M-Branes:
Lessons from M2's and Hopes for M5's

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Higher Structures in M-Theory

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Outline

- M2-Branes and three-algebras
- M5-branes and the $(2,0)$ Theory
- A $(2,0)$ System

Outline

We will discuss:

- Aspects of M2-branes and the role of three-algebras.
- M5-branes, mainly with a view to constructing the $(2,0)$-Theory in 6D

We will not discuss:

- The myriad of results arising from reduction to 4D and below (*e.g.* novel non-Lagrangian field theories, dualities, surface operators, AGT,...)[Gaiotto,...]
- Bootstrap results [Beem, Lemos, Rastelli, van Rees],...
- AdS/CFT [Heslop, Lipstein][Chester,Perlmutter],...

Gong Show Version

- M2-Branes are quite well understood but intricate
- M5-branes are hard and not well understood
- I think there is more to explore in terms of structures and relation between of M2's and M5's.

M2-Branes and Three-algebras

The M2-brane SCFT arises as the strong coupling limit of N D2-branes:

- Strong coupling IR fixed point of 3D MSYM

Lift to M-theory implies that the R-symmetry is enhanced

$$SO(1,2)_L \times SO(7)_R \to SO(1,2)_L \times SO(8)_R$$

leading to a 3D SCFT with maximal supersymmetry.

Ultimately this is a statement about gauge theory and QFT arising from M-theory. A prediction so to speak

The relevant theories have now been constructed:

With manifest maximal SUSY there is BLG [Bagger, NL][Gustavson]

- Gauge group $SU(2) \times SU(2)$ or $(SU(2) \times SU(2))/\mathbb{Z}_2$
- describes only two or three M2's on an orbifold.

For arbitrary number of M2-branes one has 3/4 manifest SUSY and ABJM [Aharony, Bergman, Jafferis, Maldacena] or ABJ

- Gauge group $U(M) \times U(N)$
- describes $N \leq M$ branes in an orbifold.

There is now a zoology of Chern-Simons Matter theories with extended SUSY $\mathcal{N} = 4, 5, 6, 8$

For $\mathcal{N} = 3$ there is no restriction on the gauge group[Kao,Lee].

The main ingredient to all these theories is a 3-algebra.

- vector space $\mathcal{V}$ with a triple product

$$[\,\cdot\,,\,\cdot\,,\,\cdot\,] : \mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}\to\mathcal{V}$$

- such that the endomorphism $\varphi(\,\cdot\,) = [\,\cdot\,, U, V] : \mathcal{V}\to\mathcal{V}$ is a derivation (the so-called fundamental identity)

$$\varphi([A,B,C]) = [\varphi(A),B,C] + [A,\varphi(B),C] + [A,B,\varphi(C)]$$

For physics we require that there is a positive definite inner-product on $\mathcal{V}$:

$$\langle\,\cdot\,,\,\cdot\,\rangle : \mathcal{V}\otimes\mathcal{V}\to\mathbb{R}$$

which induces an invariant inner-product on the space of derivations:

$$(T,\varphi) = \langle T(U), V\rangle$$

There is also a complex verion of a 3-algebra:

$$[\cdot,\cdot;\cdot]:\mathcal{V}\otimes\mathcal{V}\otimes\bar{\mathcal{V}}\to\mathcal{V}$$

with complex positive definite inner product: For physics we require that there is a positive definite inner-product on $\mathcal{V}$:

$$\langle\ \cdot\ ,\ \cdot\ \rangle:\mathcal{V}\otimes\mathcal{V}\to\mathbb{C}$$

Again the analog of adjoint map

$$\varphi_{U,\bar{V}}(X)=[X,U;\bar{V}]\qquad\varphi_{U,\bar{V}}(\bar{X})=-[\bar{X},\bar{V};U]$$

is a derivation

$$\varphi_{U,\bar{V}}([X,Y;\bar{Z}])=[\varphi_{U,\bar{V}}(X),Y;\bar{Z}]+[X,\varphi_{U,\bar{V}}(Y);\bar{Z}]+[X,Y;\varphi_{U,\bar{V}}(\bar{Z})]$$

The fundamental identity tells us that the action of φ on $\mathcal{V}$ is that of a lie-algebra $\mathcal{G}$ generated by $\varphi_{U,V}$ for all $U, V \in \mathcal{V}$

- i.e. $\mathcal{V}$ is representation of $\mathcal{G}$.
- thus a 3-algebra defines a lie-algebra $\mathcal{G}$ along with a preferred representation

In fact the reverse is also true: Given a Lie-algebra and a representation (along with invariant inner-products) one can always construct a triple product satisfying the fundamental identity (via the so-called Faulkner map).

Classified by [de Medeiros, Figueroa-O'Farrill,Ritter]...

Thus one need not think of a 3-algebra and just think of the gauge group and matter representation. However susy fixes the symmetry properties of the triple product

- and so which gauge algebras and representations arise
- leads to these rather esoteric choices (and indefinite inner-products on the lie-algebra)

Thus the amount of susy is determined by the gauge algebra and matter representations

- Whereas in super-Yang-Mills the gauge algebra is arbitrary and all fields are in the adjoint (for more than 8 susys)
- possible because in Chern-Simons theories there are no propagating gauge fields

BLG We take $\mathcal{V}$ real and $[\,\cdot\,,\,\cdot\,,\,\cdot\,]$ totally antisymmetric.

$\mathcal{N}=8$ Supersymmetry (here $\mu,\nu=0,1,2$ and $I,J=3,4,5,...,10$)

$$\begin{aligned} \delta X^I &= i\bar{\epsilon}\Gamma^I\Psi \\ \delta\Psi &= D_\mu X^I\Gamma^\mu\Gamma^I\epsilon-\frac{1}{6}[X^I,X^J,X^K]\Gamma^{IJK}\epsilon \\ \delta A_\mu(\,\cdot\,) &= i\bar{\epsilon}\Gamma_\mu\Gamma_I[\,\cdot\,,X^I,\Psi]\,. \end{aligned}$$

Lagrangian

$$\mathcal{L}=-\frac{1}{2}\langle D_\mu X^I D^\mu X^I\rangle+\frac{i}{2}\langle\bar{\Psi},\Gamma^\mu D_\mu\Psi\rangle+\frac{i}{4}\langle\bar{\Psi},\Gamma_{IJ}[X^I,X^J,\Psi]\rangle-V+\mathcal{L}_{CS}$$

Potential

$$V=\frac{1}{12}\langle[X^I,X^J,X^K],[X^I,X^J,X^K]\rangle$$

Chern-Simons term

$$\mathcal{L}_{CS}=\varepsilon^{\mu\nu\lambda}\left((A_\mu,\partial_\nu A_\lambda)+\frac{1}{3}(A_\mu,[A_\nu,A_\lambda])\right)$$

But for a positive definite choice of h^{ab} there is just one finite-dimensional solution [Gauntlett,Gutowski][Papadopoulos]:

$$[T^a, T^b, T^c] = \frac{4\pi}{k}\varepsilon^{abcd}T^d \qquad a,b,c,d = 1,2,3,4$$

The gauge algebra generated by φ is $so(4) = su(2)_+ \oplus su(2)_-$ and

$$((T^+, T^-), (W^+, W^-)) = \frac{k}{4\pi}\mathrm{tr}(T^+W^+) - \frac{k}{4\pi}\mathrm{tr}(T^-W^-)$$

so

$$\begin{aligned}\mathcal{L}_{CS} = {} & \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}\mathrm{tr}\left(A^+_\mu\partial_\nu A^+_\lambda + \frac{1}{3}A^+_\mu[A^+_\nu, A^+_\lambda]\right) \\ & - \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}\mathrm{tr}\left(A^-_\mu\partial_\nu A^-_\lambda) + \frac{1}{3}(A^-_\mu[A^-_\nu, A^-_\lambda])\right)\end{aligned}$$

Fields X^I, Ψ are in the $\mathbf{4} = \mathbf{2} + \overline{\mathbf{2}}$ = bifundamental.

A standard result tells us that $k \in \mathbb{Z}$ - no continuous parameter.

ABJM We need a little less symmetry and a complex $\mathcal{V}$.

X^I written as 4 complex scalar fields Z^A $A = 1, 2, 3, 4$ in $\mathbf{4}$ of $SU(4)$ with $U(1)$ charge 1

- $(Z^A)^\dagger = \bar{Z}_A$ in $\bar{\mathbf{4}}$ of $SU(4)$ with $U(1)$ charge -1

Ψ written as 4 complex fermions ψ_A in $\overline{\mathbf{4}}$ with $U(1)$ charge 1

- $(\psi_A)^\dagger = \psi^A$ in $\mathbf{4}$ of $SU(4)$ with $U(1)$ charge -1

The 16 components of ϵ are reduced to $\epsilon^{AB} = -\epsilon^{BA}$ in $\mathbf{6}$ of $SU(4)$ with $U(1)$ charge 0.

- $(\epsilon^{AB})^* = \epsilon_{AB} = \frac{1}{2}\varepsilon_{ABCD}\epsilon^{CD}$

$\mathcal{N}=6$ Supersymmetry:

$$
\begin{aligned}
\delta Z^A &= i\bar{\epsilon}^{AB}\psi_B \\
\delta\psi_B &= \gamma^\mu D_\mu Z^A \epsilon_{AB} + [Z^C, Z^A; \bar{Z}_C]\epsilon_{AB} + [Z^C, Z^D; \bar{Z}_B]\epsilon_{CD} \\
\delta A_\mu(\,\cdot\,) &= i\bar{\epsilon}_{AB}\gamma_\mu[\,\cdot\,, Z^A; \psi^B] - i\bar{\epsilon}^{AB}\gamma_\mu[\,\cdot\,\bar{Z}_A; \psi_B]
\end{aligned}
$$

Lagrangian

$$
\begin{aligned}
\mathcal{L} &= -\langle D^\mu \bar{Z}_A, D_\mu Z^A\rangle - i\langle\bar{\psi}^A\gamma^\mu, D_\mu\psi_A\rangle - V + \mathcal{L}_{CS} \\
&\quad -i\langle\bar{\psi}^A, [\psi_A, Z^B; \bar{Z}_B]\rangle + 2i\langle\bar{\psi}^A, [\psi_B, Z^B; \bar{Z}_A]\rangle \\
&\quad +\frac{i}{2}\varepsilon_{ABCD}\langle\bar{\psi}^A, [Z^C; Z^D; \psi^B]\rangle - \frac{i}{2}\varepsilon^{ABCD}\langle\bar{\psi}_A, [\bar{Z}_C, \bar{Z}_D; \psi_B]\rangle\ .
\end{aligned}
$$

The potential and Chern-Simons terms are

$$\begin{aligned}
V &= \frac{2}{3}\langle \Upsilon^{CD}_{B}, \bar{\Upsilon}^{B}_{CD}\rangle \\
\Upsilon^{CD}_{B} &= [Z^C, Z^D, \bar{Z}_B] - \frac{1}{2}\delta^C_B[Z^E, Z^D; \bar{Z}_E] + \frac{1}{2}\delta^D_B[Z^E, Z^C; \bar{Z}_E] \\
\mathcal{L}_{CS} &= \varepsilon^{\mu\nu\lambda}(A_\mu, \partial_\nu A_\lambda) + \frac{1}{3}\varepsilon^{\mu\nu\lambda}(A_\mu, [A_\nu, A_\lambda])
\end{aligned}$$

An infinite class of solutions are given by $M \times N$ complex matrices with $\langle A, B\rangle = \mathrm{tr}(AB^\dagger)$ and

$$[A, B; C] = \frac{4\pi}{k}(AC^\dagger B - BC^\dagger A)$$

Gauge group generated by $\delta Z^A = [Z^A, U, \bar{V}]$ is

$$\delta Z^A = MZ^A - Z^A N$$

where $M = -V^\dagger U, N = UV^\dagger$ are $M \times M$ and $N \times N$ matrices respectively and

$$(M, M') = \frac{k}{4\pi}\mathrm{tr}(\mathrm{MM}') \qquad (\mathrm{N}, \mathrm{N}') = -\frac{\mathrm{k}}{4\pi}\mathrm{tr}(\mathrm{NN}')$$

Gauge group is $U(M) \times U(N)$ with matter in the bi-fundamental.

- $M = N$ gives $SU(N) \times SU(N)$ theories
- Add by hand $U(1)$ gauge fields to get $U(N) \times U(N)$ ABJM
- $M \neq N$ gives the ABJ theories

In the special case of $SU(2) \times SU(2)$ we recover the BLG theory in complex notation.

And the list continues with less supersymmetry depending on the symmetry properties of the structure constants

$$[T^a, T^b; T_c] = f^{ab}{}_{cd} T^d$$

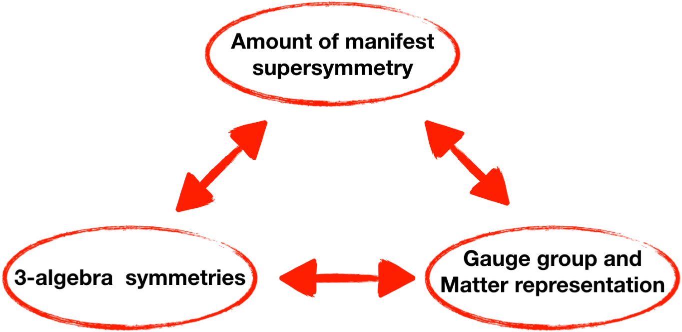
but the actions are essentially the same.

NOVELTIES: These actions ‘break’ some SUSY ‘rules’

- Gauge fields and matter fields are in the same multiplet but not in the same representation of the gauge group.
- The amount of supersymmetry is determined by the gauge group (the Lagrangians are essentially the same):

The three-algebra formalism is a neat way of encoding all this data even though in the end one is always just talking about a Chern-Simons-Matter field theory based on a gauge group and choice of representation.

- Symmetry properties of triple-product $\Longleftrightarrow$ amount of SUSY



$$f^{abcd} = f^{[abcd]} \Longleftrightarrow N = 8 \Longleftrightarrow su(2) \oplus su(2)$$

$$\begin{pmatrix} f^{ab}{}_{cd} = f^{[ab]}{}_{cd} \\ f^{ab}{}_{cd} = (f^{cd}{}_{ab})^* \end{pmatrix} \Longleftrightarrow N = 6 \Longleftrightarrow \begin{array}{c} u(N) \oplus u(M) \\ sp(N) \oplus u(1) \end{array}$$

$$\begin{pmatrix} f^{abcd} = f^{[ab]cd} \\ f^{abcd} = (f_{abcd})^* \end{pmatrix} \Longleftrightarrow N = 5 \Longleftrightarrow \begin{array}{c} sp(N) \oplus su(M) \\ so(7) \oplus su(2) \\ g_2 \oplus su(2) \end{array}$$

Physical Analysis

The first thing to look at is the vacuum moduli space. This tells us the space of all the zero-energy configurations of the M2-branes.

Consider ABJM:

$$[Z^A, Z^B; \bar{Z}_C] = 0 \longleftrightarrow Z^A \bar{Z}_C Z^B = Z^B \bar{Z}_C Z^A$$

Generically this implies that all the Z^A commute (c.f. D-branes):

$$Z^A = \mathrm{diag}(z_1^A, ..., z_n^A)$$

To see that this is all requires one to evaluate the mass formula for small fluctuations which one finds is non-zero (generically: there are special points where extra massless modes arise but are expected to be lifted by non-perturbative effects).

We must identify fields that differ by gauge transformations:

$$Z^A \to g_L Z^A g_R^{-1}$$

We could set $g_L = g_R$ so that this is an adjoint action, as with D-branes. Thus allows us to put Z^A in diagonal form (as we have already done) and in addition acts as

$$z_i^A \leftrightarrow z_j^A \qquad \text{for any } i \neq j$$

e.g. for $i, j = 1, 2$ these are generated by

$$g_L = g_R = \begin{pmatrix} 0 & i & & & \\ i & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

These generate the action of the symmetric group S_N on z_i^A.

Unlike D-branes we also have continuous gauge transformations:

$$z_i^A \to e^{i\theta_i} z_i^A$$

These arise from taking

$$g_L = g_R^{-1} = \mathrm{diag}(e^{i\theta_1/2}, ..., e^{i\theta_N/2})$$

To see the effect of this on the vacuum moduli space we must examine the Lagrangian for the moduli z_i^A, including the gauge fields:

$$\mathcal{L} = -\frac{1}{2}\sum_i D_\mu z_i^A D^\mu \bar{z}_{Ai} + \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}\sum_i A^L_{\mu i}\partial_\nu A^L_{\lambda i} - \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}\sum_i A^R_{\mu i}\partial_\nu A^R_{\lambda i}$$

where $A^L_\mu = \mathrm{diag}(A^L_{\mu 1}, ..., A^L_{\mu N})$, $A^R = \mathrm{diag}(A^R_{\mu 1}, ..., A^R_{\mu N})$ and $D_\mu z_i^A = \partial_\mu z_i^A - i(A^L_{\mu i} - A^R_{\mu i})z_i^A$.

Note that z_i^A only couples to $B_{\mu i} = A^L_{\mu i} - A^R_{\mu i}$ and not to $Q_{\mu i} = A^L_{\mu i} + A^R_{\mu i}$:

$$\mathcal{L} = -\frac{1}{2}\sum_i D_\mu z_i^A D^\mu \bar{z}_{Ai} + \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}\sum_i B_{\mu i}\partial_\nu Q_{\lambda i}$$

with $D_\mu z_i^A = \partial_\mu z_i^A - iB_{\mu i}z_i^A$.

It’s helpful to dualize $Q_{\mu i}$:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\sum_i D_\mu z_i^A D^\mu \bar{z}_{Ai} + \frac{k}{8\pi}\varepsilon^{\mu\nu\lambda}\sum_i B_{\mu i}H_{\nu\lambda i} - \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\sigma_i\partial_\mu H_{\nu\lambda i} \\ &\cong -\frac{1}{2}\sum_i D_\mu z_i^A D^\mu \bar{z}_{Ai} + \frac{k}{8\pi}\varepsilon^{\mu\nu\lambda}\sum_i B_{\mu i}H_{\nu\lambda i} + \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\partial_\mu\sigma_i H_{\nu\lambda i}\end{aligned}$$

where $H_{\nu\lambda i} = \partial_\nu Q_{\lambda i} - \partial_\lambda Q_{\nu i}$.

Integrating out $H_{\nu\lambda i}$ tells us $B_{\mu i} = -k^{-1}\partial_\mu \sigma_i$ and everything is pure gauge:

$$\mathcal{L} = -\frac{1}{2}\sum_i \partial_\mu w_i^A \partial^\mu \bar{w}_{Ai}$$

where $w_i^A = e^{i\sigma_i/k} z_i^A$ is gauge invariant.

But σ_i is periodic:

$$\begin{aligned}
\int \mathcal{L}(\sigma_i + 2\pi) - \int \mathcal{L}(\sigma_i) &= -\frac{1}{4}\sum_i \int \varepsilon^{\mu\nu\lambda}\partial_\mu H_{\nu\lambda i} \\
&= -\frac{1}{2}\sum_i \int dH \\
&= -\frac{1}{2}\sum_i \int dF^L + dF^R \\
&\in 2\pi\mathbb{Z}
\end{aligned}$$

because of the Dirac quantization rule

$$\int dF \in 2\pi\mathbb{Z}$$

and the fact that $B_i = -k^{-1} d\sigma_i$ implies $dB_i = F_i^L - F_i^R = 0$

NB This is very sensitive to the global choice of gauge group $u(N), su(N), su(N)/\mathbb{Z}_N$.

This means that (recall $w_i^A = e^{i\sigma_i/k} z_i^A$)

$$w_i^A \cong e^{2\pi i/k} w_i^A$$

Thus there is an extra orbifold action in spacetime

$$\mathbb{R}^8 \to \mathbb{C}^4/\mathbb{Z}_k$$

and the vacuum moduli space is

$$\mathcal{M} = \mathrm{Sym}^n \left(\mathbb{C}^4/\mathbb{Z}_k\right)$$

Corresponding to N M2-branes in an $\mathbb{C}^4/\mathbb{Z}_k$ transverse space.

And indeed this orbifold preserves 12 supersymmeties.

Let us return to the moduli space. It follows that we can think of

$$Z^A = \begin{pmatrix} z_i^A & & \\ & \ddots & \\ & & z_n^A \end{pmatrix}$$

as describing the positions of N M2-branes in $\mathbb{C}^4/\mathbb{Z}_k$. Furthermore the natural circle for the M-theory direction is the over-all phase.

Suppose we wanted to describe N M2-branes moving along the M-theory circle with different speeds. One might expect that this corresponds to

$$Z^A = \begin{pmatrix} z_i^A e^{i\omega_1 t} & & \\ & \ddots & \\ & & z_N^A e^{i\omega_N t} \end{pmatrix}$$

But this is pure gauge! We can un-do it by taking

$$g_L = g_R^{-1} = \begin{pmatrix} e^{-i\omega_1 t/2} & & \\ & \ddots & \\ & & e^{-i\omega_N t/2} \end{pmatrix}$$

(Note that this gauge transformation is not allowed for D-branes where the scalars are in the adjoint.) So how do the M2-branes 'explore' the full transverse space? Let us set the fermions to zero and construct the hamiltonian

$$\begin{aligned} H &= \int d^2x \, \Pi_{Z^A}\Pi_{\bar{Z}_A} + D_i Z^A D^i \bar{Z}_A + V \\ &\quad + \left(iZ^A \Pi_{Z^A} - i\Pi_{\bar{Z}_A}\bar{Z}_A - \frac{k}{2\pi}F^L_{12}\right) A^L_0 \\ &\quad + \left(i\bar{Z}_A \Pi_{\bar{Z}_A} - i\Pi_{Z^A} Z^A + \frac{k}{2\pi}F^R_{12}\right) A^R_0 \end{aligned}$$

As usual the time-components of the gauge field give constraints:

$$\begin{aligned}\frac{k}{2\pi}F^L_{12} &= iZ^A\Pi_{Z^A} - i\Pi_{\bar{Z}_A}\bar{Z}_A \\ \frac{k}{2\pi}F^R_{12} &= i\Pi_{Z^A}Z^A - i\bar{Z}_A\Pi_{\bar{Z}_A}\end{aligned}$$

Consider the vacuum moduli again:

$$Z^A = \begin{pmatrix} \frac{1}{\sqrt{2}}R^A_1 e^{i\theta^A_1} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{2}}R^A_n e^{i\theta^A_n} \end{pmatrix}$$

The constraint is

$$\frac{k}{2\pi}F^L_{12} = \frac{k}{2\pi}F^R_{12} = \begin{pmatrix} \sum_A (R^A_1)^2\partial_0\theta^A_0 & & \\ & \ddots & \\ & & \sum_A (R^A_n)^2\partial_0\theta^A_n \end{pmatrix}$$

In other words the momentum around the M-theory circle is given by the magnetic flux.

This is, in spirit, the same as dualization:

$$\partial_\mu X^{10} = \frac{1}{2}\varepsilon_{\mu\nu\lambda}F^{\nu\lambda} \qquad \longleftrightarrow \qquad \partial_0 X^{10} = F_{12}$$

This raises the next question: how do we compute quantities with 11D momentum. In particular the gauge invariant observables appear to only carry vanishing $U(1)$ charges:

$$\begin{aligned} \mathcal{O} &= (Z^A \bar{Z}_B Z^C ...) \qquad \text{OK} \\ \mathcal{O} &= (Z^A Z^B Z^C ...) \qquad \text{not OK} \end{aligned}$$

and hence don't really explore all 11 dimensions.

This brings us to monopole or 't Hooft operators: We want to create states that carry magnetic charge.

These operators are defined as a prescription for computing correlators in the path-integral. They are not constructed as a local expression of the fields.

$$< \mathcal{M}(y)\mathcal{O}(z)... >= \int_{\oint_y F=2\pi Q_M} DZD\psi DA\mathcal{O}(z)e^{-S}$$

in other words we require the fields in the path integral to have a specific singularity

$$F = \star\frac{Q_m}{2}d\left(\frac{1}{|x-y|}\right) + \text{nonsingular}$$

$Q_M \in u(n) \times u(n)$ is the magnetic flux and is subject to the standard Dirac quantization condition

$$e^{2\pi i Q_m} = 1 \ .$$

Next we note that due to the Chern-Simons term monopole operators transform locally under a gauge transformation $\delta A_\mu^{L/R} = D_\mu \omega_{L/R}$ (with $\omega \to 0$ at infinity) as

$$\begin{aligned} \mathcal{M}_{Q_M}(x) &\to e^{(ik/2\pi)\, tr \int (D\omega_L \wedge F^L - D\omega_R \wedge F^R)} \mathcal{M}_{Q_M}(x) \\ &= e^{ik\, tr((\omega_L(x) - \omega_R(x))Q_M)} \mathcal{M}_{Q_M}(x) \end{aligned}$$

Note that by construction we have broken the gauge group to $U(1)^N \times U(1)^N$. This is enough to tell us that under full gauge transformations the monopole operators transform in the representation of $U(n) \times U(n)$ whose highest weight is

$$\vec{\Lambda} = k(\vec{Q}_m, -\vec{Q}_m)$$

(actually because of the sign the second factor is the lowest weight)

This is all very abstract (and tricky to calculate with). Consider the abelian case (from the moduli space calculation and Wick rotated)

$$\mathcal{L} = -\frac{1}{2}\sum_i D_\mu z_i^A D^\mu \bar{z}_{Ai} + \frac{k}{8\pi}\varepsilon^{\mu\nu\lambda}\sum_i B_{\mu i}H_{\nu\lambda i} - \frac{i}{8\pi}\varepsilon^{\mu\nu\lambda}\sigma_i\partial_\mu H_{\nu\lambda i}$$

The monopole operators are just

$$\mathcal{M}_i(y) = e^{i\sigma_i(y)}$$

Since

$$\begin{aligned} < \mathcal{M}_i(y)\mathcal{O}(z)... > &= \int DzDBDQe^{i\sigma_i(y)}\mathcal{O}(z)e^{-\int d^3x\mathcal{L}(x)} \\ &= \int DzDBDQ\mathcal{O}(z)e^{-\int d^3x\mathcal{L}(x)-i\sigma_i(x)\delta(x-y)} \end{aligned}$$

which is the same as taking

$$\frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\partial_\mu H_{\nu\lambda i} \to \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\partial_\mu H_{\nu\lambda i} + 8\pi\delta(x-y)$$

i.e. inserting a magnetic charge at $x = y$.

Thus our gauge invariant operator on the moduli space is just

$$w_i^A = e^{i\sigma_i/k} z_i^A = (\mathcal{M}_i)^{\frac{1}{k}} z_i^A$$

and indeed $\mathcal{M}_i$ has charge $(k, -k)$ under $U(1) \times U(1)$.

Thus we see that at $k = 1$ even translations in the transverse space are not symmetries of the lagrangian.

$$P_\mu^A = \mathrm{tr}(D_\mu Z^A) \qquad \text{not OK}$$

need

$$P_\mu^A = \mathrm{tr}(\mathcal{M}_{\vec{\lambda}_1, -\vec{\lambda}_1} D_\mu Z^A) \qquad \text{OK}$$

as well as the additional two supersymmetries that enhance $\mathcal{N} = 6 \to \mathcal{N} = 8$:

$$S_\mu = \mathrm{tr}(\mathcal{M}_{2\vec{\lambda}_1, -2\vec{\lambda}_1} \Psi_A D_\mu Z^A) \qquad \text{OK}$$

How does BLG fit in? To cut a long story short [N.L,Papageorgakis],[Bashkirov,Kapustin][Agmon,Chester,Pufu]:

- BLG $(SU(2) \times SU(2))/\mathbb{Z}_2$ at $k = 1$ is dual to ABJM $U(2) \times U(2)$ at $k = 1$, i.e. 2 M2's in $\mathbb{R}^8$
- BLG $SU(2) \times SU(2)$ at $k = 2$ is dual to ABJM $U(2) \times U(2)$ at $k = 2$, i.e. 2 M2's in $\mathbb{R}^8/\mathbb{Z}_2$
- BLG $(SU(2) \times SU(2))/\mathbb{Z}_2$ at $k = 4$ is dual to ABJ $U(2) \times U(3)$ at $k = 2$, i.e. 2 M2's in $\mathbb{R}^8/\mathbb{Z}_2$ with torsion i.e. 2 M2's in $\mathbb{R}^8/\mathbb{Z}_2$
- BLG $(SU(2) \times SU(2))/\mathbb{Z}_2$ at $k = 3$ is dual to ABJM $U(3) \times U(3)$ at $k = 1$ without the centre of mass multiplet, i.e. the interacting part of 3 M2's in $\mathbb{R}^8$

So it describes 2 or 3 M2-branes in $\mathbb{R}^8$ or $\mathbb{R}^8/\mathbb{Z}_2$ with all symmetries manifest.

LESSONS and a Question:

- Too much to ask for all symmetries to be manifest
- However for two M2's more symmetries are manifest and hence the action was easier to discover
- The quantum theory can be very different than the classical one
- Must consider 'quantum' operators to see the full physics
- The role of the gauge group is non-trivial and global choices matter
- Is there a role for the general BLG theories (*i.e.* for $k > 4$)?

M5-branes and the $(2,0)$ Theory

The decoupling limit of N M5-branes leads to an interacting CFT in 5+1 dimensions.

In the abelian case $N=1$ the dynamics are known [Schwarz,Perry],[Howe,Sezgin,West],[Pasti,Sorokin,Tonin]

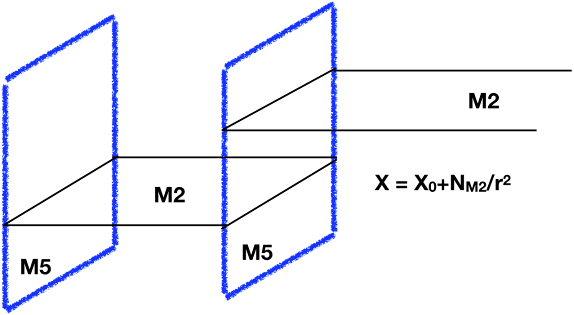
Five scalars X^I (so now $I=6,7,8,9,10$ and $\mu=0,1,2,3,4,5$), a 2-form B with self-dual field strength H and a 16-component fermion Ψ. At the linearised level we simply have

$$\partial_\mu \partial^\mu X^I = 0$$

$$H_{\mu\nu\lambda} = 3\partial_{[\mu} B_{\nu\lambda]} \qquad H_{\mu\nu\lambda} = \frac{1}{3!}\varepsilon_{\mu\nu\lambda\rho\sigma\tau} H^{\rho\sigma\tau}$$

$$i\Gamma^\mu \partial_\mu \Psi = 0$$

For $N>1$ one finds the interacting A_N $(2,0)$-Theory.



The dynamics are thought to arise from self-dual strings associated to M2-branes ending on M5-branes

- Natural BPS states
- Wilson-lines replaced by surface operators of B
- abelian case long understood [Howe,NL, West]
- non-abelian case of great interest as a higher gauge theory analogue of the Nahm transform[Saemann,...].

AdS/CFT predicts that the number of 'degrees of freedom' of N M5-branes scales as N^3[Klebanov,Tsetylin]

Reduction on S^1

Reduction of N-M5-branes on S^1 of radius R_5 gives N D4-branes in type IIA string theory with coupling $g_s = R_5/l_s$.

This is described by $U(N)$ (4+1)-D MSYM and coupling $g^2 = 4\pi^2 R_5$. So the $(2,0)$-Theory is a UV completion of 5D MSYM with enhanced Lorentz symmetry [Seiberg]

$$SO(1,4)_L \times SO(5)_R \longrightarrow SO(1,5)_L \times SO(5)_R$$

Another prediction so to speak

KK momenta are carried by instanton-solitons $F = \star F$ [Rozali]:

$$P_5 = \frac{n}{R_5} \ , \qquad n = \frac{1}{8\pi^2} \mathrm{tr} \int F \wedge F$$

associated to the topological current

$$J^\mu = \frac{1}{32\pi^2} \mathrm{tr} \int \varepsilon^{\mu\nu\lambda\rho\sigma} F_{\nu\lambda} F_{\rho\sigma}$$

Reduction on $\mathbb{T}^2$

Let us reduce again on an S^1 with radius R_4. Here we find 4D $U(N)$ MSYM with coupling $g^2 = 2\pi R_5/R_4$

S-duality swaps perturbative modes with monopoles and $R_4 \leftrightarrow R_5$.

This is a modular transformation of $\mathbb{T}^2$ which is a diffeomorphism in 6D and hence is a manifest symmetry of the $(2,0)$-Theory.

N.B. 4D MSYM is only self-dual for ADE gauge groups so the $(2,0)$-Theory can only exist for ADE gauge groups.

Indeed it was first constructed by a decoupling limit of type IIB on $K3$ with an ADE singularity [Witten]

No Action?!

There are several arguments/issues/challanges against constructing a 6D action.

1) Even without worrying about self-duality there are no ‘good’ interacting lagrangians in 6D (renormalizable, well-defined vacuum).

$$S_{6D} \sim \int d^6x H_{\mu\nu\lambda}H^{\mu\nu\lambda} + D_\mu X^I D^\mu X^I$$
$$+ \underbrace{(X)F_{\mu\nu}F^{\mu\nu} + (X)^3}_{unbounded} + non-renormalizable$$

2) How can one obtain [Witten]?

$$S_{4DMSYM} = \frac{R_4}{2\pi R_5} \int d^4x \mathcal{L}_{4DMSYM}$$

from

$$S_{6D} = \int d^6x \mathcal{L}_{6D} = 4\pi^2 R_4 R_5 \int d^4x \mathcal{L}_{6Dzero-modes}$$

3) Reduction to $\mathbb{R}^{1,1}$ on $\mathcal{M}_4$ leads to $b_2^+(\mathcal{M}_4)$ chiral bosons and $b_2^-(\mathcal{M}_4)$ anti-chiral bosons.

But it is known that there is no modular invariant partition function if $b_2^+(\mathcal{M}_4) - b_2^-(\mathcal{M}_4) \notin 8\mathbb{Z}$

So therefore no diffeomorphism invariant action in 6D [Witten].

4) The $(2,0)$-Theory exists for ADE gauge groups but reduction on S^1 with a boundary condition that twists by an outer-automorphism gives 5D MSYM with B, C gauge groups.

Take the Tachikawa Test:

Given a $SU(2n)$ $(2,0)$-Theory action add an $\mathbb{Z}_2$ twist along S^1. Does it give $SO(2n+1)$ 5D MSYM (**NB** $SO(2n+1) \nsubseteq SU(2n)$)?

Constructions

DLCQ[Aharony,Berkooz,Kashru,Seiberg,Silverstein]

Consider null-compactification: $x^{\pm} = x^0 \pm x^5, x^i$, $i = 1, 2, 3, 4$

$$x^- \cong x^- + 2\pi R_- \qquad \text{and fix} \qquad P_- = K/R_-$$

We should view this as the limit of an infinite boost $v = 1 - \epsilon^2 \to 1$ of a spacelike compactification $x^5 \cong x^5 + 2\pi R_5$

$$R_- = R_5/\epsilon$$

Key point: To keep R_- finite one must shrink $R_5 \to 0$ and hence the $(2,0)$-Theory on S^1 is well described 5D MSYM with fixed $P_5 = K/R_5$.

In this limit

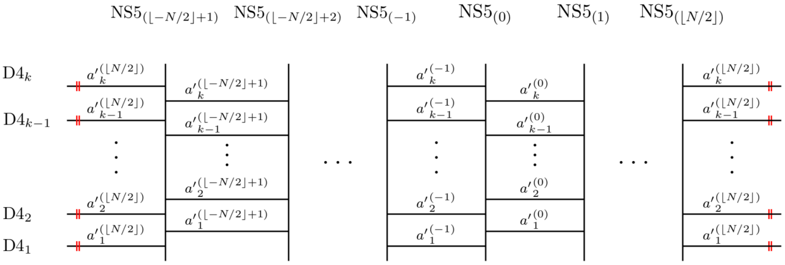
$$K = \frac{1}{8\pi^2}\mathrm{tr}\int F \wedge F$$

and we are looking at the sector of 5D MSYM with instanton number K.

Dynamics are reduced to quantum mechanics on the moduli space of $SU(N)$ instantons with instanton number K.

NB This relies heavily on the fact that $g^2 \propto R_5$ so we find weakly coupled 5D MSYM.

Deconstruction[Arkani-Hamed, Cohen, Kaplan, Karch, Motl]
Construct a quiver (moose) arising from the following brane diagram:



The D4-branes are described by $(SU(K))^N$ SYM with $N_f = 2K$ fields in the bi-fundamental of each $SU(K)$.

This gives a 4D $\mathcal{N} = 2$ SCFT.

Need to go out on the Higg's branch breaking $(SU(k))^N \to SU(K)$. A careful tuning of parameters: scalar vev's, coupling g and number of nodes N leads to a well-defined limit as $N \to \infty$.

The periodicity leads to a finite but large tower of states which for low energy look like a KK-tower.

There is an S-duality of the quiver field theory: 'KK' tower of the quiver is enhanced non-perturbatively to an $SL(2,\mathbb{Z})$ multiplet of two towers: reconstruct a 6D theory with $SO(5)$ R-symmetry.

Has recently been successfully used to make exact localization calculations [Hayling,Pomoni,Papageorgakis,Rodriguez-Gomez].

5D MSYM Maybe 5D MSYM is actually well defined non-perturbatively and is an exact description of the $(2,0)$-Theory on S^1
[Douglas],[NL,Papageorgakis,Schmidt-Sommerfeld]

It contains a complete KK tower of soliton states so any UV completion would have to remove these. Why bother?

5D momentum inserted by 'instanton' operators
[NL,Papageorgakis,Schmidt-Sommerfeld][Tachikawa]...

$$< \mathcal{I}(y)\mathcal{O}(z)... >= \int_{\mathrm{tr}\oint_y F\wedge F=8\pi^2 n} D\Psi DA\mathcal{O}(z)e^{-S}$$

Need to include zero-sized instantons but one can see N^3 behaviour [Kim,Kim,Koh,Lee,Lee],[Kallen,Zabzine]

Perturbative divergences removed by small instanton-soliton effects [Royston,Papageorgakis]

If so then 5D MSYM does provide an ‘action’ for the $(2,0)$-Theory on S^1 for any radius

But then $\mathcal{M}_4 = S^1 \times \mathcal{M}_3$ so $b_2^+(\mathcal{M}_4) = b_2^-(\mathcal{M}_4)$ and hence no-chiral modes.

Consider instead $\mathcal{M}_4$ as multi-Taub-NUT with $b_2^+(\mathcal{M}_4) \neq 0$. This is non-compact but has a nontrivial S^1 fibration.

- Reduction to IIA leads to D4-branes intersecting with D6-branes.
- two-dimensional Chiral charged modes localised at the zeros of the fibration

We can describe it by a variation of 5D MSYM with a Chern-Simons term [Linander,Ohlsson][Cordova,Jafferis].

Chiral modes exist as solitons [Ohlsson][NL, Owen]

These three descriptions are all related:

- The DLCQ description of the $(2, 0)$-Theory must also give the UV completion of 5D MSYM. But it only uses information arising from the classical IR dynamics of instanton-solitons in 5D MSYM
  - although there are singularities in the moduli space from zero-sized instantons that need regularization
- The action obtained from deconstruction is a 'lattice'-like regularization of the 5D MSYM action.
- Formally 5D MSYM only exists as the $(2, 0)$-Theory on S^1

There also exist some action proposals in the literature:

- Reduction on $\mathbb{R} \times S^5$ to 5D MSYM on $\mathbb{R} \times \mathbb{C}P^4$ with a Chern-Simons term [Kim,Lee]
- Twistor-inspired action [Saemann, Wolf],[Saemann, Schmidt]
- D5 MSYM with KK-tower [Bonetti, Grimm, Hoghenneger]
- Mixed 5D/6D action [Chu,Lo]
- $G \times G$ action [Chu]
- Non-local 6D action [Ho, Huang, Matsuo]

Some Relations

There are a few ways that we expect M5s to arise from M2's:

'T-duality' (reduction to IIA on the first S^1, T-duality to type IIB on the second S^1, T-duality back to IIA on the third S^1 and lift back up to M-theory).

- M5's on $\mathbb{T}^3$ gives M2's orthogonal to $\hat{\mathbb{T}}^3 \times \mathbb{R}^5$
- M2's orthogonal to $\mathbb{T}^3$ gives M5's on $\hat{\mathbb{T}}^3$
- But decoupling requires $R \to 0$ and $\hat{R} = l_p^3/R^2 \to \infty$

The first is rather trivial: M5 on $\mathbb{T}^3$ gives 3D MSYM and shrinking the torus goes to strong coupling. Find M2's as strong coupling IR limit of 3D MSYM.

An attempt at the second was tried in [Jeon,NL, Richmond] and gives a modified version of 5D MSYM.

Flux Background

- M2-branes in a background 3-form flux expand into M5-branes on S^3 a la Myers.
- Can construct the effective action from ABJM [Nastase,Papageorgakis] but one just finds 5D MSYM
- Monopole operators of M2-momenta map to instanton operators [NL,Nastase,Papageorgakis]

BLG with Nambu Bracket;

- It has been observed using $[X, Y, Z] = \epsilon^{ijk}\partial_i X \partial_J Y \partial_k Z$ in BLG leads to an abelian M5-brane wrapped on an auxiliary three-manifold.[Ho,Matsuo][Bandos,Townsend]

Challenges/Wish List

- Provide a field definition/construction of the $(2,0)$-Theory *i.e.* without recourse to String Theory or M-Theory
- Find the mathematical structures that best capture aspects of the $(2,0)$-Theory *e.g.* Non-abelian periods of 2-forms. Twistors, Lie-2-Groups *etc.* [Baez, Huerta, Sati, Schreiber,Saemann,Wolf,...,Everyone Here,..]
- Obtain calculable formulations of the $(2,0)$-Theory with 6D Diffeomorphisms and Lorentz!
- Construct an action (?!), Partition function(s), families of actions or something action-like.
- Better understand ‘quantum operators’ such monopole and instanton operators.
- Make S-duality manifest?
- Make the N^3 behaviour more apparent

A $(2,0)$ System

The $(2,0)$ superalgebra is [NL,Sacco][NL,Papageorgakis]

$$
\begin{aligned}
\delta X^i &= i\bar{\epsilon}\Gamma^i\Psi \\
\delta Y^\mu &= \frac{i}{2}\bar{\epsilon}\Gamma_{\lambda\rho}C^{\mu\lambda\rho}\Psi \\
\delta H_{\mu\nu\lambda} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu}D_{\lambda]}\Psi + i\bar{\epsilon}\Gamma^i\Gamma_{\mu\nu\lambda\rho}[Y^\rho, X^i, \Psi] \\
&\quad + \frac{i}{2}\bar{\epsilon}(\star C)_{\mu\nu\lambda}\Gamma^{ij}[X^i, X^j, \Psi] + \frac{3i}{4}\bar{\epsilon}\Gamma_{[\mu\nu|\rho\sigma}C^{\rho\sigma}{}_{\lambda]}\Gamma^{ij}[X^i, X^j, \Psi] \\
\delta A_\mu(\cdot) &= i\bar{\epsilon}\Gamma_{\mu\nu}[Y^\nu, \Psi, \ \cdot\ ] + \frac{i}{3!}\bar{\epsilon}C^{\nu\lambda\rho}\Gamma_{\mu\nu\lambda\rho}\Gamma^i[X^i, \Psi, \ \cdot\ ]\ , \\
\delta\Psi &= \Gamma^\mu\Gamma^i D_\mu X^i\epsilon + \frac{1}{2\cdot 3!}H_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\epsilon - \frac{1}{2}\Gamma_\mu\Gamma^{ij}[Y^\mu, X^i, X^j]\epsilon \\
&\quad + \frac{1}{3!\cdot 3!}C_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\Gamma^{ijk}[X^i, X^j, X^k]\epsilon
\end{aligned}
$$

$$
\Gamma_{012345}\epsilon = \epsilon \qquad \Gamma_{012345}\Psi = -\Psi
$$

X^I, Ψ and $H_{\mu\nu\lambda}$ are dynamical, A_μ and Y^μ are auxiliary but $C_{\mu\nu\lambda}$ is a background (abelian) 3-form

A standard (but trust me tedious) calculation shows that this system indeed closes on the following equations of motion

$$0 = \Gamma^\rho D_\rho \Psi + \Gamma_\rho \Gamma^i [Y^\rho, X^i, \Psi] + \frac{i}{2 \cdot 3!} C^{\rho\sigma\tau} \Gamma_{\rho\sigma\tau} \Gamma^{ij} [X^i, X^j, \Psi]$$

$$0 = D^2 X^i + [Y^\mu, X^j, [Y_\mu, X^j, X^i]] + \frac{1}{2 \cdot 3!} C^2 [X^j, X^k, [X^j, X^k, X^i]]$$
$$+ fermions$$

$$0 = D_{[\lambda} H_{\mu\nu\rho]} + \frac{1}{2} (\star C)_{[\mu\nu\lambda} [X^i, X^j, [Y_{\rho]}, X^i, X^j]]$$
$$+ \frac{1}{4} \varepsilon_{\mu\nu\lambda\rho\sigma\tau} [Y^\sigma, X^i, D^\tau X^i] + fermions$$

As well as constraints:

$$
\begin{aligned}
F_{\mu\nu}(\cdot) &= [Y^\lambda, H_{\mu\nu\lambda},\ \cdot\ ] - (\star C)_{\mu\nu\lambda}[X^i, D^\lambda X^i,\ \cdot\ ] + fermions \\
0 &= D_\mu Y^\nu - \frac{1}{2} H_{\mu\lambda\rho} C^{\nu\lambda\rho} \\
0 &= [Y^\mu, D_\mu(\cdot),\ \cdot'\ ] + \frac{1}{3}[D_\mu Y^\mu,\ \cdot\ ,\ \cdot'\ ] \\
0 &= C^{\mu\nu\lambda} D_\lambda(\cdot) - [Y^\mu, Y^\nu,\ \cdot\ ] \\
0 &= C \wedge Y \\
0 &= C_{\sigma[\mu\nu} C^{\sigma}{}_{\lambda]\rho}
\end{aligned}
$$

Somewhat unconventional (ugly? beautiful?).

There is a conserved supercurrent:

$$
\begin{aligned}
S^\mu = {} & 2\pi i \langle D_\nu X^i, \Gamma^\nu \Gamma^\mu \Gamma^i \Psi \rangle + \frac{2\pi i}{4} \langle H_{\nu\lambda\rho}, \Gamma^{\nu\lambda\rho} \Gamma^\mu \Psi \rangle \\
& - \frac{2\pi i}{2} \langle [Y_\mu, X^i, X^j], \Gamma^\nu \Gamma^\mu \Gamma^{ij} \Psi \rangle \\
& + \frac{2\pi i}{3!^2} C_{\nu\lambda\rho} \langle [X^i, X^j, X^k], \Gamma^{\nu\lambda\rho} \Gamma^\mu \Gamma^{ijk} \Psi \rangle
\end{aligned}
$$

and energy-momentum tensor :

$$
\begin{aligned}
T_{\mu\nu} = {} & \frac{\pi}{2} \langle H_{\mu\lambda\rho}, H_\nu{}^{\lambda\rho} \rangle + 2\pi \langle D_\mu X^i, D_\nu X^i \rangle - \pi \eta_{\mu\nu} \langle D_\lambda X^i, D^\lambda X^i \rangle \\
& - \frac{\pi}{2} \eta_{\mu\nu} \langle [Y_\lambda, X^i, X^j], [Y^\lambda, X^i, X^j] \rangle \\
& + \frac{2\pi}{3!} (C_{\mu\lambda\rho} C_\nu{}^{\lambda\rho} - \frac{1}{6} \eta_{\mu\nu} C^2) \langle [X^i, X^j, X^k], [X^i, X^j, X^k] \rangle \\
& + \frac{\pi}{3!} C_{\mu\lambda\rho} (\star C)_\nu{}^{\lambda\rho} \langle [X^i, X^j, X^k], [X^i, X^j, X^k] \rangle + fermions
\end{aligned}
$$

One can also compute the superalgebra and central charges.

Solving the Constraints: M5's

Let us start with the case $C_{\mu\nu\lambda} = 0$. Here $D_\mu Y^\nu = 0$ and can fix

$$Y^\mu = V^\mu T^4$$

where T^4 is some generator of the 3-algebra and V^μ a constant vector:

- All components of the fields along T^4 become free - 6D centre of mass (2,0) multiplet
- Remaining modes are acted on by an $su(2)$ gauge algebra.
- $[Y^\mu, D_\mu, \,\cdot\,] = 0$ so these modes only depend on the the coordinates orthogonal to V^μ.
- Can extend to any gauge group by taking a Lorentzian 3-algebra

But there are still some choices:

We can fix $V^\mu = 2\pi R_5 \delta_5^\mu$ (spacelike).

The constraints then say that the remaining dynamical fields only depend on $x^0, ..., x^4$ and

$$F_{\mu\nu} = 2\pi R_5 H_{\mu\nu 5}$$

The dynamical equations then all arise from the action

$$S = -\frac{4\pi^2}{R_5}\mathrm{tr}\int d^5x \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu X^i D^\mu X^i - \frac{1}{4}[X^i, X^j]^2 + fermions$$

i.e. 5D maximally supersymmetric Yang-Mills corresponding to M5-brane on S^1 and KK-modes are instanton-solitons:

$$P_5 = \frac{n}{R_5} \qquad\qquad n = \frac{1}{8\pi^2}\mathrm{tr}\int_{\mathbb{R}^4} F\wedge F$$

Alternatively we can set $V^\mu = 2\pi R_0 \delta^\mu_0$ with $F_{\mu\nu} = 2\pi R_0 H_{\mu\nu 0}$. The dyanmical equations then all arise from the action

$$S = \frac{4\pi^2}{R_0}\mathrm{tr}\int d^5x \frac{1}{4}F_{ab}F_{ab} - \frac{1}{2}D_aX^iD_aX^i - \frac{1}{4}[X^i, X^j]^2 + fermions$$

i.e. 5D Euclidean maximally supersymmetric Yang-Mills.

Such a Euclidean theory with compact $SO(5)$ R-symmetry was noted by [Hull][Hull,Khuri] as a time-like reduction of the M5-brane.

- Somewhat novel as typically Euclidean maximally supersymmetric Yang-Mills theories have non-compact R-symmetry. This one arises from reduction of super-Yang-Mills in $5+5$ dimensions.
- Field theory with an emergent compact time [Hull,NL]

Solving the Constraints: M2's

Let us take $C_{345} = l^3$ non-vanishing.

The constraint

$$[Y^\mu, D_\mu \cdot\,,\ \cdot'\,] + \frac{1}{3}[D_\mu Y^\mu,\ \cdot\,,\ \cdot'\,] = 0$$

suggests setting $\partial_a = 0$, $a = 3, 4, 5$ and $Y^\alpha = 0$, $\alpha = 0, 1, 2$.

In which case the constraint

$$C^{\mu\nu\lambda} D_\lambda(\cdot) - [Y^\mu, Y^\nu,\ \cdot\,] = 0$$

implies

$$A_a(\cdot) = -\frac{1}{2l^3}\varepsilon_{abc}[Y^b, Y^c,\ \cdot\,]$$

From this the remaining constraints can solved leading to

$$
\begin{aligned}
H_{abc} &= -\frac{1}{l^6}[Y_a, Y_b, Y_c] \\
H_{\alpha bc} &= -\frac{1}{l^3}\varepsilon_{bcd} D_\alpha Y^d \\
H_{\alpha\beta c} &= -\frac{1}{l^3}\varepsilon_{\alpha\beta\gamma} D^\gamma Y_c \\
H_{\alpha\beta\gamma} &= -\frac{1}{3!l^6}\varepsilon_{\alpha\beta\gamma}\varepsilon^{abc}[Y_a, Y_b, Y_c]
\end{aligned}
$$

and

$$
\begin{aligned}
F_{\alpha a}(\cdot) &= \frac{1}{l^3}\varepsilon_{abc}[Y^b, D_\alpha Y^c,\ \cdot\ ] \\
F_{ab}(\cdot) &= \frac{1}{l^6}[Y^c, [Y_a, Y_b, Y_c],\ \cdot\ ]
\end{aligned}
$$

Let us write $X^a = l^{-3/2} Y^a$ then everything is derived from the action ($I = 3, 4, 5, ..., 10$)

$$S = \int d^3x \left[\langle D_\alpha X^I, D^\alpha X^I \rangle - \frac{1}{6} \langle [X^I, X^J, X^K], [X^I, X^J, X^K] \rangle \right.$$
$$\left. + \varepsilon^{\alpha\beta\gamma} (A_\alpha, \partial_\beta A_\gamma) - \frac{1}{3} \varepsilon^{\alpha\beta\gamma} (A_\alpha, [A_\beta, A_\gamma]) \right] + fermions$$

This is the maximally supersymmetric M2-brane Chern-Simons-Matter theory [B,L][G]

This is consistent with a T-duality along the directions of $C_{\mu\nu\lambda}$:

$$M5: \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \overset{T_{345}}{\Longleftrightarrow} \quad M2: \quad 0 \quad 1 \quad 2$$

We can also take a "timelike" $C_{045} = l^3$.

Gives a Euclidean theory on x^1, x^2, x^3 with no time dependence.

This leads to a maximally supersymmetric Euclidean M2-brane theory with $SO(2,6)$ R-symmetry.

- Similar in structure to the normal maximally supersymmetric M2-brane case but with some funny signs

Consistent with [Hull],[Hull Khuri] where a time-like T-duality of M-theory leads to M*-theory with signature $(2,9)$

$$M5: \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \overset{T_{034}}{\Longleftrightarrow} \quad E3: \quad \quad 1 \quad 2 \quad \quad \quad 5$$

The Null M5

Let us return to the M5-brane case and note that we can also choose to set $Y^\mu = 2\pi R_- \delta^\mu_-$ so $D_- = 0$. Here we find

$$F_{ij} = 2\pi R_- H_{ij-}$$

and self-duality of H leads to self-duality of F_{ij}. Similarly $G_{ij} = 2\pi R_- H_{ij+}$ is anti-self-dual (but doesn't satisfy Bianchi).

The fields now depend on x^+, x^i, $i = 1,2,3,4$.

The dynamics can all be derived from the action [NL, Owen]

$$S = \frac{4\pi^2}{R_-}\mathrm{tr}\int d^4x dx^+ \Big[\frac{1}{2}F_{+i}F_{+i} - \frac{1}{2}D_iX^ID_iX^I - \frac{1}{2}F_{ij}G_{ij} \\ -\frac{i}{2}\bar\Psi\Gamma_-D_+\Psi + \frac{i}{2}\bar\Psi\Gamma_iD_i\Psi - \frac{1}{2}\bar\Psi[X^I,\Gamma_-\Gamma^I\Psi]\Big]$$

This is novel field theory in 4+1 dimensions invariant under

- 16 supersymmetries
- translations in space and time
- $SO(4)$ rotations
- $SO(5)$ R-symmetry

Note that $G_{ij} = 2\pi R_- H_{ij-}$ acts as a Lagrange multiplier imposing

$$F_{ij} = \star F_{ij}$$

This restricts the dynamics to motion on the moduli space of self-dual gauge fields.

The action reduces to a sigma model on the ADHM moduli space of fixed instanton number n:

$$S = \frac{1}{2}\int dx^+ g_{MN}(\partial_+\xi^M - L^M)(\partial_+\xi^N - L^N) - g_{MN}K^M K^N$$
$$+ fermions$$

Here L^M, K^M are vectors on moduli space determined by the vev's of A_+ and X^I.

We can view a null choice of Y^μ as a limit of an infinite boost of a spacelike Y^μ where we saw that the spatial momentum was n/R_5. Thus we are looking at an M5-brane with $P_- = n/R_-$

This reproduces the DLCQ description of the dynamics of M5-brane [Aharony,Berkooz,Seiberg][Aharony, Kachru, Seiberg,Silverstein]

The Null M2

We can also take a null $C_{04+} = l^3$ [Kucharski, NL, Owen] which leads to a rather odd system:

- Fields depend on x^+, x^1, x^2
- Y^3, Y^4, Y^- are non-zero
- Y^- joins up with X^i to form an $SO(6)$ multiplet X^I
- $H_{\mu\nu\lambda}$ is largely determined in terms of Y^3, Y^4, Y^-
- self-duality implies $Z = Y^4 + iY^3$ is holomorphic $\bar{D}Z = 0$, $z = x^1 + ix^2$
- $H = H_{+z3} = iH_{+z4}$ is undetermined

Dynamics obtained from the action [NL, Owen]

$$
\begin{aligned}
S = \int d^2x dx^+ \Bigg[& \frac{1}{4}\langle D_+ Z, D_+ \bar{Z} \rangle - \langle D X^I, \bar{D} X^I \rangle + \langle D\bar{Z}, \bar{H} \rangle + \langle \bar{D} Z, H \rangle \\
& - \frac{i}{4}\langle D_+ X^I, [Z, \bar{Z}, X^I] \rangle - \frac{1}{8}\langle [X^I, X^J, Z], [X^I, X^J, \bar{Z}] \rangle \\
& + \frac{i}{2}(A_+, F_{z\bar{z}}) + \frac{i}{2}(A_z, F_{\bar{z}+}) + \frac{i}{2}(A_{\bar{z}}, F_{+z}) + \frac{i}{2}(A_+, [A_z, A_{\bar{z}}]) \\
& + \frac{il^3}{2\sqrt{2}}\langle \Psi_+^T, D_+ \Psi_+ \rangle + i\langle \Psi_+^T, \hat{\Gamma}_z \bar{D} \Psi_- + \hat{\Gamma}_{\bar{z}} D \Psi_- \rangle \\
& - \frac{l^6}{2\sqrt{2}}\langle \Psi_+^T, \hat{\Gamma}_{Z\bar{Z}} \hat{\Gamma}^{IJ} \left[X^I, X^J, \Psi_+ \right] \rangle + \frac{1}{4\sqrt{2}}\langle \Psi_-^T, \left[Z, \bar{Z}, \Psi_- \right] \rangle \\
& + \frac{i}{2}\langle \Psi_+^T, \hat{\Gamma}^I \hat{\Gamma}_Z \left[Z, X^I, \Psi_- \right] \rangle + \frac{i}{2}\langle \Psi_+^T, \hat{\Gamma}^I \hat{\Gamma}_{\bar{Z}} \left[\bar{Z}, X^I, \Psi_- \right] \rangle \Bigg]
\end{aligned}
$$

where $\Psi_\pm = \frac{1}{2}(1 \pm \hat{\Gamma}_{034})\Psi$

This is novel field theory in 2+1 dimensions invariant under

- 16 supersymmetries
- translations in space and time
- $SO(2)$ rotations
- $SO(6)$ R-symmetry

Note that $H = H_{+z3}$ acts as a Lagrange multiplier imposing

$$\bar{D}Z = 0$$

Furthermore there is a Gauss Law constraint arising from the the A_+ equation of motion:

$$F_{z\bar{z}}(\,\cdot\,) = -\frac{1}{4}\left[X^I, \left[Z, \bar{Z}, X^I\right], \,\cdot\,\right] + \ldots$$

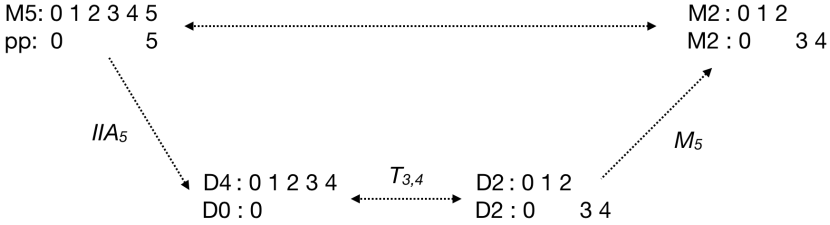
Thus the motion is constrained to the Hitchin Moduli space.

C_{34-} is the limit of an infinite boost along x^5 of the C_{345} case.

Indeed the Hitchin-system gives rise to a momentum along x^5:

$$\mathcal{P}_5 \sim \oint \langle Z, \bar{D}\bar{Z}\rangle dz + \langle \bar{Z}, DZ\rangle d\bar{z}$$

which appears as a winding of the M2-branes around x^3, x^4.



So we are looking at intersecting M2-branes that have been boosted along x^5.

16 vs 8 Supersymmetries

The field theories that we have constructed have 16 supersymmetries but their on-shell conditions lead to a sigma model on a moduli space that admits only 8 supersymmetries.

What happened to the other 8 supersymmetries?

In both cases the supersymmetries split $\mathcal{Q} \to (\mathcal{Q}_+, \mathcal{Q}_-)$ and the superalgebra is of the form

$$\begin{aligned} \{\mathcal{Q}_+, \mathcal{Q}_+\} &\sim \mathcal{P}_+ \\ \{\mathcal{Q}_+, \mathcal{Q}_-\} &\sim \mathcal{P} \\ \{\mathcal{Q}_-, \mathcal{Q}_-\} &\sim \mathcal{P}_- \end{aligned}$$

where $\mathcal{P}$ are the spatial momentum.

In both cases $\mathcal{P}_- = n/R_-$ where n is given by a topological quantity (an instanton number or a winding number) which grades the moduli space

$$\mathcal{M} = \oplus_n \mathcal{M}_n$$

Thus when we restrict to motion on $\mathcal{M}_n$ for $n \neq 0$ the $\mathcal{Q}_-$ supersymmetries are broken.

T-Duality and Doubled Field Theory?

The field theories that we obtain from this system are all consistent with the notion of 'T-duality' (really a U-duality) in M-theory on $\mathbb{T}^3$ along x^μ, x^ν, x^λ with radii R_μ, R_ν, R_λ and

$$C_{\mu\nu\lambda} = (2\pi)^3 R_\mu R_\nu R_\lambda$$

Maps M5's wrapped on $\mathbb{T}^3$ to M2's orthogonal to $\mathbb{T}^3$.

This system is reminiscent of doubled field theory:

- X^I is a position coordinate
- Y^μ is a winding coordinate
- under T-duality some Y^μ become position coordinates
- the $Y^\mu D_\mu = 0$ constraint is like a section condition

Although it should be noted that the fields are only functions of ordinary 6D coordinates x^μ (*i.e.* not winding coordinates).

Conclusions/Comments

So our representation of the $(2,0)$ superalgebra gives various field theories associated to M-branes.

- 5D SYM as the M5 on S^1
- Maximally supersymmetric M2 branes: [BL][G]
- Null M5-branes: QM on instanton moduli space
- Null M2-branes: QM on Hitchin moduli space

The later two are novel non-Lorentz invariant field theories whose on-shell dynamics reduces to one-dimensional motion on moduli space and breaks 1/2 the supersymmetry.

Consistent with T-duality in M-theory but one needs to generalise all this to more than two branes!