Geometric and topological aspects of M-branes

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Higher Structures in M-Theory

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Outline

I.Global overview

II. M-theory

III. Rational Homotopy Theory description

IIV. New connections and applications

Joint work with Urs Schreiber + Domenico Fiorenza, John Huerta, Vincent Braunack-Mayer

I. Global overview

Theme

Math *from* physics

Q1: *What new mathematical structures and constructions can we extract from studying M-theory?*

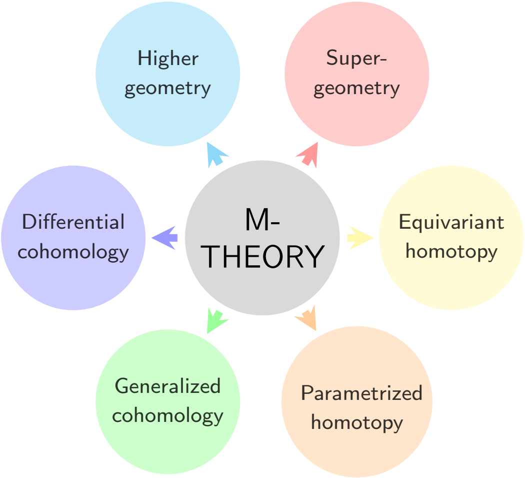
Math *in* Physics

Q2: *What mathematical structures/conditions/tools should we have in place in order to properly describe M-theory?*

Upshot

`By phrasing in context of math, the physics becomes more transparent.`

Richness of M-theory



The Physical theories and corresponding geometry

- M-theory in 11 dimensions.
  Objects ⊃ M-branes: M2-brane and M5-brane.
- Reduction to various string theories in 10 dimensions.
  Objects ⊃ string and D-branes.
- Requirements:
  - Consistent formulation.
  - No anomalies.
  - Mathematically rigorous.
  - Contain as 'much information' about the systems as possible.
- Schematically:

Physical theory	Structure	Group
Dirac theory of Spinors	Spin	$Spin(n)$
string theory	String	$String(n)$
fivebrane theory	Fivebrane	$Fivebrane(n)$

Structures associated to M-branes

Topological $\xrightarrow[\text{refinement}]{\text{Differential}}$ **Differential geometric**

Field strengths — *Potentials*

Anomalies/constraints — *Wilson loops/holonomy*

Brane	**Topological**	**Geometric**
Particle	F_2 field strength	connection A_1
String	H_3 field strength	B-field B_2
M2-brane	G_4 field strength	C-field C_3
M5-brane	G_4, H_3, G_7/H_7 dual	C-field C_3 B-field B_2 "dual field" C_6/B_6

Main setting/ingredients

String Theory	**M-Theory**
sigma model $\phi : \Sigma \hookrightarrow X^{10}$	sigma model $\Phi : M2 \hookrightarrow Y^{11}$
$\psi \in \Gamma(S\Sigma \otimes \phi^* TX^{10})$, $B_2 \rightsquigarrow$ 1-gerbe	$\psi \in \Gamma(SM2 \otimes \mathcal{N}(M2 \hookrightarrow Y^{11})$, $C_3 \rightsquigarrow$ 2-gerbe [Aschieri-Jurco, $\cdots$]
D-brane $\supset \partial\Sigma$	M5-brane $\supset \partial M2$
Freed-Witten condition $W_3 + [H_3] = 0 \in H^3(X^{10}; \mathbb{Z})$	Witten flux quantization $\frac{1}{2}\lambda + [G_4] = a \in H^4(Y^{11}; \mathbb{Z})$

Viewpoint since [Kriz-S. 04]: Physical conditions via *obstruction theory* (AHSS):

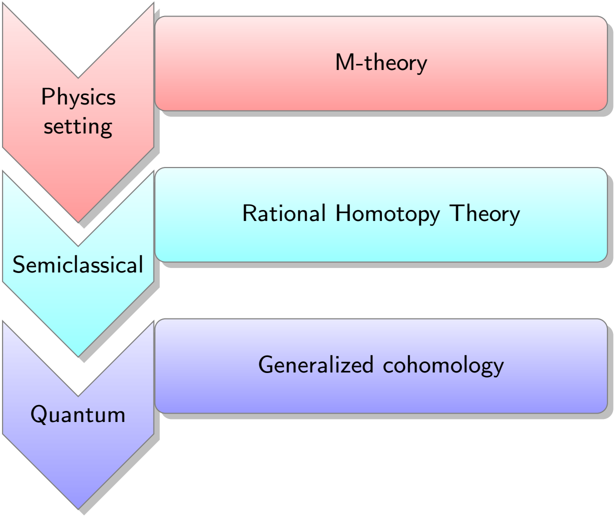
Twisted Spin^c	Twisted String [Wang, S.-Schreiber-Stasheff]
Twisted K-theory (AHSS)	Twisted elliptic cohomology [S.]

Homotopic constructions:

[Ando-S., S.-Westerland, Lind-S.-Westerland]

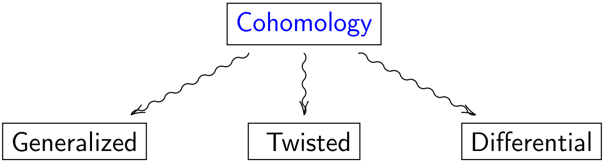
Here we step back: **Rational + geometry + other angles**.

Context


Physics setting
M-theory
Semiclassical
Rational Homotopy Theory
Quantum
Generalized cohomology

Generalities on what physics wants

Nontrivial physical entities, such as fields, charges, etc. generically take values in cohomology.



I. **Generalized**: Capture essential topological and bundles aspects.

II. **Twisting**: Account for symmetries via automorphisms.

III. **Differentially refined**: Include geometric data, such as connections, Chern character form, smooth structure, smooth representatives of maps ...

I. Generalized cohomology

Motivation from modelling of **fields** (in QFT, string theory and M-theory).

Schematically:

$$\begin{array}{ccc} \Omega^\bullet(M) & & \textit{Classical} \\ \rightsquigarrow \text{ exact complex} & & \\ H^\bullet_{\mathrm{dR}}(M) & & \\ \rightsquigarrow \text{ quantization} & & \rightsquigarrow \\ H^\bullet(M;\mathbb{Z}) & & \\ \rightsquigarrow \text{ anomaly cancellation} & & \\ E^\bullet(M) & & \textit{Quantum} \end{array}$$

$\Rightarrow$ **Partition functions** are sums/integrals over the moduli spaces of fields

$$Z = \int_{\mathcal{M}} e^{iS}$$

should take values in E.

II. Twists

- We would like to introduce automorphisms.
- These arise from geometric and physical considerations.
- Homotopy p.o.v.: moduli/family setting; bundles of spectra.

$$\Omega^\bullet(M) \circlearrowleft^{\mathit{twist}_\Omega} \xrightarrow[\text{complex}]{\text{exact}} H^\bullet_{\mathrm{dR}}(M) \circlearrowleft^{\mathit{twist}_{\mathrm{dR}}} \xrightarrow{\text{quantization}} H^\bullet(M;\mathbb{Z}) \circlearrowleft^{\mathit{twist}_H} \xrightarrow[\text{cancellation}]{\text{anomaly}} E^\bullet(M) \circlearrowleft^{\mathit{twist}_E}$$

Relations among various twists?

Example (twist_Ω)

Twisted differential forms are forms valued in the orientation line bundle. Top such form is a density (pseudo-volume form).

Example: Twisted de Rham cohomology

- The de Rham complex $(\Omega^\bullet, d)$: $\quad \ldots \xrightarrow{d} \Omega^i(X) \xrightarrow{d} \Omega^{i+1}(X) \xrightarrow{d} \ldots$
- Twist by a 1-form built out of scalar ftn: $d \rightsquigarrow d_\phi := d + d\phi\wedge$ with $d_\phi^2 = 0$.

Example (Witten's deformation of Morse theory)

For smooth $f : M \to \mathbb{R}$, the Witten differential is $d_s = e^{-sf} d e^{sf} = d + sdf\wedge$, where $s \in \mathbb{R}$. Then $d_s^2 = 0$, $d_s : \Omega^p \to \Omega^{p+1}$. The term e^{-sf} is a quasi-isomorphism

$$\begin{array}{ccccccc} \ldots & \longrightarrow & \Omega^p & \xrightarrow{d} & \Omega^{p+1} & \longrightarrow & \ldots \\ & & {\scriptstyle e^{-sf}}\downarrow & \circlearrowright & \downarrow{\scriptstyle e^{-sf}} & & \\ \ldots & \longrightarrow & \Omega^p & \xrightarrow{d_s} & \Omega^{p+1} & \longrightarrow & \ldots \end{array}$$

and d_s yields isomorphic cohomology groups.

- Twist by a closed 3-form: $d_{H_3} = d - H_3\wedge$, with $d_{H_3}^2 = 0$.

Definition

Twisted de Rham cohomology: $H^i(X, H_3) := \ker(d_{H_3})/\mathrm{im}(d_{H_3})$

Example (*The Ramond-Ramond (RR) fields in string theory*)

$F = \sum_{i\leq 5} u^{-i} F_{2i+\epsilon}$, $\epsilon = 0$ or 1 for type IIA or type IIB string theory. These are twisted by a closed 3-form, the NS-field H_3.

Higher twists?

- Mathematically: can build a differential by adding to d_H all expressions of the form $u^{-i}H_{2i+1}\wedge$, i.e.

$$d'_H = d + \sum_{i=0}^{\infty} u^{-i} H_{2i+1} \wedge \ .$$

- There is a twisted graded de Rham complex with differential $d + \sum_{i=1}^{\infty} u^{-i} H_{2i+1}\wedge$, provided the differential forms H_{2i+1} are closed.

Example (Degree seven twist in heterotic string theory [S.08])

Form $\mathcal{F} = F + *F$, with F is the abelianized Yang-Mills field & $*F$ its dual. Variation of the action $S = \int H_3 \wedge *H_3 + \int F_2 \wedge *F_2$ with respect to A and using the "Chapline-Manton coupling" $H_3 = CS_3(A)$ gives

$$\boxed{(d - H_7\wedge)\mathcal{F} = 0}.$$

This gives a twisted differential $d_{H_7} = d - H_7\wedge$ which in nilpotent, i.e. squares to zero, $d^2_{H_7} = 0$, since H_7 is closed.

revisit later in dualities →

Reverse engineering for twisted generalized cohomology

Rational twisted cohomology arises as image of some Chern character.

Example (Twisted K-theory)

Degree three twist H_3:

$$\mathrm{ch}_{H_3} : \underbrace{K^\bullet(X, H_3)}_{\text{twisted K-theory}} \longrightarrow \underbrace{H^{\mathrm{ev}}(X, H_3)}_{\textit{twisted de Rham cohomology}}$$

- Now if we are presented with higher degree twists on the left-hand-side, would they be images of some generalized Chern character whose domain is some generalized cohomology theory?

$$\mathrm{ch}^E_{\mathrm{tw}} : E^\bullet(-; \mathrm{twist}) \longrightarrow H^\bullet(-; \mathrm{twist}) .$$

III. Differential refinement

- Introduce geometric data via differential forms (connections, Chern forms, $\cdots$), i.e., retain differential form representatives of cohomology classes.

$$
\begin{array}{ccccc}
 & & \Omega^\bullet(M) & & \\
 & \swarrow^{\text{adjoin}} & \downarrow{\scriptstyle \text{adjoin}} & \searrow^{\text{adjoin}} & \\
H^\bullet_{\mathrm{dR}}(M) & & H^\bullet(M;\mathbb{Z}) & & E^\bullet(M) \\
\downarrow{\scriptstyle \text{refinement}} & & \downarrow{\scriptstyle \text{refinement}} & & \downarrow{\scriptstyle \text{refinement}} \\
\widehat{H^\bullet}_{\mathrm{dR}}(M) & & \widehat{H^\bullet}(M;\mathbb{Z}) & & \widehat{E^\bullet}(M)
\end{array}
$$

- Amalgam of an underlying (topological) cohomology theory and the data of differential forms:

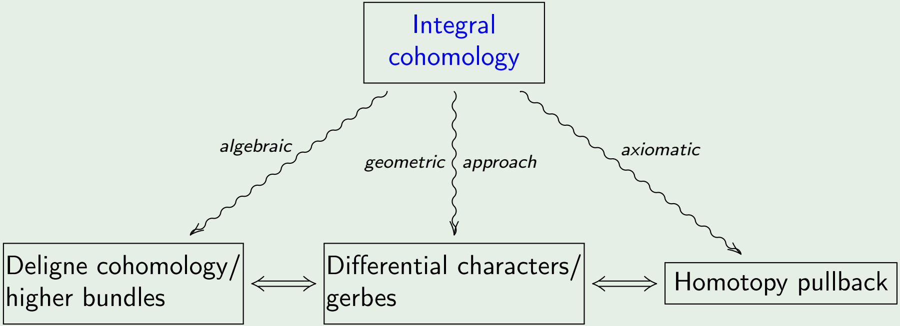
$$
\begin{array}{ccc}
\text{Differential gen. cohomology} & \longrightarrow & \text{Forms} \\
\downarrow & & \downarrow \\
\text{Gen. cohomology} & \longrightarrow & \text{de Rham cohomology}
\end{array}
$$

- That is, we have a fiber product or twisted product

$$\text{“Differential cohomology} = \text{Cohomology} \times_{\text{de Rham}} \text{Forms”}$$

Example (Differential refinement of integral cohomology)

Various approaches to differential integral cohomology:



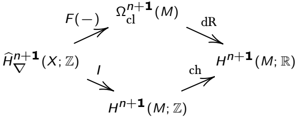
Consider the truncated de Rham complex

$$[\ \Omega^{\mathbf{0}} = \mathcal{O} \xrightarrow{d} \Omega^{\mathbf{1}} \xrightarrow{d} \Omega^{\mathbf{2}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n}\ ]$$

Replace the structure sheaf $\mathcal{O}$ with the multiplicative group $\mathcal{O}^{\times}$ under the exponential map to get the Deligne complex

$$[\ \mathcal{O}^{\times} \xrightarrow{d} \Omega^{\mathbf{1}} \xrightarrow{d} \Omega^{\mathbf{2}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n}\ ]$$

Deligne cohomology $H_{\mathcal{D}}^{n+\mathbf{1}}(X)$ in degree $n+\mathbf{1}$ is the hypercohomology for this complex of sheaves of abelian groups, i.e. abelian sheaf cohomology with coefficients in this chain complex.



[Schreiber ...]

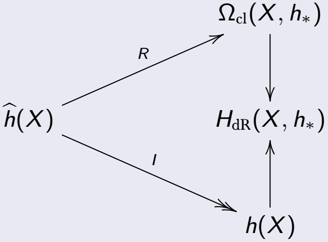
Differential generalized cohomology

- Start with a generalized cohomology theory h
- $\Omega(X, h_*) := \Omega(X) \otimes_{\mathbb{Z}} h_*$ Smooth differential forms with coefficients in $h_* := h(*)$
- $\Omega_{\mathrm{cl}}(X, h_*) \subseteq \Omega(X, h_*)$ closed forms
- $H_{\mathrm{dR}}(X, h_*)$ cohomology of the complex $\big(\Omega(X, h_*), d\big)$

Definition

A smooth extension of h is a contravariant functor

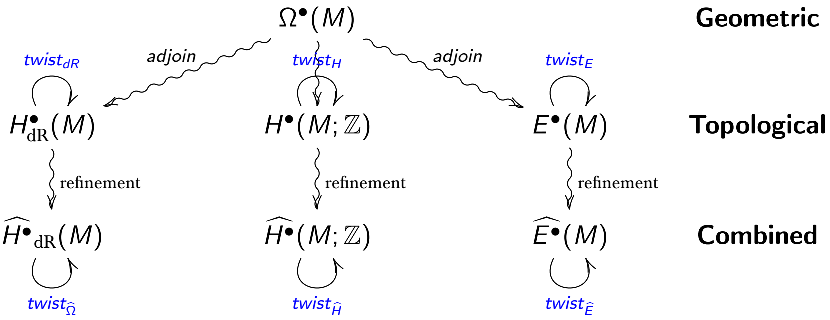
$$\widehat{h} : \textbf{Compact Smooth Manifolds} \longrightarrow \textbf{Graded Abelian Grps}$$



[Chern-Simons, Cheeger-Simons, Simons-Sullivan, Hopkins-Singer, Bunke-Schick, Schreiber, ...]

Full structure

Twisted $\cap$ Differential $\cap$ Generalized



[Bunke-Nikolaus, Grady-S.]

Examples

1. Twisted Movara K-theory K(2) & E-theory E(2) [S.-Westerland], twisted tmf [with Ando].
2. Twisted K-theories of n-vector bundles. e.g. K(K(KU)) [S.-Lind-Westerland].
3. Differential refinements of twisted cohomology theories including above [Grady-S.].

Application: What is needed to describe RR fields?

Recall approach [Kriz-S] : Physical conditions as obstructions, to orientation or as differentials in the AHSS spectral sequence.
The spectral sequences can be extended to the differential refinements, that is we can discuss theory E by adjoining geometric data to it.

Theorem (Grady-S.)

We have the differential refinement of the following:

1. *Primary cohomology operations: Steenrod Sq.*
2. *Secondary cohomology operations: Massey* $\langle \cdot, \cdot, \cdots \rangle_{\text{Massey}}$.
3. *AHSS with a concrete identification of the differentials .*

$$
\begin{array}{ccc}
\boxed{\text{Untwisted theory } \mathrm{E}} & \xrightarrow{\text{twist}} & \boxed{\text{Twisted theory } \mathrm{E}_{\text{tw}}} \\
\Big\downarrow \text{refinement} & & \Big\downarrow \text{refinement} \\
\boxed{\text{Differential theory } \widehat{E}} & \xrightarrow{\text{twist}} & \boxed{\text{Twisted differential theory } \widehat{E}_{\text{tw}}}
\end{array}
$$

$$
\begin{array}{ccc}
d & \xrightarrow{\text{twist}} & d_{\text{twist}} = d + \textit{secondary operation} \\
\Big\downarrow \text{refinement} & & \Big\downarrow \text{refinement} \\
d_{\text{refined}} = \widehat{d} & \xrightarrow{\text{twist}} & \widehat{d}_{\text{twist}} = \widehat{d} + \widehat{\textit{secondary operation}}
\end{array}
$$

Higher tangential structures

Proper description of fields requires some extra tangential structure

$$\begin{array}{ccc} & & BSO\langle n\rangle \\ & \overset{\bar{f}}{\nearrow} & \downarrow \\ X & \xrightarrow{\;f\;} & BSO \end{array}$$

Spin structure: $n = 2$

$$\begin{array}{ccccc} & & B\mathrm{Spin} & & \\ & \overset{\bar{f}}{\nearrow} & \downarrow & & \\ X & \xrightarrow{\;f\;} & BSO & \xrightarrow{\;w_2\;} & K(\mathbb{Z}_2, 2) \end{array}$$

String structure $n = 4$

$$\begin{array}{ccccc} & & BString & & \\ & \overset{\bar{f}}{\nearrow} & \downarrow & & \\ X & \xrightarrow{\;f\;} & B\mathrm{Spin} & \xrightarrow{\;\frac{1}{2}p_1\;} & K(\mathbb{Z}, 4) \end{array}$$

- Even higher structures: *Fivebrane* $(n = 9)$, *Ninebrane* $(n = 12)$.

k		7	8	9	10	11	12
$\pi_k(\mathrm{O}(n))$		$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
$\mathrm{O}(n)\langle k\rangle$	$\mathrm{String}(n)$	$\mathrm{Fivebrane}(n)$	$\mathrm{O}\langle 9\rangle(n)$	$\mathrm{O}\langle 10\rangle(n)$		$\mathrm{Ninebrane}(n)$	
		kill π_7	kill π_8	kill π_9		kill π_{11}	

At the infinitesimal level:

Examples

1. $\mathfrak{string}$ is a Lie 2-algebra
2. $\mathfrak{fivebrane}$ is a Lie 6-algebra
3. $\mathfrak{ninebrane}$ is a Lie 10-algebra.

- These are truncations of L_∞-algebras.
- Characteristic classes of L_∞-algebra bundles [S.-Schreiber-Stasheff].

Remark (Variations)

- **Twisted:** All the above can be twisted and differentially refined, e.g. *twisted differential String structures* etc. [S.-Schreiber-Stasheff].
- **Stacky:** Via stacks and higher bundles [Fiorenza-Schreiber-Stasheff].
- **Indefinite:** Above structures can be defined for the indefinite (Lorentzian case) via Spin(p, q) [S.-Shim].
- **Rational:** Explicit characterizations at the level of rational homotopy and cohomology [S.-Wheeler].

... ***more later*** ...

Why stacks? (in a nutshell)

G a Lie group $\rightsquigarrow$ Classifying space BG is a topological space

$$\boxed{[X, BG]} \simeq \boxed{\text{equivalence classes of } G\text{-principal bundles on } X}$$

- Shortcoming: BG does not know about:
  1. the smooth gauge transformations: G-valued functions,
  2. actual gauge fields: connections on G-principal bundles.
- Remedy: There is a smooth groupoid/smooth stack $\mathbf{B}G$:

$$\boxed{\text{maps of smooth stacks } X \to \mathbf{B}G} \simeq \boxed{G\text{-bundles on } X},$$

$$\{\text{homotopies of such maps}\} \simeq \{\text{ smooth gauge transformations}\}.$$

- *Differential refinement* to a richer smooth stack $\mathbf{B}G_\nabla$:

$$\boxed{\text{maps } X \to \mathbf{B}G_\nabla} \simeq \boxed{G\text{-Yang-Mills gauge fields on } X},$$

- True configuration space: smooth mapping stack $[X, \mathbf{B}G_\nabla]$:
  - elements are gauge fields on X,
  - morphisms are gauge transformations.

Higher $U(1)$-bundles

Machine: Chain complexes $\xrightarrow[\text{+stackification}]{\text{Dold-Kan}}$ Stacks

Definition (Fiorenza-Schreiber-Stasheff)

1. The *n-stack of $U(1)$-n-bundles* (without connection) $\boxed{\mathbf{B}^n U(1)}$ is obtained via "Dold-Kan" + "stackification" from the sheaf of chain complexes
$$\underline{U}(1)[n] = \left(\underline{U}(1) \to 0 \to \cdots \to 0 \right) ,$$
with $C^\infty(-; U(1))$ in degree n.

2. The *n-stack of $U(1)$-n-bundles with connections* $\boxed{\mathbf{B}^n U(1)_\nabla}$ is obtained by "Dold-Kan" + "stackification" to the $(n+1)$-term Deligne complex
$$\underline{U}(1)[n]^\infty_D = \left(\underline{U}(1) \xrightarrow{\frac{1}{2\pi i} d\log} \Omega^1(-;\mathbb{R}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(-;\mathbb{R}) \right) ,$$
where $\underline{U}(1)$ is the sheaf of smooth functions with values in $U(1)$, and with $\Omega^n(-;\mathbb{R})$ in degree zero.

- Equivalence classes of $U(1)$-n-bundles on X are in natural bijection with
$$H^{n+1}(X;\mathbb{Z}) \cong H^n(X;\underline{U}(1)) \cong \mathbb{H}^0(X;\underline{U}(1)[n]) \cong \pi_0 \mathbf{H}(X;\mathbf{B}^n U(1)) .$$
- Equivalence classes of $U(1)$-n-bundles with connection on smooth manifold X
$$\hat{H}^{n+1}(X;\mathbb{Z}) \cong \mathbb{H}^0(X;\underline{U}(1)[n]^\infty_D) \cong \pi_0 \mathbf{H}(X;\mathbf{B}^n U(1)_\nabla) .$$

Geometric realizations and smooth refinement

Obvious morphism of chain complexes of sheaves $\underline{U}(\mathbf{1})[n]_D^\infty \to \underline{U}(\mathbf{1})[n]$ induces:

– the forget the connection morphism $\mathrm{B}^n U(\mathbf{1})_\nabla \to \mathrm{B}^n U(\mathbf{1})$,

– Level of equivalence classes: natural morphism $\hat{H}^{n+\mathbf{1}}(X;\mathbb{Z}) \to H^{n+\mathbf{1}}(X;\mathbb{Z})$ from differential cohomology to integral cohomology.

While smooth higher stacks have richer structure than topological spaces, there is a map called geometric realization $|-|$ that sends any smooth higher stack to the topological space which is the "best approximation" to it, in a precise sense.

Examples

1. The geometric realization of the n-stack $\mathbf{B}^n\mathrm{U}(1)$ is the *Eilenberg-MacLane space* $K(\mathbb{Z}, n+1)$ (notice the degree shift) which classifies integral cohomology $|\mathbf{B}^n\mathrm{U}(1)| \simeq K(\mathbb{Z}, n+1)$.
2. The geometric realization of the moduli stack $\mathbf{B}\mathrm{Spin}$ of Spin-principal bundles is the ordinary classifying space $B\mathrm{Spin}$: $|\mathbf{B}\mathrm{Spin}| \simeq B\mathrm{Spin}$ (all up to weak homotopy equivalence).

II. M-theory

Bosonic 11D supergravity

- **Bosonic Lagrangian:** given by the eleven-form [Cremmer-Julia-Scherk]

$$\boxed{\mathsf{L}_{11}^{\text{bos}} = R *\mathbb{1} - \tfrac{1}{2} G_4 \wedge * G_4 - \tfrac{1}{6} G_4 \wedge G_4 \wedge C_3} \,. \tag{1}$$

- **Equations of motion:** The variation $\frac{\delta L_{(\mathbf{11}),\text{bos}}}{\delta C_3} = 0$ for C_3 gives the corresponding equation of motion

$$\boxed{\mathrm{d}{*}\mathrm{G}_4 + \tfrac{1}{2} G_4 \wedge G_4 = 0} \,. \tag{2}$$

- **Bianchi identity:**

$$\boxed{\mathrm{dG}_4 = 0} \,. \tag{3}$$

- The second order equation (2) can be written in a first order form, by first writing $d\left(*G_4 + \frac{1}{2} C_3 \wedge G_4\right) = 0$ so that

$$\boxed{{*}\mathrm{G}_4 = G_7 := dC_6 - \tfrac{1}{2} C_3 \wedge G_4} \,, \tag{4}$$

where C_6 is the potential of G_7, the Hodge dual field strength to G_4 in 11 dimensions.

The effect of the fermions

- The femionic field $\psi \in \Gamma(S \otimes TM)$ (the gravitino) satisfies the generalized Dirac equation, the generalized Rarita-Schwinger equation

$$\boxed{\mathrm{D}_{RS}\psi = 0\ , \qquad \psi \in \Gamma(S \otimes T^*M)}\ .$$

(involves mixing of terms).

- The fields themselves are in fact combinations of bosonic and fermionic fields. Physics literature usually writes:

$$G_4^{\text{super}} = \underbrace{G_4}_{\rightsquigarrow\textbf{topology/geometry}} + \underbrace{\overline{\psi}\Gamma_2\psi}_{\rightsquigarrow\textbf{topology/geometry}}$$

- Similarly for the connections

$$\omega^{\text{super}} = \omega + \text{fermion-bilinears}$$

[See Duff-Nilsson-Pope]

Strategy: Extract topology/higher geometry from bosons and fermions separately.

The M-theory gauge algebra

- Action $S_{\text{bos}} = \int_Y d\text{vol}(Y) L_{\text{bos}}$, and hence the EOMs, are invariant under the abelian gauge transformation

$$\delta C_3 = d\lambda_2,$$

Alternatively write the gauge parameter as $\Lambda_3 = d\lambda_2$.

- First order eqn (4) is invariant under the infinitesimal gauge transformations

$$\delta C_3 = \Lambda_3, \qquad \delta C_6 = \Lambda_6 - \tfrac{1}{2}\Lambda_3 \wedge C_3,$$

where Λ_6 is the 6-form gauge parameter satisfying $d\Lambda_6 = 0$.

- Applying two successive gauge transformations:

$$\boxed{\begin{aligned} \left[\delta_{\boldsymbol{\Lambda_3}}, \delta_{\boldsymbol{\Lambda'_3}}\right] &= -\delta_{\boldsymbol{\Lambda''_6}} \ , \\ \left[\delta_{\boldsymbol{\Lambda_3}}, \delta_{\boldsymbol{\Lambda_6}}\right] &= 0 \ , \\ \left[\delta_{\boldsymbol{\Lambda_6}}, \delta_{\boldsymbol{\Lambda'_6}}\right] &= 0 \ , \end{aligned}}$$

with the new parameter $\Lambda''_6 = \Lambda_3 \wedge \Lambda'_3$.

- Nonlinear due to Chern-Simons form.

[See Kalkkinen-Stelle]

- Introduce generators v_3 and v_6 for Λ_3 and Λ_6 gauge transformations, resp.
- On the generators, we get the graded Lie algebra

$$\begin{aligned} \{v_3, v_3\} &= -v_6 \ , \\ [v_3, v_6] &= 0 \ , \\ [v_6, v_6] &= 0 \ . \end{aligned}$$

- Use graded commutators:

$$\boxed{\begin{aligned} &[v_3, v_3] = -v_6 \ , \\ &[v_3, v_6] = 0 \ , \\ &[v_6, v_6] = 0 \ . \end{aligned}}$$

- **Properties:**
  1. Constant: $dv_3 = 0 = dv_6$.
  2. Grading on the generators v_3 and v_6 follow that of the potentials C_3 and C_6.
- Maurer-Cartan flatness: Total uniform degree field strength $\mathcal{G} = d\mathcal{V}\mathcal{V}^{-1}$ with

$$\mathcal{V} = e^{C_3 \otimes v_3} e^{C_6 \otimes v_6} . \qquad (5)$$

The equation of motion for C_3 (= Bianchi identity for C_6) and the Bianchi identity for C_3 are obtained together from

$$d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0 \ .$$

[See Cremmer-Julia-Lu-Pope, S09]

$\longrightarrow$ ***We will offer two (related) interpretations.***

Rational degree four twists

Three-form C_3 with $G_4 = dC_3$. We can build a differential with G_4 as $d_{G_4} = d + v_3^{-1} G_4 \wedge$

Proposition (S.09)

The de Rham complex can be twisted by a differential of the form $d + v_{2i-1}^{-1} G_{2i} \wedge$ provided that G_{2i} is closed and v_{2i-1} is Grassmann algebra-valued.

Form a graded uniform degree form $G = v_3^{-1} G_4 + v_6^{-1} G_7$. This expression can now be used to twist the de Rham differential, leading to

$$d_G = d + G\wedge = d + v_3^{-1} G_4 \wedge + v_6^{-1} G_7 \wedge \ . \tag{6}$$

Proposition (S.09)

The de Rham complex can be twisted by the differential d_G provided $\{v_3, v_3\} = v_6$ and $dG_7 = -\frac{1}{2} G_4 \wedge G_4$.

- The first condition is the **M-theory gauge algebra** and the second is the **equation of motion**.

Massey products in M-theory

Main Points

[Kriz-S.]: Lifted Chern-Simons term can be written as a Massey triple product and the one-loop term can be explained as being a part of the Massey product indeterminacy.

Chern-Simons term and the one-loop gravitational correction term,

$$\frac{1}{6}\int_{Y^{11}} C_3 \wedge G_4 \wedge G_4 - C_3 \wedge I_8^{\mathrm{dR}} \tag{7}$$

where I_8^{dR} is a polynomial in the curvature of Y^{11} whose class is

$$I_8 = \frac{1}{48}(p_2 - \lambda^2)\,, \qquad \lambda := \frac{1}{2}p_1$$

View the EOM

$$d * G_4 = -\frac{1}{2} G_4 \wedge G_4. \tag{8}$$

as a trivialization of the cup product $[G_4] \cup [G_4] = 0$, by writing

$$G_4 \wedge G_4 = -2d * G_4$$

Once the cup product is trivial as a primary cohomology operation, one can introduce a secondary cohomology operation, the Massey product on the kernel of the first operation.

- A differential graded algebra (DGA) is a graded algebra A with a map $d : A \to A$ of degree $+1$ which satisfies the relations

$$\begin{aligned} dd &= 0, \\ d(\alpha\beta) &= (d\alpha)\beta + (-1)^{\dim \alpha}\alpha(d\beta). \end{aligned}$$

- Then the cohomology $H(A)$ of A with respect to d is a graded algebra.
- It has further certain operations called Massey products, as a correspondence

$$H(A) \otimes H(A) \otimes H(A) \to H(A) \tag{9}$$

which is denoted by $[\alpha, \beta, \gamma]$, where $\alpha, \beta, \gamma \in H(A)$. It is defined only when $\alpha\beta = \beta\gamma = 0 \in H(A)$, and the dimension of the result is

$$\dim(\alpha) + \dim(\beta) + \dim(\gamma) - 1. \tag{10}$$

- It is also not well defined, it is only defined modulo terms of the form $\alpha x + y\beta$ where x, y are some elements of $H(A)$.

Definition (Massey product)

With $\alpha\beta = dy, \beta\gamma = dz$ for $y, z \in A$, set

$$\langle \alpha, \beta, \gamma \rangle = y\gamma + (-1)^{\dim \alpha + 1}\alpha z. \tag{11}$$

This is a cocycle and the cohomology class is defined modulo the indeterminacy given above.

Apply to M-theory:

Main Points

- The EOMs define a triple Massey product $\langle G_4, G_4, G_4 \rangle$ as a coset in H^{11}.
- If we view $*G_4$ as an independent field G_7 then we can write the Lagrangian itself as a Massey product.
- The one-loop term can be explained as a part of the indeterminacy. So the Massey product predicts its existence.
- More precisely, the term is of the form

$$G_4 \wedge I_7 \tag{12}$$

  where I_7 is a 7-dimensional cohomology class in Y^{11}.
- We can view I_7 as a flat potential for I_8.
- From the structural point of view, this hints at underlying rich homotopic structures.

Differential refinement:

[Grady-S.] Refinement of Massey products to differential cohomology.

M-branes and nonabelian Chern-Simons

- When a class is trivial in cohomology $[\beta_i] = 0 \in H^i(X; \mathbb{R})$ then the corresponding differential form is exact $\beta_i = d\gamma_{i-1}$.
  $\Rightarrow$ This allows us to consider boundaries.
- For the M2-brane: Trivially we essentially have Chern-Simons theory. This arises from the trivialization of the String structure, $p_1(\omega) \sim dCS_3$.
- What about the M5-brane? A trivialization of the Fivebrane structure $p_2(\omega) \sim dCS_7$.
  In fact, it will be a 'composite' (cup-product) nonabelian Chern-Simons theory: $CS_{M5} \sim CS_7 + CS_3 \wedge p_1$.

  $\Rightarrow$ Capture aspects of the nonabelian gerbe theory on the (extended) M5-brane worldvolume via 7d Chern-Simons [FSS].

The abelian CS-theory. Conformal blocks of $(0, 2)$-SCFT are identified with the geometric quantization of a 7d CS-theory

$$C_3 \mapsto \int_{X^7 \times S^4} C_3 \wedge G_4 \wedge G_4 \quad = N \int_{X_7} C_3 \wedge dC_3\,, \tag{13}$$

where $N := \int_{S^4} G_4$. This induces on its 6-dimensional boundary the self-dual 2-form.

The nonabelian CS-theory. One-loop term via M5-branes:

$$(\omega, C_3) \mapsto \int_{X^7 \times S^4} C_3 \wedge \left(\tfrac{1}{6} G_4 \wedge G_4 - I_8^{\mathrm{dR}}(\omega)\right)\;, \tag{14}$$

So we pick up another 7-dimensional Chern-Simons term, now one which depends on *nonabelian* fields. Locally,

$$S_{7d\mathrm{CS}} : (\omega, C_3) \mapsto \frac{N}{6} \int_{X^7} C_3 \wedge dC_3 - N \int_{X^7} \mathrm{CS}_{I_8}(\omega)\;, \tag{15}$$

where $\mathrm{CS}_{I_8}(\omega)$ is a Chern-Simons form for $I_8^{\mathrm{dR}}(\omega)$

$$d\mathrm{CS}_{I_8}(\omega) = I_8^{\mathrm{dR}}(\omega)\,. \tag{16}$$

Boundaries.

Recall flux quantization condition

$$\boxed{2[G_4] = \tfrac{1}{2}p_1 + 2a \;\in\; H^4(X,\mathbb{Z})}\,. \tag{17}$$

On an asymptotic neighborhood of the asymptotic boundary ∂X:

$$\boxed{\tfrac{1}{2}p_1 + 2a = 0 \;\in\; H^4(\partial X,\mathbb{Z})}\,. \tag{18}$$

Notice that $[G_4] = 0$ at the boundary means that the C field is still there, but given by a globally defined differential 3-form C_3.
Imposing condition (18) in a gauge equivariant way involves refining it from an equation between cohomology classes (hence gauge equivalence classes) to a *choice of coboundary* between cocycles for $\frac{1}{2}p_1$ and $2a$.
$\Rightarrow$

Main Points

locally: Spin connection $\rightsquigarrow$ globally: *2-connection on a twisted* String*-principal 2-bundle*, or equivalently a *twisted differential String structure*, where the twist is given by the class $2a$.

Example of "cup-product Chern-Simons theories" [FSS].

A dictionary

between stacky notions and the corresponding bundle structures appearing in relation to M5-branes and M-theory:

symbol	(higher) moduli stack of...
$\mathbf{B}\mathrm{U}(1)$	circle bundles / Dirac magnetic charges
$\mathbf{B}\mathrm{U}(1)_{\nabla}$	$\mathrm{U}(1)$-connections / abelian Yang-Mills fields
$\mathbf{B}\mathrm{Spin}_{\nabla}$	Spin connections / field of gravity
$\mathbf{B}E_8$	E_8-instanton configurations
$(\mathbf{B}E_8)_{\nabla}$	E_8-Yang-Mills fields
$\mathbf{B}^2\mathrm{U}(1)_{\nabla}$	B-field configurations (without twists)
$\mathbf{B}^3\mathrm{U}(1)_{\nabla}$	C-field configurations (without twists)
$\mathbf{B}\mathrm{String}_{\nabla}$	String 2-connections / nonabelian 2-form connections
$\mathbf{B}\mathrm{String}^{2\mathbf{a}}$	E_8-twisted String-2-connections
CField	bulk configurations of supergravity C-fields (and gravity)

The moduli stack of supergravity C-field configurations [FSS]

- Locally: 3-form (C-field), $\mathfrak{so}$-valued 1-form (vierbein), and on boundary: $\mathfrak{e}_8$-valued 1-form (the gauge field), and B-field.
- Globally: these fields are interrelated and arrange to certain nonabelian twisted differential cocycles.
- The analogy with $\mathbf{dd} : \mathbf{B}\mathrm{PU}(\mathcal{H}) \to \mathbf{B}^2 U(1)$ is the canonical map

$$\mathbf{a} : \mathbf{B}E_8 \to \mathbf{B}^3 U(1) \tag{19}$$

  from the moduli stack of E_8-bundles to that of circle 3-bundles / bundle 2-gerbes, constructed as a morphism of smooth 3-stacks.
- Under **geometric realization**: morphism $a : BE_8 \to K(\mathbb{Z}, 4)$ of topological spaces representing a generating degree-4 integral cohomology class in $H^4(BE_8) \simeq \mathbb{Z}$. Higher connectedness of E_8 this a is an equivalence on 15-coskeleta.

Main Points

So, while nonabelian E_8-gauge fields have a very different differential geometry than abelian 3-form connections, the instanton sectors on both sides may be identified.

- $M \to Y$ a 5-brane worldvolume embedded into spacetime $Y = Y^{11}$.
- A corresponding cocycle in **a**-twisted relative differential cohomology is a homotopy commuting diagram of higher stacks

$$\begin{array}{ccc} Q & \xrightarrow{\hat{B}} & \mathbf{B}(E_8)_\nabla \\ \downarrow & \swarrow_{\simeq} & \downarrow^{\hat{\mathbf{a}}} \\ X & \xrightarrow[\hat{C}]{} & \mathbf{B}^3 U(1)_\nabla \end{array} \tag{20}$$

- For fixed bulk field $\hat{C}$, this is equivalently of $\hat{C}|_Q$-twisted differential $\mathrm{String}(E_8)$-structures [SSS], which are twisted $\mathrm{String}(E_8)$-2-connections on Q.

Main Points

Therefore, where the restriction of the abelian B-field on a D-brane gives rise to a nonabelian 1-form gauge field, the restriction of the C-field relative the **a**-class gives rise to a nonabelian 2-form gauge field.

The moduli 3-stack of C-field configurations for $\frac{1}{2}p_1$ divisible by 2 is then the homotopy pullback

$$\begin{array}{ccc} \mathbf{CField} & \longrightarrow & \mathbf{B}^3\mathrm{U}(1) \\ \downarrow & & \downarrow \cdot 2 \\ \mathbf{B}\mathrm{Spin}_\nabla \times \mathbf{B}E_8 & \xrightarrow{\frac{1}{2}\mathbf{p_1}+2\mathbf{a}} & \mathbf{B}^3\mathrm{U}(1) \end{array} \quad , \qquad (21)$$

where $\frac{1}{2}\mathbf{p}_1$ is the smooth refinement of $\frac{1}{2}p_1$.

Main Points

A field configuration $\phi : \Sigma \to \mathbf{CField}$ has an underlying circle 3-connection $\hat{C}$, a Spin connection $\hat{F}_\omega$, an E_8-principal bundle with class a, and a choice of gauge transformation

$$H : G \xrightarrow{\simeq} a - \frac{1}{4}p_1 \qquad (22)$$

between the underlying circle 3-bundle of $\hat{G}$ and the difference between the Chern-Simons circle 3-bundles of the Spin- and the E_8-bundle.

Boundary moduli [FSS]

Two stages of boundary conditions for this data, exhibited by a sequence of maps

$$\mathbf{CField}^{\mathrm{bdr}_{\mathfrak{o}}} \to \mathbf{CField}^{\mathrm{bdr}} \to \mathbf{CField}\,. \tag{23}$$

- Restriction to M5: For boundary field configurations $\phi : \Sigma \to \mathbf{CField}^{\mathrm{bdr}}$ the integral cohomology class of $\hat{G}_4$ is required to vanish and a differential 3-form part may remain.
- Restriction to heterotic boundary: while for $\mathbf{CField}^{\mathrm{bdr}_{\mathfrak{o}}}$ the full differential cohomology class of $\hat{G}_4$ is required to vanish.
- Both cases: E_8-bundle picks up a connection over boundary $\rightsquigarrow$ *dynamical*.

Enter RHT

Connection to rational homotopy theory

Definition

The field equations of (a limit) of M-theory on an 11-dimensional manifold Y^{11} are

$$
\begin{aligned}
d * G_4 &= \tfrac{1}{2} G_4 \wedge G_4 \\
dG_4 &= 0
\end{aligned}
$$

- **Q. What topological & geometric information can the above system provide us?**
  - Rational structures: Differential forms, rational cohomology, rational homotopy theory ...
  - More refined structures: (twisted) 2-gerbes, (twisted) String structures, orientations ...

- A priori, G_4 should be described by a map $f : Y \to K(\mathbb{Z}, 4)$.
- Differential refinement $\widehat{G}_4$ corresponds to $Y \to B^2U(1)_\nabla$.
- Product structure on Eilenberg-MacLane spaces is cup product, with no a priori information about *trivialization*.
- Need (G_4, G_7) satisfying above $\leftrightarrow$ $Y \to ?$.
- Need $(\widehat{G}_4, \widehat{G}_7)$ satisfying above $\leftrightarrow$ $Y \to \widehat{?}$.

Observation (The **Sullivan model** as the equations of motion (S13))

The above equations correspond to the Sullivan CDGA model of the 4-sphere S^4

$$\mathcal{M}(S^4) = \big(\wedge(y_4, y_7);\ dy_7 = y_4^2,\ dy_4 = 0\big)$$

What about the factor of $\frac{1}{2}$?

- Whitehead bracket $[\iota_4, \iota_4]_W : S^7 \to S^4$ generates $\mathbb{Z}$ ($\mathbb{Q}$)-summand in $\pi_7(S^4)$.
- There is an extra symmetry as we are in the dimension of a Hopf fibration, i.e. σ the $\mathbb{H}$-Hopf map and so the generator is $\sigma = \frac{1}{2}[\iota_4, \iota_4]_W$.

Observation (The **Quillen model** as the M-theory gauge algebra (Fiorenza-S.-Schreiber))

The Sullivan model for S^{2n} is given by the DGCA

$$\mathcal{M}(S^{2n}) = \big(\wedge(x_{2n}, x_{4n-1});\ dx_{2n} = 0,\ dx_{4n-1} = {x_{2n}}^2\big)\ ,$$

so that imposing the Maurer-Cartan equation on the degree 1 element $x_{2n}\xi_{1-2n} + x_{4n-1}\xi_{2-4n}$ we find the Lie bracket dual to the differential is given by

$$[\xi_{1-2n}, \xi_{1-2n}] = 2\xi_{2-4n}$$

with all the other brackets zero.

Example ($n = 2$)

The graded Lie algebra $\mathbb{R}\xi_{-3} \oplus \mathbb{R}\xi_{-6}$ with bracket $[\xi_{-3}, \xi_{-3}] = 2\xi_{-6}$ (Quillen model) can be identified with the M-theory gauge Lie algebra.

- What comes out of this?

Proposal (S13)

Higher gauge fields in M-theory are cocycles in cohomotopy.

- $[Y, S^4_{\mathbb{Q}}] = \pi^4_{\mathbb{Q}}(Y)$ rational cohomotopy.
- Ultimately interested in full $\mathrm{Map}(Y, S^4) \ni f$.
- Geometry + physics $\Rightarrow$ interested in differential cohomology, i.e., differential cohomotopy [Fiorenza-S.-Schreiber]

Formulate in stacks/chain complexes.

Preview:

1. Reduction via a circle bundle $\Rightarrow$ new functors formalizing dimensional reduction via loop (and mapping) spaces.
2. The rational data of S^4 on the total space Y^{11} of a circle bundle $S^1 \to Y^{11} \to X^{10}$ leads exactly to rational data of twisted K-theory on base X^{10}.
3. Even if we take *flat + rational* we can still see a lot of structure: Study of cocycles in Super-Minkowski space recovers cocycles in rational twisted K-theory.
4. Furthermore, T-duality can be derived at the level of supercocycles.

Differential cohomotopy [Fiorenza-S.-Schreiber]

- $\mathbb{H}$-Hopf fibration:

$$S^3 \longrightarrow S^7 \longrightarrow S^4 \longrightarrow B\mathrm{SU}(2) \xrightarrow{c_2} K(\mathbb{Z}, 4). \qquad (24)$$

- Rationalize: $S^3_{\mathbb{Q}} \longrightarrow S^7_{\mathbb{Q}} \longrightarrow S^4_{\mathbb{Q}} \longrightarrow (BS^3)_{\mathbb{Q}}$ which is equivalent to

$$K(\mathbb{Q}, 7) \longrightarrow S^4_{\mathbb{Q}} \longrightarrow K(\mathbb{Q}, 4)$$

- Rational homotopy of spaces can be modelled using L_∞-algebras.
- The Eilenberg-MacLane spaces $K(\mathbb{Q}, n) = B^n\mathbb{Q}$ can be modelled using algebras via chain complexes: $b^n\mathbb{Q} = \mathbb{Q}[n]$.
- Lie 7- algebra $\mathfrak{s}^4$ is defined by $\mathrm{CE}(\mathfrak{s}^4) = \mathbb{R}[g_4, g_7]$ with g_k in degree k and with the differential defined by $dg_4 = 0, dg_7 = g_4 \wedge g_4$.
- Has a natural structure of infinitesimal $\mathbb{R}[2]$-quotient of $\mathbb{R}[6]$, i.e., there exists a natural homotopy fiber sequence of L_∞-algebras

$$\begin{array}{ccc} \mathbb{R}[6] & \longrightarrow & \mathfrak{s}^4 \\ \downarrow & & \downarrow{\scriptstyle p} \\ 0 & \longrightarrow & \mathbb{R}[3] \end{array} . \qquad (25)$$

- Define the stack S^4_∇ by analogy with that of the stack $\mathbf{B}^{p+1}U(1)_\nabla$.

$$\begin{array}{ccc} S^4_\nabla & \longrightarrow & \int(S^4) \\ \downarrow & & \downarrow \\ \mathbf{B}^3 U(1)_\nabla & \longrightarrow & \mathbf{B}^4\mathbb{Z} \end{array}$$

exhibits S^4_∇ as a differential refinement of the homotopy type of S^4 in analogy to how $\mathbf{B}^3U(1)_\nabla$ is a differential refinement of the homotopy type $K(\mathbb{Z};4)$.

$$\begin{array}{ccc} \text{Differential 4-cohomotopy} & \longrightarrow & S^4 \text{ homotopy type} \\ \downarrow & & \downarrow \\ \text{Differential } K(\mathbb{Z},4) \text{ with connection} & \longrightarrow & \text{Differential } K(\mathbb{Z},4) \end{array}$$

Theorem (FSS)

The system $(\widehat{G}_4, \widehat{G}_7)$ forms a cocycle in differential cohomotopy.

Let's go super

Super-Minkowski spacetimes

In terms of the (super-)L_∞-algebras/semifree differential (bi-)graded commutative algebras duality, the algebra $\mathrm{CE}(\mathbb{R}^{d-1,1|N})$ precisely encodes the super-Lie algebra structure of the left-translation in the super-Minkowski spacetime $\mathbb{R}^{d-1,1|N}$.

Example (Eleven dimensions)

- Super Minkowski spacetime $\mathbb{R}^{10,1|N}$, where N is a real Spinor representation of $\mathrm{Spin}(10,1)$.
- $\overline{(-)}\Gamma(-) : N \otimes N \to \mathbb{R}^d$ is a symmetric bilinear Spin-equivariant pairing.
- $\{e^a\}_{a=1}^{d}$, $\{\psi^\alpha\}_{\alpha=1}^{\dim N}$ basis of left-invariant 1-forms on $\mathbb{R}^{10,1|N}$ satisfy

$$d\psi^\alpha = 0, \qquad de^a = \overline{\psi} \wedge \Gamma\psi \,. \tag{26}$$

- $\mathrm{CE}(\mathbb{R}^{10,1|N})$ the differential $(\mathbb{N}, \mathbb{Z}/2)$-bigraded commutative algebra of left-invariant polynomial differential forms. Algebraically, $\mathbb{R}[e^a, \psi^\alpha]$ on the generators $\{e^a, \psi^\alpha\}$ in bidegree $(1, \mathrm{even})$, $(1, \mathrm{odd})$, with d as in (26) of degree $(1, \mathrm{even})$.
- This encodes the super-Lie algebra structure of *left-translations on* $\mathbb{R}^{10,1|N}$.

For M-theory: $N = 32$, so we consider $\mathbb{R}^{10,1|32}$.

The supercocycles in M-theory [Fiorenza-S.-Schreiber]

$\mathfrak{l}(X)$:=L_∞**-algebra dual to given Sullivan model** (A_X, d_X) **for rationalization of** X**, i.e,** $\mathrm{CE}(\mathfrak{l}(X)) := (A_X, d_X)$.

Observation

There are elements μ_4 and μ_7 in $\mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}})$ which satisfy

$$d\mu_4 = 0 \,, \qquad d\mu_7 = \mu_4 \wedge \mu_4 \,.$$

The pair (μ_4, μ_7) equivalently constitutes components of an L_∞-morphism

$$\mu := (\mu_4, \mu_7) : \mathbb{R}^{10,1|\mathbf{32}} \longrightarrow \mathfrak{l}(S^4) \,,$$

namely, dually, the components of a dg-algebra homomorphism

$$\begin{array}{ccc} \mathrm{CE}(\mathfrak{l}S^4) & \longrightarrow & \mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}}) \\ g_4 & \longmapsto & \mu_4 \\ g_7 & \longmapsto & \mu_7 \end{array}$$

The morphism $\mu = (\mu_4, \mu_7)$ is actually induced by an equivariant 7-cocycle on the $\mathfrak{m2brane}$ extension of the super-Minkowski space $\mathbb{R}^{10,1|\mathbf{32}}$.

Definition

Write $\mathfrak{m2brane}$ for the super L_∞-algebra which is the homotopy fiber of μ_4, i.e. sitting in a homotopy pullback diagram of the form

$$\begin{array}{ccc} \mathfrak{m2brane} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{R}^{10,1|\mathbf{32}} & \xrightarrow{\mu_4} & b^3\mathbb{R} \,. \end{array}$$

Observation

(i) There is a super L_∞-cocycle of the form

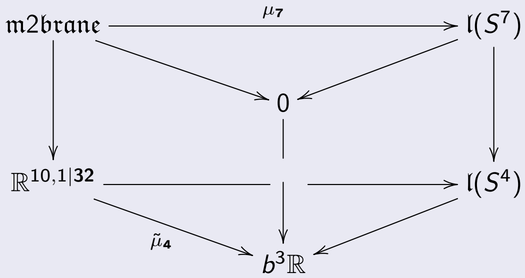
$$\mathfrak{m2brane} \xrightarrow{\mu_7} \mathfrak{l}(S^7) = b^6\mathbb{R} \ .$$

(ii) Both $\mathfrak{m2brane}$ and $\mathfrak{l}S^7$ are naturally $b^2\mathbb{R}$-principal bundles, and the morphism μ_7 is $b^2\mathbb{R}$-equivariant.

The ingredients arrange themselves according to the quaternionic Hopf fibration

Proposition

Starting with cocycle μ_4, there is commutative diagram of L_∞-algebras of the form



where the two front faces of the prism are homotopy pullbacks.

Applications

Application 1: Reduction on a circle

Q. *If fields in M-theory are modelled by S^4, what would fields in type IIA string theory be modelled by?*

Example (Cyclified loop space)

Let $X = S^4$ be the 4-sphere, with

$$\mathrm{CE}(\mathfrak{l}(S^4)) = (\wedge^\bullet \langle g_4, g_7 \rangle, dg_4 = 0\,, dg_7 = \tfrac{1}{2} g_4 \wedge g_4)\,.$$

The free loop space of S^4 is modeled by

$$\mathrm{CE}(\mathfrak{l}(\mathcal{L}S^4)) = \left(\mathbb{R}[\omega_4, \omega_6, h_3, h_7]\,; \begin{array}{ll} d\omega_4 = 0, & d\omega_6 = h_3 \wedge \omega_4, \\ dh_3 = 0, & dh_7 = -\frac{1}{2}\omega_4 \wedge \omega_4 \end{array} \right)\,.$$

and the homotopy quotient by S^1 is modeled as

$$\mathrm{CE}(\mathfrak{l}(\mathcal{L}S^4)/S^1) = \left(\mathbb{R}[\omega_2, \omega_4, \omega_6, h_3, h_7]\,; \begin{array}{l} d\omega_2 = 0, d\omega_4 = h_3 \wedge \omega_2, d\omega_6 = h_3 \wedge \omega_4 \\ dh_3 = 0, \quad dh_7 = -\frac{1}{2}\omega_4 \wedge \omega_4 + \omega_6 \wedge \omega_2 \end{array} \right)\,. \tag{27}$$

Observation

1. Relations in (27) correspond to EOMs of fields in type IIA string theory.
2. These in turn correspond to rationalization (via the Chern character) of twisted K-theory classes.

Formalization of dimensional reduction

Observation

The (M-theory) super-Minkowski spacetime $\mathbb{R}^{10,1|\mathbf{32}}$ is rationally a $S^1_{\mathbb{R}}$-principal bundle over the (type IIA) spacetime $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$.

- There is a homotopy fiber sequence of L_∞-algebras

$$\begin{array}{ccc} \mathbb{R}^{10,1|\mathbf{32}} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \xrightarrow{\overline{\Psi}\Gamma^{11}\Psi} & b\mathbb{R}\,. \end{array}$$

  which exhibits $\mathbb{R}^{10,1|32}$ as the central extension of super Lie algebras classified by an explicit 2-cocycle (D0-brane's $\overline{\Psi}\Gamma^{11}\Psi$).
- There is an *equivalence* between two rational cohomology theories:

Theorem (FSS)

There is an isomorphism of hom-sets

$$\mathrm{Hom}_{L_\infty}(\mathbb{R}^{10,1|32}, S^4_{\mathbb{Q}}) \xrightarrow{\cong} \mathrm{Hom}_{L_\infty}(\mathbb{R}^{10,1|16+\overline{16}}, \mathcal{L}S^4_{\mathbb{Q}})$$

(one can refine further ...)

Application 2: Relation to twisted K-theory

Observation

1. This is exactly the data for rational twisted K-theory at the level of field equations.
2. There is a correspondence between Massey products and differentials in the AHSS for twisted K-theory.

- In string theory we have 3-cocycles in stead of 4-cocycles in M-theory.
- To start connecting to twisted K-theory, we form:

Definition

(i) The super Lie 2-algebra $\mathfrak{string}_{\mathrm{IIA}}$ is the the super Lie 2-algebra extension of $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ classified by a certain 3-cocycle μ_{F1}, the string cocycle.
Equivalently, it is the homotopy fiber (in super L_∞-algebras) of the 3-cocycle μ_{F1}:

$$\begin{array}{ccc} \mathfrak{string}_{\mathrm{IIA}} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \xrightarrow{\mu_{F1}} & b^2\mathbb{R} \end{array} .$$

(ii) For $p \in \{0, 2, 4, 6, 8\}$ there is Dp-brane cocycle $\mu_{Dp} \in \mathrm{CE}(\mathfrak{string}_{\mathrm{IIA}})$.

Definition

Define $\mathfrak{l}(\mathrm{KU})$ to be the L_∞-algebra $\mathfrak{l}(\mathrm{KU}) = \bigoplus_{p\ \text{even}} b^{p+1}\mathbb{R}$ as the minimal Sullivan model for the rationalization of the K-theory spectrum.

- Notice that the Chevalley-Eilenberg algebra of $\mathfrak{l}(\mathrm{KU})$ is

$$\mathrm{CE}(\mathfrak{l}(\mathrm{KU})) = \left(\mathbb{R}[\{\omega_{2p}\}_{p=1,2,\ldots}]\,;\, d\omega_{2p} = 0\right) \,,$$

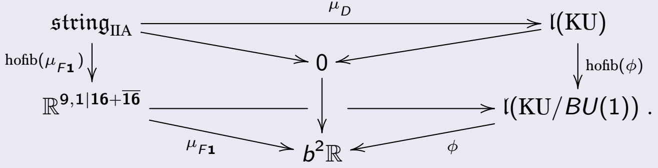
i.e., the even closed forms, as appropriate for rationalization of K-theory, via the Chern character, with target even rational cohomology.

- The direct sum of cocycles $\mu_D = \bigoplus_{p=0,2,4,6,8} \mu_{D_p}$ defines an L_∞-morphism

$$\mu_D : \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \longrightarrow \bigoplus_{p=0,2,4,6,8} b^{p+1}\mathbb{R} \hookrightarrow \mathfrak{l}(\mathrm{KU}). \qquad \textit{truncated}$$

Proposition

The brane cocycles of type IIA fit into a comm. diagram of super L_∞-algebras



Both front faces of the prism are homotopy pullbacks, and

$$\mathrm{CE}\big(\mathfrak{l}(\mathrm{KU}/BU(1))\big) := \left\{\mathbb{R}[\{\omega_{2p}, h_3\}_{p=1,2,\ldots}]\,;\, dh_3 = 0,\ d\omega_{2(p+1)} = h_3 \wedge \omega_{2p}\right\}.$$

Derivation of twisted K-theory from M-theory

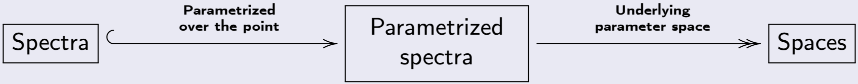
How to enhance the cyclification of the $\mathrm{M2}/\mathrm{M5}$-cocycle from 6-truncated to un-truncated rational twisted K-theory?

Earlier approaches: [Diaconescu-Moore-Witten] (untwisted)
[Moore-Saulina] (trivially twisted)
[Mathai-Sati] (twisted)

assume that the fields are already described by (twisted) K-theory and check the behavior of the partition function of the C-field is compatible with the a priori K-theory classification of D-branes. Here we provide an actual **derivation**.

Main Points (Parametrized spectra)

- A *spectrum* is a kind of linearized version of a topological space.
- By *Brown representability theory* maps into spectra represent cocycles in abelian generalized cohomology theories, such as K-theory.
- Moreover, a *parametrized spectrum* is a bundle of spectra over some base space, and maps into these represent cocycles in twisted generalized cohomology theories.



- Homotopy theory supplied in thesis of [V. Braunack-Mayer] leading to full solution in [Braunack-Mayer-S.-Schreiber]

Answers quest since [S06]: *Which cohomology theory describes fields in M-theory.*

- (Rational) *cohomotopy* in degree-4 is the generalized cohomology theory that is represented by the (rationalized) 4-sphere, meaning that the joint $\mathrm{M2}/\mathrm{M5}$-brane cocycle is a morphism in the rational super homotopy category of the form [Fiorenza-S.-Schreiber]

$$\mathbb{R}^{10,1|\mathbf{32}} \xrightarrow{\;\mu_{M2/M5}\;} S^4 \qquad \in \mathrm{Ho}\left(\mathrm{SuperSpaces}_{\mathbb{R}}\right) \,.$$

- Compatible circle action

$$\underset{\circlearrowright^{S^1}}{S^4} \;:=\; S(\mathbb{R} \oplus \underset{\circlearrowright^{\mathrm{U}(1)}}{\mathbb{C}^2})$$

which is the suspension of the circle action on the $\mathbb{C}$-Hopf fibration.
The projection to the corresponding (rational) homotopy quotient is hence identified with the M-theory circle fibration:

M2/M5-brane cocycle

11d super-spacetime

10d super-spacetime

$$\begin{array}{ccc} \mathbb{R}^{10,1|\mathbf{32}} & \xrightarrow{\;\mu_{M2/M5}\;} & S^4 \\ \downarrow & & \downarrow \\ \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \simeq \mathbb{R}^{10,1|\mathbf{32}} /\!/ S^1 & & S^4 /\!/ S^1 \end{array}$$

- The identification

$$\overset{\mathrm{SU}(2)_L}{\circlearrowright} \qquad S^4 \;\simeq\; S(\,\mathbb{R} \oplus \mathbb{C}^2\,)$$

  induces an action of $\mathrm{SU}(2)$ on the 4-sphere, where on the right we have the defining linear representation of $\mathrm{SU}(2)$ on $\mathbb{C}^2$.
- Along the canonical inclusion $S^1 \simeq U(1) \hookrightarrow \mathrm{SU}(2)$ this restricts to a circle action on the 4-sphere.
- We call the corresponding homotopy quotient (Borel construction)

$$S^4 /\!\!/ S^1 \;\simeq\; S^4 \underset{S^1}{\times} ES^1$$

  the *A-type orbispace of the 4-sphere*.
- The ordinary topological quotient of the above circle action is the 3-sphere:

$$S^4/S^1 \;\simeq\; S^3\,.$$

- The fixed point space of the circle action is the 0-sphere, included as two antipodal points

$$S^0 \;=\; \big(S^4\big)^{S^1} \lhook\joinrel\longrightarrow S^4\,.$$

- In summary, we have the following system of spaces over S^3:

$$\begin{array}{ccccccc}
\underbrace{\text{Fixed points}} & & \underbrace{\text{4-sphere}} & & \underbrace{\text{Homotpy quotient}} & & \underbrace{\text{Naive quotient}} \\
S^0 = \big(S^4\big)^{S^1} & \hookrightarrow & S^4 & \longrightarrow & S^4/\!\!/S^1 & \longrightarrow & S^4/S^1 \\
& \searrow & & \searrow & \downarrow & \swarrow\!\!= & \\
& & & & S^3 & &
\end{array} \qquad (28)$$

Application 3: T-duality

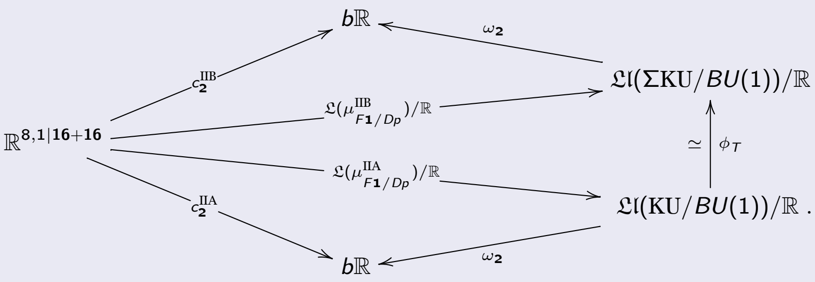
Theorem (Fiorenza-S.-Schreiber)

1. *There is an L_∞-isomorphism of the form*

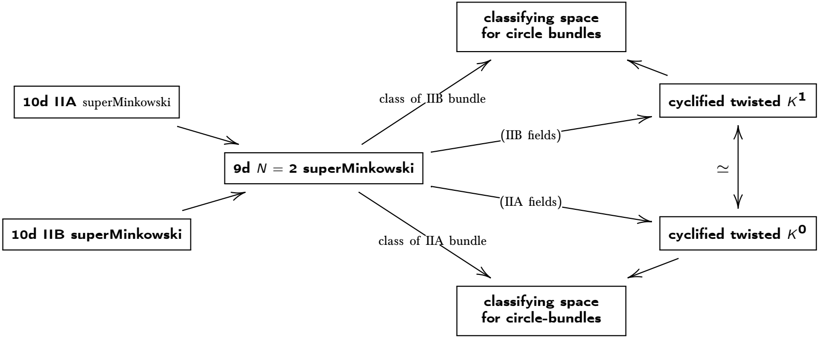
$$\phi_T \ : \ \mathfrak{l}(\mathcal{L}(\mathrm{KU}/BU(1))/S^1) \xrightarrow[\simeq]{\phi_T} \mathfrak{l}(\mathcal{L}(\Sigma\mathrm{KU}/BU(1))/S^1)$$

relating the cyclifications of the rational twisted KU-coefficients.

2. *This takes the dimensionally reduced type IIA F1/Dp-cocycle $\mathfrak{L}(\mu^{\mathrm{IIA}}_{F\mathbf{1}/Dp})/\mathbb{R}$ to the dimensionally reduced IIB cocycle $\mathfrak{L}(\mu^{\mathrm{IIB}}_{F\mathbf{1}/Dp})/\mathbb{R}$, making the following diagram of super L_∞-algebras commute:*



- There is a corresponding explicit pull-push (Fourier-Mukai) formula.



classifying space for circle bundles
10d IIA superMinkowski
cyclified twisted K^1
class of IIB bundle
(IIB fields)
9d $N = 2$ superMinkowski
$\simeq$
(IIA fields)
10d IIB superMinkowski
cyclified twisted K^0
class of IIA bundle
classifying space for circle-bundles

Higher T-duality in M-theory

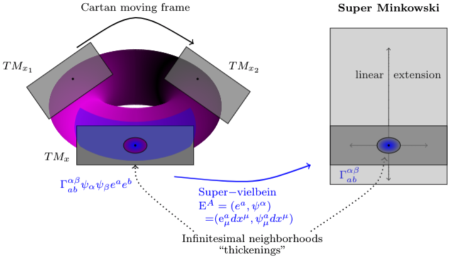
Higher T-duality in M-theory

Main Points

- By analyzing super-torsion and brane super-cocycles, we derive a new duality in M-theory, which takes the form of a higher version of T-duality in string theory.
- This involves a new topology change mechanism abelianizing the 3-sphere associated with the C-field topology to the 517-torus associated with exceptional-generalized super-geometry.
- This explains parity symmetry in M-theory within exceptional-generalized super-spacetime at the same level of spherical T-duality, namely as an isomorphism on 7-twisted cohomology.
- Generalizes "topological T-duality" and "spherical topological T-duality".

This is a derivation of M-theoretic structures from *first principles*, not involving any extrapolation from perturbative string theory nor any conjectures or informal analogies from other sources.

- Replacing the string with the M5-brane, we uncover a duality fully within M-theory, which may be formulated as a *higher-structural analog of T-duality in string theory*.
- Uses constraints on super p-brane fluxes due to local supersymmetry (supergravity) [Achúcarro-Evans-Townsend-Wiltshire, de Azcárraga-Townsend]
- Reduces in string theory to a generalization of "topological T-duality"
- The fermionic component of the brane charges (fluxes) restricted to any of these infinitesimal neighborhoods is constrained to be a non-trivial solution to the supersymmetric Gauss law (mathematically: a non-trivial cocycle in the Chevalley-Eilenberg CE-complex of the supersymmetry algebra) [Bergshoeff-Sezgin-Townsend, Bandos-Lechner-Nurmagambetov-Pasti-Sorokin-Tonin]



- The full implication of these constraints has perhaps not been fully appreciated until recently. Indeed, in [FSS16], we showed that these constraints already imply the Buscher rules for the F1/Dp-brane charge sector of T-duality.

3-spherical T-duality for M5-branes

Rationally: Higher bundles of odd degree are equivalent to higher spheres of odd dimension:

$$K(\mathbb{Z}, 2n+1) \sim_{\mathbb{Q}} K(\mathbb{Q}, 2n+1) \sim_{\mathbb{Q}} S^{2n+1}_{\mathbb{Q}}$$

Example ($n = 1$)

A 2-gerbe or $B^2U(1)$-principal bundle is rationally equivalent to S^3-principal bundle.

Concept	T-duality in string theory	Higher T-duality in M-theory
Maurer-Cartan	θ 1-form	C_3
Curvature	F_2 of circle bundle	G_4 of 3-bundle
T-dualizable flux	$\tilde{H}_3 = H_3 + \theta \wedge F_2'$	$\tilde{G}_7 = G_7 + \frac{1}{2} C_3 \wedge G_4$
Integration	$\int_T \tilde{H} = F_2'$	$\int_{S^3} \tilde{G}_7 = G_4$
Poincaré form	$\mathcal{P}_2$ on $S^1 \times S^1$ bundle	$\mathcal{P}_6$ on $S^3 \times S^3$ bundle
Fux transformation	$e^{B_2'} \wedge \mathcal{C}' = \int_T e^{\mathcal{P}_2} \wedge \pi^* \left(e^{B_2} \wedge \mathcal{C}\right)$	$e^{C_6} \wedge \mathcal{F}' = \int_{S^3} e^{\mathcal{P}_6} \wedge \pi^* \left(e^{C_6} \wedge \mathcal{F}\right)$
Isomorphism	$H^{\bullet + \tilde{H}_3}(X^{10}_{\mathrm{IIA}}) \xrightarrow[\simeq]{\int_T e^{\mathcal{P}_2} \wedge \pi^*(-)} H^{\bullet + \tilde{H}_3'}(X^{10}_{\mathrm{IIB}})$	$H^{\bullet + \tilde{G}_7}(Y^{11}_M) \xrightarrow[\simeq]{\int_{S^3} e^{\mathcal{P}_6} \wedge \pi^*(-)} H^{\bullet + \tilde{G}_7}(Y^{11}_M)$

How to interpret the 7-twisted M-theoretic flux $e^{C_6} \wedge \mathcal{F}$?

- new effect seen in M-theory.
- should couple to whatever it is on which M5-branes may end.
- *String theory limit*: We invoke Hořava-Witten theory. On this boundary:

$$\left.\begin{matrix} \tilde{G}_7 \\ e^{C_6} \wedge \mathcal{F} \end{matrix}\right\} \xmapsto{\text{M/HET duality}} \left\{\begin{matrix} H_7 \\ e^{B_6} \wedge \mathcal{F}^{\text{het}} \end{matrix}\right. \quad (29)$$

- Precisely this had been identified in [S09]: $\mathcal{F}_2^{\text{het}}$ must be the heterotic gauge field strength and $\mathcal{F}_8^{\text{het}}$ its 10-dimensional Hodge dual. These jointly form an H_7-twisted cocycle due to the Green-Schwarz anomaly cancellation mechanism, i.e. from the twisted Bianchi identity

$$dH \propto \mathrm{tr}\left(\mathcal{F}_2^{\text{het}} \wedge \mathcal{F}_2^{\text{het}}\right)$$

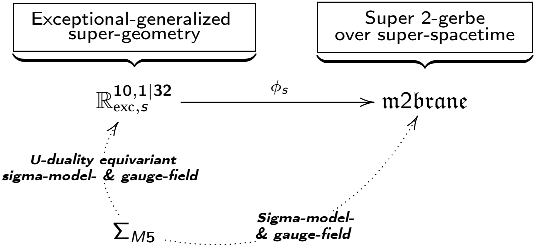
for the NS 3-form flux.

T-duality	**Higher T-duality**	
Torus bundle MC 1-form θ	3-sphere bundle MC 3-form C_3	517-torus bundle decomposable C_3^{exc}
String $H_3 + \theta \wedge F_2'$	M5-brane $G_7 + \frac{1}{2} C_3^{(\text{exc})} \wedge G_4$	
D-branes $e^{B_2} \wedge \mathcal{C}$	Exotic M(9)-branes? $e^{C_6} \wedge \mathcal{F}$	

A new topology change mechanism

- Transmutation of 3-spherical fiber to a 517-torus fiber via exceptional supergeometry
- Subtlety: spherical T-duality of M5-branes acts not on internal spaces within super-spacetime, but rather on the M2-gerbe *over* the total spacetime.

1. we ask for parameterization of that super 2-gerbe by an *ordinary* super-space, such that it still serves as a target space for the M5-brane sigma-model.



2. super-geometric refinement $\mathbb{R}^{10,1|32}_{\mathrm{exc},s}$ of the M-theoretic exceptional generalized geometry of [Hull07b]:

$$\mathbb{R}^{10,1|32}_{\mathrm{exc},s} \simeq \underbrace{\underbrace{\underbrace{\mathbb{R}^{10,1|32}}_{\textbf{Spacetime}} \oplus \mathbf{32}}_{\textbf{Super-spacetime}} \oplus \wedge^2(\mathbb{R}^{10,1})^* \oplus \wedge^5(\mathbb{R}^{10,1})^* \oplus \mathbf{32}}_{\textbf{Exceptional-generalized spacetime}}$$

- **Effect on C-field**: comparison map ϕ_s pulls back the universal C-field c_3 to a decomposable form:

$$c_{\mathrm{exc},s} = \phi_s^*(c_3)\,. \tag{30}$$

- This decomposition is indeed what makes the idea of exceptional-generalized geometry work: Each choice of section (linear splitting) of the exceptional tangent bundle

$$\begin{array}{lc} \textbf{Moduli space} & \mathbb{R}^{\mathbf{10,1|32}}_{\mathrm{exc},s} \\ \textbf{Classifying map} & \sigma \uparrow \;\; \downarrow \pi_{\mathrm{exc},s} \\ \textbf{Spacetime} & \mathbb{R}^{\mathbf{10,1|32}} \end{array}$$

allows to pull-back the universal decomposed C-field and thus induce an actual C-field configuration satisfying the M2-brane super-torsion constraint [BST2]:

$$\underbrace{d\,\underbrace{\left(\sigma^* c_{\mathrm{exc},s}\right)}_{\substack{\textbf{C-field}\\ \textbf{configuration}}} = \tfrac{i}{2}\overline{\psi}\Gamma_{ab}\psi \wedge e^a \wedge e^b}_{\textbf{Torsion constraint}}\,. \tag{31}$$

This leads to the following characterizations:

1. *Each of the fermionic extensions $\mathbb{R}^{\mathbf{10,1|32}}_{\mathrm{exc},s}$ of the $\mathbb{R}^{\mathbf{10,1|32}}$, for each nonzero real parameter s, serves as a moduli space for C-field configurations.*

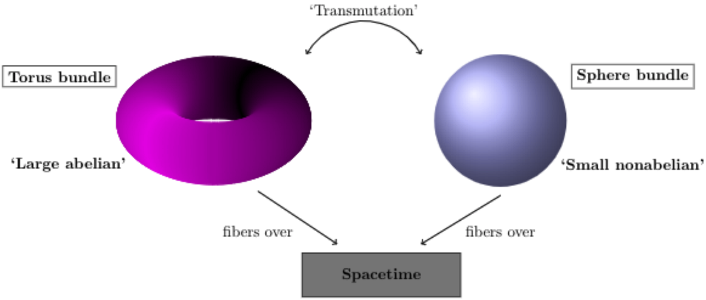
2. *The decomposed C-field $c_{\mathrm{exc},s}$ in $\mathrm{CE}(\mathbb{R}^{\mathbf{10,1|32}}_{\mathrm{exc},s})$ is the corresponding universal field on the moduli space, whose pullback along classifying maps σ yield the actual C-field configurations on super-Minkowski spacetime.*

- Using the new concept of C-cohomology

$$\text{C-cohomology} = \frac{\mathrm{kernel}(c_{\mathrm{exc},s} \wedge (-))}{\mathrm{image}(c_{\mathrm{exc},s} \wedge (-))} . \tag{32}$$

and corresponding spectral sequence analysis, one can show that the spherical T-duality of M5-branes passes along the decomposition map to the exceptional-generalized superspacetime. cf. [Ševera]

- If we think of the latter as compactified, this means to trade the original rational 3-sphere for a 517-torus ($517 = 528 - 11$) with tangent space $\wedge^2(\mathbb{R}^{10,1})^* \oplus \wedge^5(\mathbb{R}^{10,1})$



Parity symmetry

- Lifting of parity symmetry on exceptional generalized spacetime to the level of spherical T-duality.
- 11D supergravity is invariant under an odd number of spacetime-reflections if these are accompanied by sending the C-field to its negative [Duff-Nilsson-Pope]

$$C_3 \mapsto -C_3 \,. \tag{33}$$

- This operation lifts to an equivalence ρ^* of M5-brane 7-flux-twisted cohomology on exceptional-generalized super-spacetime, on par with the spherical T-duality equivalence,

$$H^{\bullet + \tilde{\mu}_{M\mathbf{5},s}}\big(\mathbb{R}^{\mathbf{10,1|32}}_{\mathrm{exc},s}\big) \underset{\simeq}{\xrightarrow{\rho^*}} H^{\bullet + \tilde{\mu}_{M\mathbf{5},s}}\big(\mathbb{R}^{\mathbf{10,1|32}}_{\mathrm{exc},s}\big) \,. \tag{34}$$

- This means that we should view parity and 3-spherical T-duality as generators that *jointly* induce a larger M-theoretic duality group which also contains their composite operation:

$$\begin{array}{ccc} & H^{\bullet+\tilde{\mu}_{M\mathbf{5}}} & \\ {\scriptstyle \int_{S^{\mathbf{3}}} e^{\mathcal{P}_{\mathbf{6}}} \wedge \pi^*(-)} \nearrow \simeq & & \simeq \searrow {\scriptstyle \rho^*} \\ H^{\bullet+\tilde{\mu}_{M\mathbf{5}}} & \underset{\substack{\text{'Paritized'} \\ \text{3-spherical T-duality}}}{\xrightarrow{\quad\simeq\quad}} & H^{\bullet+\tilde{\mu}_{M\mathbf{5}}} \,. \end{array} \tag{35}$$

- Compatible with the results of parity in the topological sector in [Diaconescu-Freed-Moore].

From global geometric and topological perspective, M-theory is parity invariant, and so should in principle be formulated in a way that makes sense on unoriented, and possibly non-orientable manifolds. We hope that our formulation provides some insight into this problem.

Example (Parity as rational T-duality of E_8 bundles)

Since the homotopy group of E_8 are concentrated in degrees $(3, 15, \cdots)$, the group E_8 has the same homotopy type as $K(\mathbb{Z}, 3)$ up to degree 14. E_8 and $SU(2)$ have the same rational homotopy and cohomology in the above range. Overall:

$$\text{“}E_8 \simeq_{14,\mathbb{Q}} K(\mathbb{Q}, 3) \simeq_{\mathbb{Q}} SU(2) \simeq_{\mathbb{Q}} S^3\text{”} \ .$$

Equivalence in rational homotopy theory:

$$\begin{array}{ccccc} S^3_{\mathbb{Q}} \simeq K(\mathbb{Q},3) \simeq_{14,\mathbb{Q}} E_8 \longrightarrow E & & & & E' \longleftarrow E_8 \simeq_{\mathbb{Q}} K(\mathbb{Q},3) \simeq S^3_{\mathbb{Q}} \\ & \searrow^{\pi} & & {}^{\pi'}\swarrow & \\ & & Y & & \end{array}$$

Taking the class of the bundle E to be a and the class of the bundle E' to be $-a$ then puts the two bundles as a parity dual pair, which fits into our discussion of T-duality for rational sphere bundle as a special case.

Let's go equivariant

M-branes at singularities

- Use equivariant homotopy theory.
- Unexpexted equivalence via

Theorem (*Elmendorf's theorem*)

Homotopy theory for equivariant homotopies is equivalent to another homotopy theory where no equivariance on homotopies exists anymore, but where instead extra structure appears on singularities, namely on the fixed point strata of the original group action.

Physics impact:

- For *fundamental* M2/M5-brane: deg 4 cohomotopy on superspaces [FSS].
- Here the open question is: *which enhancement of rational cohomotopy also captures the black M-brane located at real ADE-singularities?*
- Proposal [Huerta-S.-Schreiber]: ADE-equivariant cohomotopy on superspaces the equivalence of homotopy theories that is given by Elmendorf's theorem translates into a duality in string/M-theory that makes the black branes at real ADE-singularities appear from the equivariance of the super-cocycle of the fundamental M2/M5-brane:

G-equivariance	Elmendorf's theorem $\longleftrightarrow$	G-fixed points
Fundamental M2/M5-branes on 11d superspacetime with real ADE-equivariant sigma-model	[HSS] $\longleftrightarrow$	Fundamental F1/M2/M5-branes on intersecting black M-branes at real ADE-singularities

Definition (Actions on the 4-sphere)

Regard the 4-sphere as the unit sphere:

$$S^4 \simeq S(\mathbb{R} \oplus \underbrace{\overbrace{\mathbb{R} \oplus \mathbb{R}^2}^{\mathrm{Im}(\mathbb{H})} \oplus \mathbb{R}}_{\mathbb{H}}) . \quad (36)$$

This decomposition induces the following group actions on the 4-sphere:

1. Multiplication by unit quaternions on $\mathbb{H}$ preserves the 4-sphere and hence yields two actions $\mathrm{SU}(2)$, which we denote by $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$, respectively.
2. These two actions manifestly commute with each other, and hence we have the corresponding action of the Cartesian product of $\mathrm{SU}(2)$ with itself, which we denote by $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$.
3. We denote the action induced from this via the diagonal homomorphism $\mathrm{SU}(2) \xrightarrow{\Delta} \mathrm{SU}(2) \times \mathrm{SU}(2)$ by $\mathrm{SU}(2)_\Delta$.
4. There is then an inclusion $S^1 \hookrightarrow \mathrm{SU}(2)$ such that the corresponding restriction of the diagonal action fixes the second coordinate in (36). This induced action we accordingly denote by S^1_Δ.
5. The $\mathbb{Z}_2$-action induced by the involution given by reflection of the last coordinate in (36) (i.e. multiplication by -1 on the real part of the quaternionic coordinate in (36)) we denote by $(\mathbb{Z}_2)_{\mathrm{HW}}$.
6. This commutes with the $\mathrm{SU}(2)_\Delta$-action, so that there is the corresponding action of the Cartesian product group, which we accordingly denote by $\mathrm{SU}(2)_\Delta \times (\mathbb{Z}_2)_{\mathrm{HW}}$.

Remark (Summary of actions)

In terms of the decomposition (36):

$$S^4 \simeq S\Big(\mathbb{R} \;\oplus\; \underbrace{\overbrace{\mathbb{R} \oplus \underset{\circlearrowright\, S^1_\Delta}{\mathbb{R}^2}}^{\mathrm{Im}(\mathbb{H})\;\circlearrowleft\;\mathrm{SU}(2)_\Delta} \;\oplus\; \overset{(\mathbb{Z}_2)_{\mathrm{HW}}\;\circlearrowright}{\mathbb{R}}}_{\mathrm{SU}(2)_L \;\circlearrowright\; \mathbb{H} \;\circlearrowleft\; \mathrm{SU}(2)_R}\Big) \tag{37}$$

which may by collected into two actions of Cartesian products as

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \;\circlearrowright\; S^4 \qquad \text{and} \qquad \mathrm{SU}(2)_\Delta \times (\mathbb{Z}_2)_{\mathrm{HW}} \;\circlearrowright\; S^4 \,.$$

Example (Suspended Hopf action)

Under the canonical inclusion $S^1 \simeq U(1) \hookrightarrow \mathrm{SU}(2)_L$ of the circle group into the special unitary group (as the subgroup of diagonal matrices) the induced action S^1_L on the 4-sphere, by above, is the image under topological suspension of the S^1-action that exhibits the complex Hopf fibration $S^3 \to S^2$ as an S^1-principal bundle.

- Which enhancement of rational cohomotopy also captures the *black* M-brane located at real ADE-singularities?
- We set up *equivariant cohomotopy on* **superspaces**.

Theorem (Equivariant enhancements of the fundamental brane cocycles (Huerta-S.-Schreiber))

Enhancement of the fundamental M2/M5-brane cocycle from (rational) cohomotopy of superspaces to (rational) equivariant cohomotopy exists, and the possible choices correspond to fundamental branes propagating on intersecting black M-branes at real ADE-singularities.

- Part of this statement is a full classification of finite group actions on super Minkowski super spacetime $\mathbb{R}^{10,1|32}$ by isometries.
- Enhancements of the $\mathrm{M2}/\mathrm{M5}$-brane cocycle to *equivariant* cohomotopy

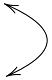
$$\underset{\mathbb{R}^{10,1|32}}{\overset{G_{\mathrm{ADE}}\times G_{\mathrm{HW}}}{\circlearrowright}} \xrightarrow{\widehat{\mu}_{\mathbf{M2}/\mathbf{M5}}} \underset{S^4}{\overset{G_{\mathrm{ADE}}\times G_{\mathrm{HW}}}{\circlearrowright}} \quad \in \mathrm{Ho}\big((G_{\mathrm{ADE}} \times G_{\mathrm{HW}})\, \mathrm{SuperSpaces}_{\mathbb{R}} \big) \,.$$

- It is shown in [Huerta-S.-Schreiber] that this equivariant enhancement makes the *black branes* at ADE-singularities appear, and unifies them with the fundamental M-branes.

More on K-theory from M-theory

Recall that we provided a derivation of twisted K-theory from M-theory. Can this be enhanced to *equivariant*?
Point is that *equivariant* rational homotopy theory will see a bit of finite information, thereby going a bit beyond rational towards full cohomology.

Objects	**Cohomology theory**
M-branes	Real ADE-equivariant Cohomotopy
D-branes	Real K-theory



stabilized
Ext/Cyc-adjunction

To support this, we have seen:

(1) a homotopy-theoretic formulation of "compactifying M-theory on a circle", s.t.
(2) under this operation the cohomology theory *degree-4 cohomotopy* transmutes into the cohomology theory *K-theory*, matching how the M-branes are supposed to reduce to F1/Dp-branes under **double dimensional reduction**.

- [FSS16a][FSS16b]: Rationally, **(1)** is exhibited by the *Ext/Cyc-adjunction* and then **(2)** follows, since 6-truncated twisted K-theory appears, rationally, in the cyclic loop space of the 4-sphere.
- [B-MSS]: the **gauge enhancement** of this result to the full, untruncated, twisted K-theory spectrum.

The fundamental brane scan

- *Green–Schwarz-type* Lagrangians shown in [Heurta-S.-Schreiber] to be just the *super* volume forms (hence the "super Nambu-Goto Lagrangians")

$$\mathbf{L}^{\mathrm{GS}}_{p+1} \;:=\; \underbrace{\underbrace{\mathrm{vol}_{p+1}}_{\mathrm{NG}} + \underbrace{\Theta_{p+1}}_{\mathrm{WZW}}}_{\substack{\mathrm{svol}_{p+1} \\ \textbf{supersymmetric volume}}} \quad \overset{d}{\longmapsto} \quad \underbrace{\mu_{p+2}}_{\substack{\textbf{super-} \\ \textbf{cocycle}}} \;. \qquad (38)$$

- These cocycles μ_{p+2} are what the (old) *brane scan* [Duff] classifies: the non-trivial Spin-invariant super $(p+2)$-cocycles on super Minkowski spacetimes, for $p \geq 1$.

p / **d + 1**		1	2	3	4	5	6	7	8	9	10
10 + 1			μ_{M2}								
9 + 1		$\mu_{F1}^{H/I}$				μ_{NS5}					
8 + 1					$*$						
7 + 1				$*$							
6 + 1			$*$								
5 + 1		$*$		$*$							
4 + 1			$*$								
3 + 1		$*$	$*$								
2 + 1		$\mu_{F1}^{D=3}$									

- These cocycles correspond to those fundamental super p-branes that do *not* carry (higher) gauge fields on their worldvolume.

- [FSS13]: Improve to include also all these further branes, if one passes from super Lie algebras to their homotopy theoretic incarnation: super *L_∞-algebras*.
- Every *2-cocycle* μ_2 on a (super) Lie algebra classifies a central extension.

Example

Type IIA superspacetime carries a Spin-invariant 2-cocycle $\mu_{D0} = \overline{\psi}\Gamma^{10}\psi$, whose central extension is $D = 11$, $\mathcal{N} = 1$ super Minkowski spacetime:

$$\begin{array}{c} \mathbb{R}^{10,1|\mathbf{32}} \\ \Big\downarrow {\scriptstyle \text{central extension by } \mu_{D0} = \overline{\psi}\Gamma^{10}\psi} \\ \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \end{array} .$$

- In rational super homotopy theory, a 2-cocycle as above is equivalently a map, namely a map of the form $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \xrightarrow{\mu_2} B\mathbb{R}$, and for every map in homotopy there is the corresponding homotopy fiber.

Example (D0-brane Lie superalgebra continued)

$$\begin{array}{ccc} \mathbb{R}^{10,1|\mathbf{32}} & & \\ \Big\downarrow {\scriptstyle \text{homotopy fiber of } \mu_2 = \overline{\psi}\Gamma^{10}\psi} & & \\ \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \xrightarrow{\mu_2 = \overline{\psi}\Gamma^{10}\psi} & B^2\mathbb{R} \end{array} .$$

- Now we go higher:

Example (The *superstring Lie 2-algebra*)

$$
\begin{array}{ccc}
\mathfrak{string}_{\mathrm{IIA}} & & \\
\Big\downarrow{\scriptstyle \text{homotopy fiber of } \mu_{F1}} & & \\
\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \xrightarrow{\ \mu_{F1} = \frac{i}{2}(\overline{\psi}\Gamma_a\psi)\wedge e^a\ } & B^3\mathbb{R}\,.
\end{array}
$$

- **Q**: *Does this carry further Spin-invariant cohomology classes?*
  Indeed it does carry non-trivial Spin-invariant cocycles precisely for all the previously missing branes:

Example (Super D-branes of type IIA)

$$
\begin{array}{ccc}
\mathfrak{d}2p\mathfrak{brane} & & \\
\Big\downarrow{\scriptstyle \text{homotopy fiber of } \mu_{D(\mathbf{2}p)}} & & \\
\mathfrak{string}_{\mathrm{IIA}} & \xrightarrow{\ \mu_{D(\mathbf{2}p)}\ } & B^{2p+2}\mathbb{R}\,.
\end{array}
$$

The global picture

The completion of the old brane scan to the remaining branes and various further details are encoded in the fundamental brane bouquet: