M-theory from the superpoint

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Prologue

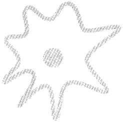

Figure: $\mathbb{R}^{0|1}$

Prologue

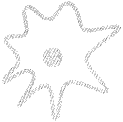

Figure: $\mathbb{R}^{0|1}$

$\mathbb{R}^{0|1}$ has a single odd coordinate θ, and $\theta^2 = 0$, so a power series terminates immediately:

$$f(\theta) = f(0) + f'(0)\theta.$$

In essence, this means we should regard θ as infinitesimal. Thus $\mathbb{R}^{0|1}$ is a single point with an infinitesimal neighborhood, as depicted above.

Prologue

We will peer into the superpoint using *homotopy theory*.

Inside, we will find all the super Minkowski spacetimes of string theory and M-theory, going up to dimension 11.

Then we will find the strings, Dp-branes and M-branes themselves, thanks to the brane bouquet of Fiorenza, Sati and Schreiber.

$$
\begin{tikzcd}
 & \mathbb{R}^{10,1|\mathbf{32}} \arrow[dr] & \\
\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
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\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & \mathfrak{m2brane} \arrow[d] & & \\
 & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & & \\
\mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] \\
 & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \\
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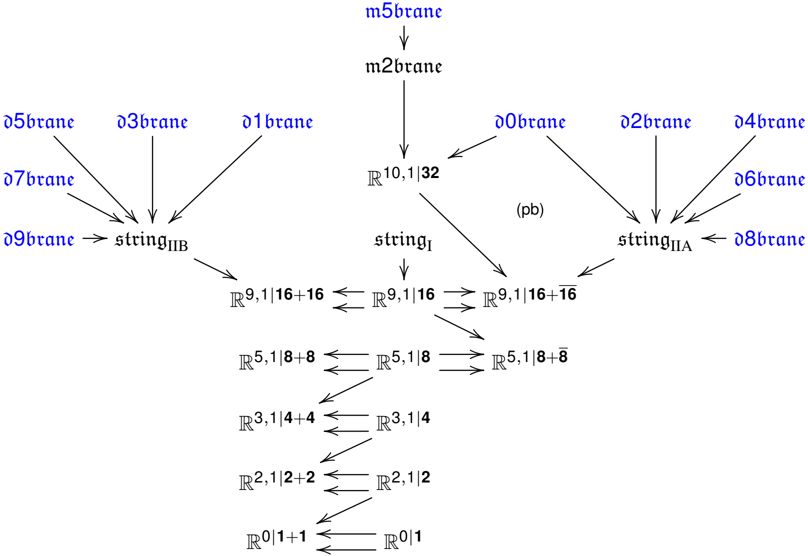

$\mathfrak{m5brane}$
$\mathfrak{m2brane}$
$\mathfrak{d5brane}$
$\mathfrak{d3brane}$
$\mathfrak{d1brane}$
$\mathfrak{d0brane}$
$\mathfrak{d2brane}$
$\mathfrak{d4brane}$
$\mathfrak{d7brane}$
$\mathbb{R}^{10,1|\mathbf{32}}$
$\mathfrak{d6brane}$
(pb)
$\mathfrak{d9brane}$
$\mathfrak{string}_{\mathrm{IIB}}$
$\mathfrak{string}_{\mathrm{I}}$
$\mathfrak{string}_{\mathrm{IIA}}$
$\mathfrak{d8brane}$
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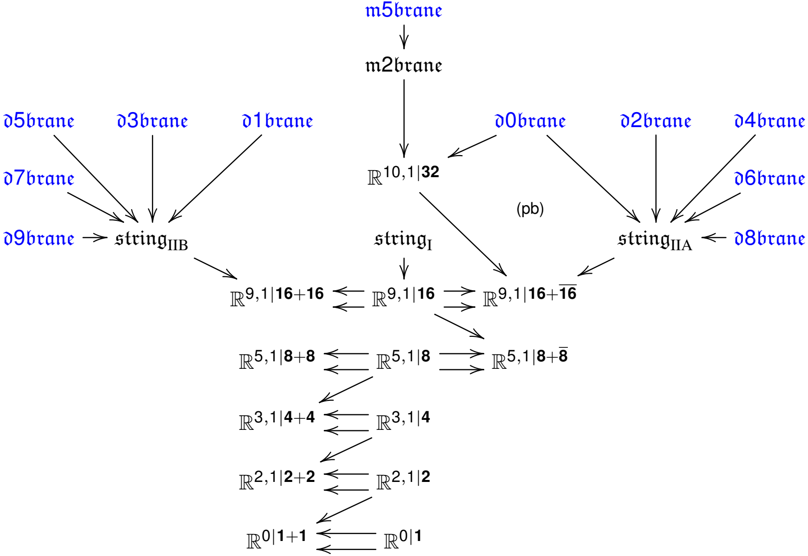

The brane bouquet.

M-theory

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Witten christened this topic

M-theory

The M arguably stands for “membrane”.

M-theory

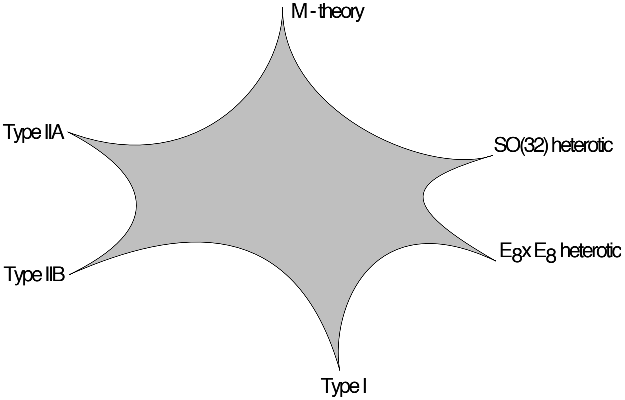

Figure: Polchinski's schematic.

In this highly schematic picture, M-theory unites the five 10d string theories (and 11d supergravity, not shown).

M-theory

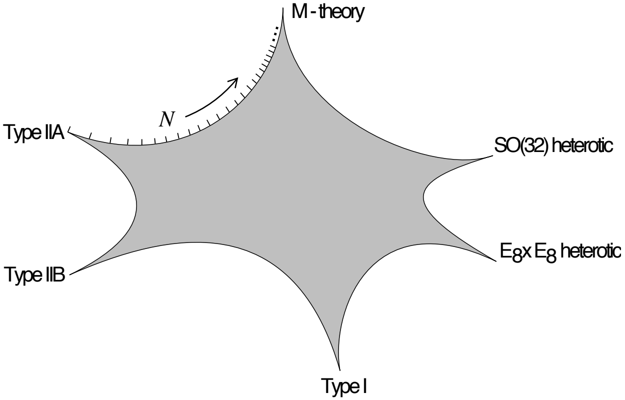

Figure: Polchinski's schematic.

Most directly, M-theory is a limit of type IIA string theory which "grows an extra dimension".

M-theory

10d spacetime becomes 11d:

$$\text{type IIA string theory} \quad M^{10} \rightsquigarrow N^{11} \quad \text{M-theory}.$$

Infinitesimally, 10d Minkowski spacetime becomes 11d:

$$\mathbb{R}^{9,1} \rightsquigarrow \mathbb{R}^{10,1}.$$

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We will see this is mathematically natural and beautiful: it is a central extension!

Super Minkowski spacetime

- $\mathbb{R}^{d-1,1|\mathbf{N}}$ is the 'super version' of $\mathbb{R}^{d-1,1}$.
  - Which is $\mathbb{R}^{d}$ with the metric $\eta(u,v) = -u^0v^0 + u^1v^1 + \cdots + u^{d-1}v^{d-1}$.

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    $\eta(u, v) = -u^0 v^0 + u^1 v^1 + \cdots + u^{d-1} v^{d-1}$.
- $\mathbb{R}^{d-1,1|\mathbf{N}}$ is a **super Lie algebra.**
- Meaning it is a super vector space:

$$\mathbb{R}^{d-1,1|\mathbf{N}}_{\text{even}} = \mathbb{R}^{d-1,1}, \quad \mathbb{R}^{d-1,1|\mathbf{N}}_{\text{odd}} = \mathbf{N}$$

- Equipped with a Lie bracket:

$$[-,-] \colon \mathbb{R}^{d-1,1|\mathbf{N}} \otimes \mathbb{R}^{d-1,1|\mathbf{N}} \to \mathbb{R}^{d-1,1|\mathbf{N}}.$$

Super Minkowski spacetime

This structure is dictated by representation theory.

- $\mathrm{Spin}(d-1,1)$ is the double cover of the connected Lorentz group $\mathrm{SO}_0(d-1,1)$.
- $\mathbb{R}^{d-1,1}$ is a representation of $\mathrm{Spin}(d-1,1)$.
- **N** is a choice of a real spinor representation of $\mathrm{Spin}(d-1,1)$.
- The bracket is a choice of a $\mathrm{Spin}(d-1,1)$-equivariant map.

Super Minkowski spacetime

Concretely, the bracket on $\mathbb{R}^{d-1,1|\mathbf{N}}$ is:

- The only nonzero part of the bracket is the spinor-to-vector pairing:
$$[-,-]\colon \mathbf{N} \otimes \mathbf{N} \to \mathbb{R}^{d-1,1}.$$
- If $\mathbf{N}$ is irreducible, this map is unique up to rescaling. If $\mathbf{N}$ is reducible, there is more choice involved.
- Physicists write this bracket using gamma matrices:
$$[Q_\alpha, Q_\beta] = -2\Gamma^\mu_{\alpha\beta} P_\mu.$$
and call it an “anticommutator”, because Q_α and Q_β are odd.

Central extensions

Remember that, physically:

- Type IIA string theory lives on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$.
- M-theory lives on $\mathbb{R}^{10,1|\mathbf{32}}$.
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Question

What is this process mathematically?

Central extensions

It's a **central extension**!

Given

- $\mathfrak{g}$ a super Lie algebra,
- $\omega \colon \Lambda^2\mathfrak{g} \to \mathbb{R}$ a 2-cocycle, meaning:

$$\omega([X, Y], Z) \pm \omega([Y, Z], X) \pm \omega([Z, X], Y) = 0,$$

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we can form the central extension:

$$\mathfrak{g}_\omega = \mathfrak{g} \oplus \mathbb{R}c,$$

with one extra generator c, even and central, and modified Lie bracket:

$$[X, Y]_\omega = [X, Y] + \omega(X, Y)c.$$

Central extensions

In particular:

- $\mathbb{R}^{10,1|\mathbf{32}}$ is a central extension of $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$.
- The 2-cocycle is

$$\omega = d\theta^\alpha \wedge \Gamma^{01\cdots 9}_{\alpha\beta} d\theta^\beta,$$

where $\Gamma^{01\cdots 9} = \Gamma^0\Gamma^1\cdots\Gamma^9$, and (x^μ, θ^α) are the even and odd coordinates on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$.

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- It is left-invariant (as a form on the super Lie group).
- $d\omega = 0$, by the naive calculation.

Moreover, it really does give $\mathbb{R}^{10,1|\mathbf{32}}$, by the usual “yoga” of gamma matrices.

Central extensions

Notation

Every central extension comes with a projection map:

$$\mathfrak{g}_\omega \to \mathfrak{g}$$

that sets c to zero; we will often write this map to indicate central extension. For example:

$$\mathbb{R}^{10,1|\mathbf{32}} \to \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}.$$

The superpoint

This prompts a number of questions.

Question

What singles out the 2-cocycle

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It is invariant under $\mathrm{Spin}(9,1)$.

Question

Are any other dimensions of spacetime due to central extension?

Answer

All of them! This is our main result.

The superpoint

At the extreme end, we could start with the superpoint $\mathbb{R}^{0|1}$, and study its central extensions.

Definition

The **superpoint** $\mathbb{R}^{0|1}$ is the super vector space consisting of $\mathbb{R}$ in odd degree:

$$\mathbb{R}^{0|1}_{\text{even}} = 0, \quad \mathbb{R}^{0|1}_{\text{odd}} = \mathbb{R}.$$

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$$\mathbb{R}^{0|1}_{\text{even}} = 0, \quad \mathbb{R}^{0|1}_{\text{odd}} = \mathbb{R}.$$

- It has no Lie bracket;
- It has no metric;
- It has no spin structure.

We will discover all structure through central extension.

The superpoint

$\mathbb{R}^{0|1}$ exactly one 2-cocycle:

$$d\theta \wedge d\theta$$

Extending by this 2-cocycle gives $\mathbb{R}^{1|1}$, the superline, the worldline of the superparticle:

$$\mathbb{R}^{1|1} \to \mathbb{R}^{0|1}.$$

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Can we find more dimensions?

Maximal invariant central extensions

This is a game with two moves:

- We can extend by all 2-cocycles satisfying a suitable invariance condition.
- We can double the number of spinors.

This will lead us from the superpoint up to 11 dimensions and beyond.

Maximal invariant central extensions

We want that “suitable invariance condition” to be invariance under $\mathrm{Spin}(d-1,1)$, but we don’t have a metric!

Maximal invariant central extensions

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Proposition (H.–Schreiber, folklore)

For a super Minkowski spacetime $\mathbb{R}^{d-1,1|\mathbf{N}}$, its connected automorphism group is:

$$\mathrm{Aut}_0(\mathbb{R}^{d-1,1|\mathbf{N}}) \simeq \mathbb{R}^+ \times \mathrm{Spin}(d-1,1) \times \text{R-group}$$

where the R-group acts trivially on $\mathbb{R}^{d-1,1}$.

Thus, we can recover the group $\mathrm{Spin}(d-1,1)$ by considering the automorphisms of the Lie bracket alone.

First, we will double the number of fermionic dimensions:

$$\mathbb{R}^{0|2}$$

We will write this operation as follows:

$$\mathbb{R}^{0|2} \leftleftarrows \mathbb{R}^{0|1}$$

Now, $\mathbb{R}^{0|2}$ has two odd generators, θ_1 and θ_2, and there are three 2-cocycles:

$$d\theta_1 \wedge d\theta_1, \quad d\theta_1 \wedge d\theta_2, \quad d\theta_2 \wedge d\theta_2.$$

Extending by all three we get:

$$\mathbb{R}^{3|2} \longrightarrow \mathbb{R}^{0|2}.$$

Now something remarkable happens: a metric appears!

$$\mathrm{Aut}_0(\mathbb{R}^{3|2}) = \mathbb{R}^+ \times \mathrm{Spin}(2,1).$$

Thanks to this metric, we can look for $\mathrm{Spin}(2,1)$-invariant 2-cocycles on $\mathbb{R}^{2,1|2}$. There are none, because the only $\mathrm{Spin}(2,1)$-invariant map:

$$\mathbf{2} \otimes \mathbf{2} \to \mathbb{R}$$

is antisymmetric.

The dimensional ladder

Dimension 4

Double the number of spinors again:

$$\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} \leftleftarrows \mathbb{R}^{2,1|\mathbf{2}}$$

There is precisely one $\mathrm{Spin}(2,1)$-invariant 2-cocycle, and extending by this gives:

$$\mathbb{R}^{3,1|\mathbf{4}} \longrightarrow \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}}$$

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$$\mathbb{R}^{3,1|\mathbf{4}} \longrightarrow \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}}$$

Again, the metric is not a choice:

$$\mathrm{Aut}_0(\mathbb{R}^{3,1|\mathbf{4}}) = \mathbb{R}^+ \times \mathrm{Spin}(3,1) \times \mathrm{U}(1).$$

$\mathrm{U}(1)$ is the R-group.

There are no further $\mathrm{Spin}(3,1)$-invariant 2-cocycles.

Double the number of spinors again:

$$\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} \leftleftarrows \mathbb{R}^{3,1|\mathbf{4}}$$

Now there are two $\mathrm{Spin}(3,1)$-invariant 2-cocycles.

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$$\mathrm{Aut}_0(\mathbb{R}^{5,1|\mathbf{8}}) = \mathbb{R}^+ \times \mathrm{Spin}(5,1) \times \mathrm{Sp}(1).$$

$\mathrm{Sp}(1)$ is the R-group.

There are no further $\mathrm{Spin}(5,1)$-invariant 2-cocycles.

Now we have a choice of two different ways to double the spinors, a type IIA and type IIB:

$$\mathbb{R}^{5,1|\mathbf{8}+\bar{\mathbf{8}}} \leftleftarrows \mathbb{R}^{5,1|\mathbf{8}}$$

and

$$\mathbb{R}^{5,1|\mathbf{8}+\mathbf{8}} \leftleftarrows \mathbb{R}^{5,1|\mathbf{8}}$$

There are no $\mathrm{Spin}(5,1)$-invariant 2-cocycles in type IIB, but on type IIA there are four:

$$\mathbb{R}^{9,1|\mathbf{16}} \longrightarrow \mathbb{R}^{5,1|\mathbf{8}+\bar{\mathbf{8}}}.$$

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There are no $\mathrm{Spin}(9,1)$-invariant 2-cocycles in type IIB, but on type IIA there is one, the one we started with:

$$\mathbb{R}^{10,1|\mathbf{32}} \longrightarrow \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}.$$

Theorem (H.–Schreiber)

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The brane scan

We have seen that 2-cocycles give central extensions.

Fact

The 2nd **Chevalley–Eilenberg cohomology group**

$$H^2(\mathfrak{g})$$

classifies central extensions of $\mathfrak{g}$.

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Question

What do higher degree cocycles in $H^\bullet(\mathfrak{g})$ classify?

Answer (Physics)

Invariant $(p+2)$-cocycles on $\mathbb{R}^{d-1,1|\mathbf{N}}$ classify some of the p-branes.

Answer (Mathematics)

Higher degree cocycles classify extensions to L_∞-algebras.

The physical answer

The Lie algebra cohomology of $\mathbb{R}^{d-1,1|\mathbf{N}}$ gives rise to particular p-branes called **Green–Schwarz p-branes.**

- Write a generating set of left-invariant forms:

$$e^\mu = dx^\mu - \theta\Gamma^\mu d\theta, \quad d\theta^\alpha.$$

- Find the $\mathrm{Spin}(d-1,1)$-invariant combinations:

$$\mu_p = e^{\nu_1} \wedge \cdots \wedge e^{\nu_p} \wedge d\bar{\theta}\Gamma_{\nu_1 \cdots \nu_p} d\theta.$$

- This is $(p+2)$-cocycle if and only if it is closed:

$$d\mu_p = 0.$$

- This happens only for special values of $d, \mathbf{N}$ and p.

The brane scan

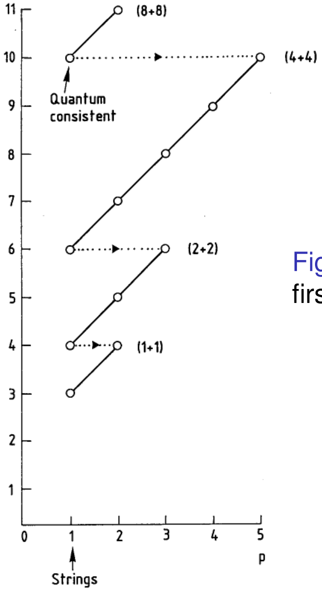

Figure: M. Duff - Supermembranes: the first fifteen weeks, 1988

The brane scan

This figure is called **the old brane scan**.

It fails to show many examples of branes that would be important later:

- D-branes and the M5-brane.
- Black branes from supergravity.
- Brane intersections.

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Where can we find these? To answer, we use some homotopy theory!

The mathematical answer

- The brane scan $(p+2)$-cocycles on $\mathbb{R}^{d-1,1|\mathbf{N}}$:

$$\mu_p = e^{\nu_1} \wedge \cdots \wedge e^{\nu_p} \wedge d\bar{\theta}\Gamma_{\nu_1 \cdots \nu_p} d\theta.$$

- Extending by these $(p+2)$-cocycles, we get the **brane scan algebras**:

$$\mathfrak{string}_{\mathrm{I}} = \mathbb{R}^{9,1|\mathbf{16}}_{\mu_{\mathrm{I}}}, \ \mathfrak{string}_{\mathrm{IIA}} = \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}_{\mu_{\mathrm{IIA}}}, \ \mathfrak{string}_{\mathrm{IIB}} = \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}}_{\mu_{\mathrm{IIB}}},$$

$$\mathfrak{m2brane} = \mathbb{R}^{10,1|\mathbf{32}}_{\mu_{\mathrm{M2}}}.$$

- Because these are not 2-cocycles, the resulting extensions are not super Lie algebras—they are **super L_∞-algebras.**

The brane bouquet

A super L_∞-algebra $\mathfrak{g}$ is like a Lie algebra, defined on a chain complex of super vector spaces:

$$\mathfrak{g}_0 \xleftarrow{\partial} \mathfrak{g}_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} \mathfrak{g}_n \xleftarrow{\partial} \cdots$$

But the Jacobi identity *does not hold*:

$$[[X, Y], Z] \pm [[Y, Z], X] \pm [[Z, X], Y] \neq 0.$$

Instead, it holds up to *coherent homotopy*: we get infinitely many identities like this:

$$[[X, Y], Z] \pm [[Y, Z], X] \pm [[Z, X], Y] = \partial[X, Y, Z] + [\partial(X \wedge Y \wedge Z)].$$

This says the Jacobi identity holds up to a chain homotopy, given by a trilinear bracket:

$$[-.-, -] \colon \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g},$$

satisfying its own Jacobi-like identity up to a 4-linear bracket . . .

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A super Lie algebra is a super L_∞-algebra concentrated in degree 0:

$$\mathfrak{g}_0 \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots$$

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Given any $(p+2)$-cocycle $\omega\colon \Lambda^{p+2}\mathfrak{g} \to \mathbb{R}$, we can construct an L_∞-algebra $\mathfrak{g}_\omega$ as follows:

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where

- $\mathfrak{g}$ is in degree 0, $\mathbb{R}$ is in degree p.
- $[-,-]$ is the Lie bracket.
- The $(p+2)$-linear bracket, $[-,\cdots,-] = \omega$, is the cocycle.
- All other brackets are 0.

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In homotopy theory, this operation is called 'taking the homotopy fiber' of ω.

The brane bouquet

$$
\begin{tikzcd}
 & & \mathfrak{m2brane} \arrow[d] & & \\
 & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & & \\
\mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] \\
 & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} &
\end{tikzcd}
$$

The brane bouquet

Thanks to $\mathfrak{string}_{\mathrm{I}}$, $\mathfrak{string}_{\mathrm{IIA}}$, $\mathfrak{string}_{\mathrm{IIB}}$ and $\mathfrak{m2brane}$, we can find some of the branes missing from the brane scan.

Fact

The left-invariant forms on $\mathfrak{g}_{\omega}$ are generated by the left-invariant forms on $\mathfrak{g}$ with one additional $(p+1)$-form b such that $db = \omega$.

For example:

- On $\mathfrak{string}_{\mathrm{IIA}} = \mathbb{R}_{\mu_{\mathrm{IIA}}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$, the left-invariant forms are
- from $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$:

$$e^{\nu} = dx^{\nu} - \theta\Gamma^{\nu} d\theta, \quad d\theta^{\alpha}$$

- and a 2-form F such that

$$dF = \mu_{\mathrm{IIA}}.$$

The brane bouquet

Thanks to F, there are new cocycles on $\mathfrak{string}_{\mathrm{IIA}}$.

$$\mu_{\mathrm{D}p} = \sum_{k=0}^{(p+2)/2} c_k^p e^{\nu_1} \wedge \cdots \wedge e^{\nu_{p-2k}} \wedge d\overline{\theta} \wedge \Gamma_{\nu_1 \cdots \nu_{p-2k}} d\theta \wedge F \wedge \cdots \wedge F.$$

- c_k^p are some coefficients chosen to make $d\mu_{\mathrm{D}p} = 0$.
- With some theoretical machinery due to Fiorenza–Sati–Schreiber, we can turn this cocycle into the Dp-brane action.
- Similarly, we can find a cocycle for the M5-brane on $\mathfrak{m2brane}$.

The brane bouquet

$$
\begin{tikzcd}
 & & & & \mathfrak{m5brane} \arrow[d] & & & & \\
 & & & & \mathfrak{m2brane} \arrow[dd] & & & & \\
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & & \mathfrak{d0brane} \arrow[dl] \arrow[ddrr] & & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] \\
\mathfrak{d7brane} \arrow[dr] & & & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddrr] & & & & \mathfrak{d6brane} \arrow[dl] \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[drr] & & & \mathfrak{string}_{\mathrm{I}} \arrow[d] & \text{(pb)} & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] \\
 & & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[rr, shift left] \arrow[rr, shift right] & & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & \mathfrak{m5brane} \arrow[d] & & & \\
 & & & \mathfrak{m2brane} \arrow[dd] & & & \\
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[dl] \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] \\
\mathfrak{d7brane} \arrow[dr] & & & \mathbb{R}^{10,1\vert\mathbf{32}} \arrow[ddr] & \text{(pb)} & & \mathfrak{d6brane} \arrow[dl] \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] \\
 & & \mathbb{R}^{9,1\vert\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1\vert\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1\vert\mathbf{16}+\overline{\mathbf{16}}} & & \\
 & & \mathbb{R}^{5,1\vert\mathbf{8}+\mathbf{8}} & \mathbb{R}^{5,1\vert\mathbf{8}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1\vert\mathbf{8}+\overline{\mathbf{8}}} & & \\
 & & \mathbb{R}^{3,1\vert\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1\vert\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & \\
 & & \mathbb{R}^{2,1\vert\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1\vert\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & \\
 & & \mathbb{R}^{0\vert\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0\vert\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & &
\end{tikzcd}
$$

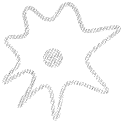

Figure: $\mathbb{R}^{0|1}$

MANY THANKS

References I

The use of L_∞-algebras in physics originates with the work of D'Auria and Fré, who call them 'free differential algebras'.

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The discovery that the WZW terms for Dp-branes and the M5-branes live on the ‘extended superspacetimes’ $\mathfrak{string}_{\mathrm{IIA}}$, $\mathfrak{string}_{\mathrm{IIB}}$ and $\mathfrak{m2brane}$ appears in two articles. The case of the type IIA Dp-branes and the M5-brane is in:

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while the type IIB Dp-branes are in section 2 of:

- M. Sakaguchi, IIB-branes and new spacetime superalgebras, *JHEP* 04 (2000), pp. 019, arXiv:hep-th/9909143.

References III

Later, Fiorenza, Sati and Schreiber placed this into the context of the homotopy theory of L_∞-algebras, discovering the brane bouquet:

- D. Fiorenza, H. Sati, U. Schreiber, Super Lie *n*-algebra extensions, higher WZW models, and super *p*-branes with tensor multiplet fields, *Intern. J. Geom. Meth. Mod. Phys.* **12** (2015), 1550018 (35 pages). arXiv:1308.5264.

Finally, Schreiber and I derived the brane bouquet from the superpoint:

- J. Huerta and U. Schreiber, M-theory from the superpoint. To appear in *Lett. Math. Phys*. arXiv:1702.01774