L_∞ algebras in double and exceptional field theory

Olaf Hohm

- O.H, Zwiebach, 1701.08824
- O.H., Samtleben, 1707.06693 & 1805.03220
- work in progress

Higher Structures in M-Theory, Durham, August 2018

Road Map

- Duality covariant formulation in 1) gauged supergravity (‘embedding tensor formalim’) and 2) double/exceptional field theory requires redundant or unphysical objects $\Rightarrow$ ‘higher equivalences’

- analogous features in algebraic topology and homotopy theory, where ‘∞-algebras’ allow one “to live with slightly false algebraic identities in a new world where they become effectively true.” [D. Sullivan]

- Features of physical theories usually taken for granted
  [e.g.: “continuous symmetries $\equiv$ Lie algebras”]
  hold only ‘up to homotopy’, which quite likely provides deep pointers for (so far) elusive underlying mathematical structure of DFT/ExFT

Overview

- Strongly Homotopy (sh) or ∞-Algebras

- Field Theories and L_∞ Algebras $\rightarrow$ weakly constrained DFT?

- Leibniz (or Loday) Algebras and their Chern-Simons Gauge Theory

- Embedding tensor formalism:

  Leibniz algebras as coadjoint action of Lie algebras

- General Remarks and Outlook

Strongly Homotopy Lie or L$_\infty$ Algebras

An L$_\infty$ algebras is a graded vector space [Zwiebach (1993), Lada & Stasheff (1993)]

$$X = \bigoplus_{n\in\mathbb{Z}} X_n \,,$$

equipped with *multilinear and graded antisymmetric* brackets or maps

$$x_1,\dots,x_n \;\mapsto\; \ell_n(x_1,\dots,x_n) \;\in\; X_{n-2+\sum_i |x_i|} \,,$$

satisfying, for each $n = 1,2,3,\dots$, the *generalized Jacobi identities*

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} (-1)^{\sigma} \epsilon(\sigma;x)\, \ell_j\big(\ell_i(x_{\sigma(1)},\dots,x_{\sigma(i)}), x_{\sigma(i+1)},\dots,x_{\sigma(n)}\big) = 0$$

with the sum over all permutations of n objects with partially ordered arguments (‘unshuffles’), $\sigma(1) \leqslant \cdots \leqslant \sigma(i)$, $\sigma(i+1) \leqslant \cdots \leqslant \sigma(n)$,

and *Koszul sign* $\epsilon(\sigma;x)$, determined for any graded algebra with $x_i x_j = (-1)^{x_i x_j}\, x_j x_i$ by $x_1 \cdots x_k = \epsilon(\sigma;x)\, x_{\sigma(1)} \cdots x_{\sigma(k)}$

Explicit L_∞-relations

For $n = 1$ we learn that $\ell_1 \equiv Q$ is nil-potent:

$$\ell_1(\ell_1(x)) \ = \ 0$$

For $n = 2$ we learn that ℓ_1 is a derivation of $\ell_2 \equiv [\cdot,\cdot]$:

$$\ell_1(\ell_2(x_1,x_2)) \ = \ \ell_2(\ell_1(x_1),x_2) + (-1)^{x_1}\ell_2(x_1,\ell_1(x_2))$$

For $n = 3$ we learn that $\ell_2 \equiv [\cdot,\cdot]$ satisfies Jacobi only 'up to homotopy'

$$\begin{aligned} 0 \ = \ & \ell_2(\ell_2(x_1,x_2),x_3) + 2 \text{ terms} \\ & + \ell_1(\ell_3(x_1,x_2,x_3)) \\ & + \ell_3(\ell_1(x_1),x_2,x_3) + 2 \text{ terms} \end{aligned}$$

For $n = 4$ we learn that $\ell_2\ell_3 + \ell_3\ell_2$ is zero 'up to homotopy', i.e., up to the the failure of ℓ_1 to act as a derivation on ℓ_4

plus infinitely more relations

Field Theories & Weakly Constrained DFT

Dictionary L_∞ algebra $\longleftrightarrow$ field theory:

$$\begin{array}{ccccccccc} \cdots \to & X_1 & \xrightarrow{\ell_1} & X_0 & \xrightarrow{\ell_1} & X_{-1} & \xrightarrow{\ell_1} & X_{-2} & \to \cdots \\ & \chi & & \xi & & \Psi & & \mathrm{EOM} & \end{array}$$

Gauge transformations and field equations:

$$\begin{aligned} \delta_\xi \Psi &= \ell_1(\xi) + \ell_2(\xi, \Psi) - \tfrac{1}{2}\,\ell_3(\xi, \Psi, \Psi) + \cdots \\ 0 &= \ell_1(\Psi) - \tfrac{1}{2}\,\ell_2(\Psi, \Psi) - \tfrac{1}{3!}\,\ell_3(\Psi, \Psi, \Psi) + \cdots \end{aligned}$$

gauge algebra closes 'up to homotopy': trivial parameters $\xi = \ell_1(\chi)$

Example: Courant algebroid/gauge structure of DFT, with $\ell_2 = [\cdot, \cdot]_c$,
defines L_∞ algebra with $\ell_4 = 0$ [Roytenberg & Weinstein (1998)]

$\to$ generalization to weakly constrained? Indeed, in general L_∞ non-trivial

$$\ell_2(\chi_1, \chi_2) \;=\; \langle \mathcal{D}\chi_1, \mathcal{D}\chi_2 \rangle \; (= \; \partial^M \chi_1 \, \partial_M \chi_2 \;=\; 0 \,)$$

$\to$ still *very non-trivial* (non-local projected product needed)

[A. Sen (2016)]

Leibniz Algebras and their Chern-Simons Theory

Leibniz (or Loday) algebra: vector space with product $\circ$, satisfying

$$x \circ (y \circ z) \;=\; (x \circ y) \circ z + y \circ (x \circ z)$$

If $\circ$ antisymmetric $\Rightarrow$ Lie algebra

Defines symmetry variations: $\delta_x y \;=\; \mathcal{L}_x y \;\equiv\; x \circ y$ that close:

$$\begin{aligned}[\mathcal{L}_x, \mathcal{L}_y] z \;&\equiv\; \mathcal{L}_x(\mathcal{L}_y z) - \mathcal{L}_y(\mathcal{L}_x z) \;=\; x \circ (y \circ z) - y \circ (x \circ z) \\ &=\; (x \circ y) \circ z \;=\; \mathcal{L}_{x \circ y} z\end{aligned}$$

(Anti-)symmetrizing in x, y:

$$[\mathcal{L}_x, \mathcal{L}_y] z \;=\; \mathcal{L}_{[x,y]} z \;, \qquad \mathcal{L}_{\{x,y\}} z \;=\; 0$$

Thus, $\{,\}$ defines 'trivial vector'. Jacobiator is trivial:

$$\sum_{\text{antisym}} 3[[x_1, x_2], x_3] - \{x_1 \circ x_2, x_3\} \;=\; 0$$

'Trivial space' forms ideal of bracket: $[\cdot, \{\,,\}] = \{\cdot, \cdot\}$. Thus:

<u>Theorem:</u> Any Leibniz algebra defines L_∞ algebra with $\ell_2 = [\cdot, \cdot]$

[O.H., Kupriyanov, Lüst, Traube, 1709.10004]

Leibniz-valued Gauge Fields and Chern-Simons Action

Leibniz-valued one-form with gauge transformations

$$\delta_\lambda A_\mu \;=\; D_\mu\lambda \;\equiv\; \partial_\mu\lambda - A_\mu \circ \lambda$$

This closes up to ‘higher gauge transformations’ (c.f. trivial parameters).

Generalized Chern-Simons action

$$S_{\mathrm{CS}} \;\equiv\; \int \mathrm{d}^3x\,\epsilon^{\mu\nu\rho}\big\langle A_\mu\,,\,\partial_\nu A_\rho - \tfrac{1}{3}A_\nu \circ A_\rho\big\rangle$$

is gauge invariant provided the inner product $\langle\,,\rangle$ is invariant and

$$\langle x, \{\cdot,\cdot\}\rangle \;=\; 0 \quad \forall x$$

$\Rightarrow$ situation in 3D gauged SUGRA in embedding tensor formalism

[de Wit, Nicolai & Samtleben (2001–2002)]

$\Rightarrow$ any Leibniz algebra with $\langle\,,\rangle$ as above defines Chern-Simons theory

$\Rightarrow$ general dimensions: tensor hierarchy (& corresponding L_∞ algebra)

Leibniz algebras via coadjoint action of Lie algebras I

Embedding tensor in 3D: Global Lie algebra $\mathfrak{g}$: $[t^M, t^N] = f^{MN}{}_K t^K$ defines structure constants of *gauge algebra*

$$X_{MN}{}^K \equiv \Theta_{ML} f^{LK}{}_N \qquad (A \circ B)^M \equiv X_{NK}{}^M A^N B^K \tag{1}$$

where Θ_{MN} is the embedding tensor, and $D_\mu \equiv \partial_\mu - A_\mu{}^M \Theta_{MN} t^N$, satisfying *quadratic constraint/Leibniz algebra*.

Invariantly: Consider 'covectors' $A \in \mathfrak{g}^*$ with pairing $A(v) \equiv A^M v_M$.
Coadjoint action of $\zeta \in \mathfrak{g}$ on $\mathfrak{g}^*$:

$$(\mathrm{ad}^*_\zeta A)(v) \equiv -A([\zeta, v]) \,, \qquad (\mathrm{ad}^*_\zeta A)^M = f^{MN}{}_K \zeta_N A^K$$

Embedding tensor map Θ : $\mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathbb{R}$, and (1) yields

$$(A \circ B)(v) = \Theta(A, \mathrm{ad}^*_v B) \,.$$

Leibniz algebras via coadjoint action of Lie algebras II

Alternative viewpoint: embedding tensor is map

$$\vartheta \; : \quad \mathfrak{g}^* \quad \rightarrow \quad \mathfrak{g}\,, \qquad \vartheta(\tilde{t}_M) \;=\; -\Theta_{MN}\, t^N$$

Invariantly: Θ is related to ϑ by $\Theta(A,B) = -A(\vartheta(B))$

One may then prove: Leibniz algebra given by

$$A \circ B \;\equiv\; \mathsf{ad}^*_{\vartheta(A)}\, B$$

More generally: any $\mathfrak{g}$ representation R becomes representation of Leibniz algebra via $\delta_\Lambda \equiv R_{\vartheta(\Lambda)}$

Invariance of Θ (quadratic constraint) $\;\Rightarrow\;$ Leibniz algebra

Embedding tensor of $\mathsf{E}_{8(8)}$ generalized diffeomorphisms

Starting point: global Lie algebra of decompactification limit $R \to \infty$, internal diffeomorphisms and Y-dependent $\mathsf{E}_{8(8)}$ rotations, $\zeta = (\lambda^M, \sigma_M)$,

$$\big[\zeta_1, \zeta_2\big] \;=\; \big(2\,\lambda_{[1}{}^N \partial_N \lambda_{2]}{}^M\,,\; 2\,\lambda_{[1}{}^N \partial_N \sigma_{2]M} + f^{KL}{}_M \sigma_{1K} \sigma_{2L}\big)\;.$$

Pairing between $v = (p^M, q_M) \in \mathfrak{g}$ and $\mathcal{A} \equiv (A^M, B_M) \in \mathfrak{g}^*$ given by

$$\mathcal{A}(v) \;\equiv\; \int \mathrm{d}^{248}Y \big(A^M q_M \;+\; B_M p^M\big)$$

Coadjoint action determined by invariance.

With embedding tensor

$$\Theta(\mathcal{A}_1, \mathcal{A}_2) \;\equiv\; -2 \int \mathrm{d}Y \big(A_{(1}{}^M B_{2)M} - \tfrac{1}{2} f^M{}_{NK} A_1{}^N \partial_M A_2{}^K\big)$$

one obtains Chern-Simons term of $\mathsf{E}_{8(8)}$ ExFT.

The map $\vartheta :\; \mathfrak{g}^* \to \mathfrak{g}$, satisfying $\Theta(\mathcal{A}_1, \mathcal{A}_2) = -\mathcal{A}_1(\vartheta(\mathcal{A}_2))$, yields generalized Lie derivative via $\mathcal{L}_\Upsilon \mathcal{A} = \mathrm{ad}^*_{\vartheta(\Upsilon)} \mathcal{A}$.

Outlook & Remarks

- algebraic structures beyond Lie arise naturally in string/M-theory
- tensor hierarchy of gauged SUGRA & ExFT suggests ∞-algebra, difficult/unnatural in terms of Lie algebra
- natural Chern-Simons theories beyond Lie algebras

  $\rightarrow$ complete topological sector of $E_{8(8)}$ ExFT including 3D gravity

  generalizing Achucarro & Townsend (1986) and Witten (1988)
- 3D superconformal field theories with infinite-dimensional gauge groups

  [work in progress]
- unifying algebraic structure of M-theory?

  affine $E_{9(9)}$ works analogously to $E_{8(8)}$ $\rightarrow$ Henning's talk

  $\Rightarrow$ Lie algebra theory may be the "slightly wrong" framework