i) necessary and sufficient condition for SRLP

(ii) sufficient condition for SRLP

(iii) probabilistic interpretation

3. SRLP and asymptotic aperiodicity

(i) asymptotic period

(ii) birth-death processes

(iii) Markov chains

setting

time-homogeneous, discrete-time Markov chain

$$\mathcal{X} \equiv \{X(n),\, n = 0, 1, \ldots\}$$

state space $S := \{0, 1, 2, \ldots\}$

matrix of one-step transition probabilities

$$P \equiv (P(i,j),\, i,j \in S)$$

n-step transition probabilities

$$P^{(n)}(i,j) \equiv \Pr\{X(m+n) = j \,|\, X(m) = i\}$$

$$P^{(n)} \equiv (P^{(n)}(i,j),\, i,j \in S) = P^n$$

assumption: P *irreducible, aperiodic, (sub)stochastic*

definition (Orey (1961)): $\mathcal{X}$ *recurrent*

$\mathcal{X}$ has strong ratio limit property (SRLP) if there exist positive constants $\mu(i)$, $i \in S$, such that

$$\lim_{n\to\infty} \frac{P^{(n+m)}(i,j)}{P^{(n)}(k,l)} = \frac{\mu(j)}{\mu(l)}, \quad i,j,k,l \in S,\ m \in \mathbb{Z}$$

definition (Pruitt (1965)): $\mathcal{X}$ *recurrent* or *transient*

$\mathcal{X}$ has SRLP if there exist positive constants ρ, $\mu(i)$, $i \in S$, and $f(i)$, $i \in S$, such that

$$\lim_{n\to\infty} \frac{P^{(n+m)}(i,j)}{P^{(n)}(k,l)} = \rho^m \frac{f(i)\mu(j)}{f(k)\mu(l)}, \quad i,j,k,l \in S,\ m \in \mathbb{Z}$$

problems: (i) give conditions on P for SRLP
(ii) identify constants ρ, $\mu(i)$ and $f(i)$

strong ratio limit property

SRLP: there exist positive constants ρ, $\mu(i)$ and $f(i)$ such that

$$\lim_{n\to\infty} \frac{P^{(n+m)}(i,j)}{P^{(n)}(k,l)} = \rho^m \frac{f(i)\mu(j)}{f(k)\mu(l)}, \quad i,j,k,l \in S,\ m \in \mathbb{Z}$$

SRLP prevails if and only if there exist positive constants ρ, $\mu(i)$ and $f(i)$, such that

$$\lim_{n\to\infty} \frac{P^{(n+1)}(i,j)}{P^{(n)}(i,j)} = \rho, \quad i,j \in S \tag{1}$$

$$\lim_{n\to\infty} \frac{P^{(n)}(i,j)}{P^{(n)}(i,l)} = \frac{\mu(j)}{\mu(l)}, \quad i,j,l \in S \tag{2}$$

$$\lim_{n\to\infty} \frac{P^{(n)}(i,j)}{P^{(n)}(k,j)} = \frac{f(i)}{f(k)}, \quad i,j,k \in S \tag{3}$$

theorem (Vere-Jones (1962)): the power series

$$P_{ij}(z) \equiv \sum_{n=0}^{\infty} P^{(n)}(i,j) z^n, \quad i,j \in S$$

have common radius of convergence R, $1 \le R < \infty$, and converge or diverge together

definition: P is R-transient if

$$P_{ij}(R) < \infty$$

and R-recurrent if

$$P_{ij}(R) = \infty$$

theorem (Kingman (1963)):

$$\lim_{n\to\infty} \left(P^{(n)}(i,j)\right)^{1/n} = \frac{1}{R}, \quad i,j \in S$$

hence, if

$$\lim_{n\to\infty} \frac{P^{(n+1)}(i,j)}{P^{(n)}(i,j)} \quad \text{exists}$$

then

$$\lim_{n\to\infty} \frac{P^{(n+1)}(i,j)}{P^{(n)}(i,j)} = \frac{1}{R}$$

so ρ in SRLP satisfies

$$\rho = \frac{1}{R}$$

ρ is decay parameter

SRLP: there exist positive constants ρ, $\mu(i)$ and $f(i)$ such that

$$\lim_{n\to\infty} \frac{P^{(n+m)}(i,j)}{P^{(n)}(k,l)} = \rho^m \frac{f(i)\mu(j)}{f(k)\mu(l)}, \quad i,j,k,l \in S, \ m \in \mathbb{Z}$$

SRLP prevails if and only if

$$\lim_{n\to\infty} \frac{P^{(n+1)}(i,j)}{P^{(n)}(i,j)} \text{ exists}, \quad i,j \in S \tag{1}$$

and there exist positive constants $\mu(i)$ and $f(i)$, such that

$$\lim_{n\to\infty} \frac{P^{(n)}(i,j)}{P^{(n)}(i,l)} = \frac{\mu(j)}{\mu(l)}, \quad i,j,l \in S \tag{2}$$

$$\lim_{n\to\infty} \frac{P^{(n)}(i,j)}{P^{(n)}(k,j)} = \frac{f(i)}{f(k)}, \quad i,j,k \in S \tag{3}$$

theorem (Pruitt (1965)): P *(sub)stochastic* and R-*recurrent*;

$$\lim_{n\to\infty} \frac{P^{(n+1)}(0,0)}{P^{(n)}(0,0)} \text{ exists}$$

$\iff$ P has SRLP:

$$\lim_{n\to\infty} \frac{P^{(n+m)}(i,j)}{P^{(n)}(k,l)} = \rho^m \frac{f(i)\mu(j)}{f(k)\mu(l)}, \quad i,j,k,l \in S,\ m \in \mathbb{Z}$$

where $\rho = R^{-1}$ and, up to constant factors, μ is *unique* ρ-invariant measure:

$$\sum_{i\in S} \mu(i)P(i,j) = \rho\mu(j), \quad j \in S$$

and f is *unique* ρ-harmonic function (or ρ-invariant vector):

$$\sum_{j\in S} P(i,j)f(j) = \rho f(i), \quad i \in S$$

if P is strictly substochastic (coffin state ∂), $\rho < 1$ and μ a ρ-invariant measure, that is,

$$\sum_{i \in S} \mu(i) P(i,j) = \rho \mu(j), \quad j \in S$$

then absorption at ∂ is certain and μ constitutes a (minimal) quasistationary distribution, that is

$$\mathbb{P}_\mu(X(n) = j \,|\, T > n) = \mu_j, \quad j \in S$$

with T denoting the absorption time

theorem (Pruitt (1965)): P *(sub)stochastic* and R-*recurrent*;

$$P \text{ has SRLP} \iff \lim_{n\to\infty} \frac{P^{(n+1)}(0,0)}{P^{(n)}(0,0)} \text{ exists}$$

sufficient conditions for SRLP:

- P is R-recurrent and symmetrizable (Pruitt (1965))
- P is R-recurrent and $P^{(n)}(i,i) \geq \varepsilon > 0$ for some n and all $i \in S$ (extension of Kingman & Orey (1964))

problems: (i) can we do better if P is R-recurrent?
(ii) what can be said if P is R-transient?

strong ratio limit property

setting: P *irreducible, aperiodic, (sub)stochastic* (but not necessarily R-recurrent)

theorem (Kesten (1995)): if for each n sufficiently large there exists a constant $\varepsilon \equiv \varepsilon(n) > 0$ such that $P^{(n)}(i,i) \geq \varepsilon$ for all $i \in S$ (= condition K) then

$$\lim_{n\to\infty} \frac{P^{(n+1)}(i,j)}{P^{(n)}(i,j)} = \rho, \quad i,j \in S$$

theorem (Handelman (1999)): assume condition K
SRLP $\iff$ there exist *unique* ρ-invariant measure μ and *unique* ρ-harmonic function f, in which case

$$\lim_{n\to\infty} \frac{P^{(n)}(i,j)}{P^{(n)}(i,l)} = \frac{\mu(j)}{\mu(l)} \quad \text{and} \quad \lim_{n\to\infty} \frac{P^{(n)}(i,j)}{P^{(n)}(k,j)} = \frac{f(i)}{f(k)}, \quad i,j,k,l \in S$$

Handelman (2002): "Your e-mail brought back painful memories – struggling through the details of the arguments in the paper – which I had put completely out of my mind."

conclusion: condition K + existence of unique ρ-invariant measure and unique ρ-harmonic function $\Rightarrow$ SRLP

remark: without condition K existence of unique ρ-invariant measure and unique ρ-harmonic function is *not necessary* for SRLP, so existence of

$$\lim_{n\to\infty} \frac{P^{(n+1)}(i,j)}{P^{(n)}(i,j)}, \quad i,j \in S$$

per se is not sufficient for Handelman's conclusions

problem: find condition weaker (and more elegant) than condition K for SRLP to prevail, *assuming existence of a unique ρ-invariant measure and unique ρ-harmonic function*

approach: first look at birth-death chains, then try to generalize

setting:

$$P = \begin{pmatrix} r_0 & p_0 & 0 & 0 & \cdots \\ q_1 & r_1 & p_1 & 0 & \cdots \\ 0 & q_2 & r_2 & p_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

matrix of 1-step transition probabilities of birth-death chain $\mathcal{X}$ on $\{0, 1, 2, \dots\}$

assumption: P irreducible, aperiodic, (sub)stochastic

recall: decay parameter

$$\rho = \frac{1}{R} \leq 1$$

with $R =$ radius of convergence of $\sum_{n=0}^{\infty} P^{(n)}(i,j) z^n$

letting

$$p_i Q_{i+1}(x) = (x - r_i)Q_i(x) - q_i Q_{i-1}(x), \quad i > 0$$
$$p_0 Q_1(x) = x - r_0, \quad Q_0(x) = 1$$

and

$$\pi_0 := 1, \quad \pi_i := \frac{p_0 \cdots p_{i-1}}{q_1 \cdots q_i}, \quad i > 0$$

we have (up to constant factors) *unique* ρ-harmonic function f

$$\sum_{j \in S} P(i,j) f(j) = \rho f(i) \quad \Longleftrightarrow \quad f(i) = c Q_i(\rho)$$

and *unique* ρ-invariant measure μ

$$\sum_{j \in S} \mu(j) P(j,i) = \rho \mu(i) \quad \Longleftrightarrow \quad \mu(i) = c \pi_i Q_i(\rho)$$

note: $\{Q_i\}$ orthogonal polynomial sequence with respect to (unique) Borel measure ψ on $(-1, 1]$

recall: for Markov chain condition K + existence of unique ρ-invariant measure and unique ρ-harmonic function implies SRLP

birth-death chain has unique ρ-harmonic function f and ρ-invariant measure μ, but we do *not* assume condition K

fact: P is symmetrizable so, by Pruitt's (1965) result, P has SRLP if P is R-recurrent

assumptions in what follows (wlog):

- P is *stochastic* and $\rho = 1$, so that $f(i) = Q_i(1) = 1$
- P is transient

theorem (Papangelou (1967)): P has SRLP (involving μ and f)

$$\iff \quad \lim_{n\to\infty} \frac{P^{(n+1)}(0,0)}{P^{(n)}(0,0)} \text{ exists}$$

results (vD & Schrijner (1995)):

$$\lim_{n\to\infty} \frac{P^{(n+1)}(0,0)}{P^{(n)}(0,0)} \text{ exists} \iff \lim_{n\to\infty} \frac{\int_{-1}^{0}(-x)^n \psi(dx)}{\int_0^1 x^n \psi(dx)} = 0$$

$$\lim_{n\to\infty} |Q_n(-1)| = \infty \Rightarrow \lim_{n\to\infty} \frac{\int_{-1}^{0}(-x)^n \psi(dx)}{\int_0^1 x^n \psi(dx)} = 0$$

hence $\lim_{n\to\infty} |Q_n(-1)| = \infty \quad \Rightarrow \quad$ SRLP

and under mild regularity conditions on ψ:

$$\lim_{n\to\infty} |Q_n(-1)| = \infty \iff \text{SRLP}$$

result:

$$\lim_{n\to\infty} |Q_n(-1)| = \infty \quad \Rightarrow \quad \text{SRLP}$$

with $\boldsymbol{Q}(x) := (Q_0(x), Q_1(x), \dots)$ we have $P\boldsymbol{Q}(x) = x\boldsymbol{Q}(x)$, and hence

$$P^2\boldsymbol{Q}(x) = x^2\boldsymbol{Q}(x)$$

while

$$Q_n(1) = 1, \quad |Q_n(-1)| \geq 1 \text{ and increasing}$$

so $\boldsymbol{Q}(1)$ and $\boldsymbol{Q}(-1)$ are two distinct solutions of $P^2\boldsymbol{y} = \boldsymbol{y}$, and hence any solution of $P^2\boldsymbol{y} = \boldsymbol{y}$, that is, any 1-harmonic function for P^2, is a linear combination of $\boldsymbol{Q}(1)$ and $\boldsymbol{Q}(-1)$

result: the constant function is the only bounded 1-harmonic function for P^2 $\quad \Rightarrow \quad$ P has SRLP

birth-death chains ($\rho = 1$)

result: the constant function is the only bounded 1-harmonic function for P^2 $\Rightarrow$ P has SRLP

recall: P (and hence P^2) is transient

boundary theory: the constant function is the only bounded 1-harmonic function for P^2 $\iff$ P^2 has exactly one escape route to infinity

birth-death chains ($\rho = 1$)

summary: assume (wlog) P *stochastic, transient* and $\rho = 1$, and define

$$p_i Q_{i+1}(x) = (x - r_i) Q_i(x) - q_i Q_{i-1}(x), \quad i > 0$$
$$p_0 Q_1(x) = x - r_0, \quad Q_0(x) = 1$$

(orthogonal polynomials w.r.t. measure ψ on $(-1, 1]$), then

SRLP prevails $\iff \lim_{n\to\infty} \dfrac{P^{(n+1)}(0,0)}{P^{(n)}(0,0)}$ exists

$$\iff \lim_{n\to\infty} \frac{\int_{-1}^{0} (-x)^n \psi(dx)}{\int_0^1 x^n \psi(dx)} = 0$$

$$\Leftarrow \quad \lim_{n\to\infty} |Q_n(-1)| = \infty \qquad \text{(conjecture: } \iff \text{)}$$

$\iff$ P^2 has exactly one escape route to ∞

asymptotic period

setting: Markov chain $\mathcal{X} \equiv \{X(n), n = 0, 1, \ldots\}$ on countable S with irreducible, aperiodic, stochastic transition matrix P

let $\quad \beta(\mathcal{X}) := \#$ almost closed sets for $\mathcal{X}$

(≈ # escape routes to infinity if $\mathcal{X}$ is transient)

$\mathcal{X}^{(m)} \equiv \{X(mn), n = 0, 1, \ldots\}$ $\quad m$-step chain

assumptions:

- $\mathcal{X}$ is *transient* and $\rho = 1$
- constant function is only bounded 1-harmonic function for $\mathcal{X}$ $(\beta(\mathcal{X}) = 1)$

definition: asymptotic period of $\mathcal{X}$:

$$d(\mathcal{X}) := \sup\{\beta(\mathcal{X}^{(m)}) \,|\, m \geq 1\} \quad (1 \leq d(\mathcal{X}) \leq \infty)$$

$\mathcal{X}$ is asymptotically aperiodic if $d(\mathcal{X}) = 1$

asymptotic period: birth-death chain

results: $\mathcal{X}$ is birth-death chain $\Rightarrow$ $d(\mathcal{X}) = 1, 2$ or ∞

$$d(\mathcal{X}) = 2 \text{ or } d(\mathcal{X}) = \infty \iff \beta(\mathcal{X}^{(2)}) = 2$$

hence

$$\beta(\mathcal{X}^{(2)}) = 1 \iff \mathcal{X} \text{ is asymptotically aperiodic}$$

recall: $\beta(\mathcal{X}^{(2)}) = 1 \Rightarrow \mathcal{X}$ has SRLP (conjecture: $\iff$)

conclusion:

$$\mathcal{X} \text{ is asymptotically aperiodic} \Rightarrow \mathcal{X} \text{ has SRLP}$$

conjecture (valid under mild regularity conditions):

$$\mathcal{X} \text{ is asymptotically aperiodic} \iff \mathcal{X} \text{ has SRLP}$$

setting: irreducible, aperiodic, (sub)stochastic Markov chain $\mathcal{X}$

asymptotic period $d(\mathcal{X})$

$$1 \leq d(\mathcal{X}) = \sup\{\beta(\mathcal{X}^{(m)}) \mid m \geq 1\} \leq \infty$$

birth-death setting:

- asymptotic aperiodicity of related birth-death process is sufficient (and, under mild conditions, necessary) for SRLP

general setting, assuming existence of unique ρ-harmonic function and ρ-invariant measure:

- asymptotic aperiodicity of two related Markov chains is not sufficient, but conjectured to be necessary for SRLP

generalization?

setting: Markov chain $\mathcal{X}$ on $S = \{0, 1, 2, \ldots\}$ with irreducible, aperiodic, (sub)stochastic transition matrix P

assumption: P has *unique* ρ-invariant measure μ and *unique* ρ-harmonic function f

let

$$\mu_D := \text{diag}(\mu(i),\ i \in S) \quad \text{and} \quad f_D := \text{diag}(f(i),\ i \in S)$$

and define

$$P_\mu := \frac{1}{\rho}\mu_D^{-1}P^T\mu_D \quad \text{and} \quad P_f := \frac{1}{\rho}f_D^{-1}Pf_D$$

then P_μ and P_f are nonnegative and *stochastic*, hence matrices of 1-step transition probabilities of Markov chains $\mathcal{X}_\mu$ and $\mathcal{X}_f$

generalization?

$$P_\mu := \frac{1}{\rho}\mu_D^{-1}P^T\mu_D \quad \text{and} \quad P_f := \frac{1}{\rho}f_D^{-1}Pf_D$$

P_μ and P_f are matrices of 1-step transition probabilities of (stochastic) Markov chains $\mathcal{X}_\mu$ and $\mathcal{X}_f$

also: P_μ and P_f are irreducible, aperiodic, $\rho(P_\mu) = \rho(P_f) = 1$

furthermore:

P_μ and P_f have unique 1-harmonic function $g(i) = 1$
so that $\mathcal{X}_\mu$ and $\mathcal{X}_f$ are simple

P_μ and P_f have unique 1-invariant measure $\nu(i) = \mu(i)f(i)$

P has SRLP $\iff$ P_μ and P_f have SRLP

and

$$P_\mu = P_f \iff P \text{ is symmetrizable}$$

generalization?

$$P_\mu := \frac{1}{\rho}\mu_D^{-1} P^T \mu_D \quad \text{and} \quad P_f := \frac{1}{\rho} f_D^{-1} P f_D$$

P_μ and P_f are matrices of 1-step transition probabilities of (stochastic) Markov chains $\mathcal{X}_\mu$ and $\mathcal{X}_f$

result: $\mathcal{X}$ satisfies condition K $\Rightarrow$ $\mathcal{X}_\mu$ and $\mathcal{X}_f$ asymptotically aperiodic

but asymptotic aperiodicity of $\mathcal{X}_\mu$ and $\mathcal{X}_f$ is not, in general, sufficient for the SRLP since

$\mathcal{X}$ is R-recurrent $\Rightarrow$ $\mathcal{X}_\mu$ and $\mathcal{X}_f$ asymptotically aperiodic

while example exists of recurrent chain not satisfying the SRLP

conjecture: $\mathcal{X}$ has SRLP $\Rightarrow$ $\mathcal{X}_\mu$ and $\mathcal{X}_f$ are asymptotically aperiodic