Symmetries in field theories from an NQ-manifold perspective

Andreas Deser[1], Christian Sämann[2], Jim Stasheff[3]
Ann. Henri Poincaré **19** (2018) and Commun.Math.Phys. **339** (2015)

[1]Istituto Nazionale di Fisica Nucleare, Sezione di Torino
[2] Department of Mathematics, Heriot-Watt University, Edinburgh
[3] Department of Mathematics, University of Pennsylvania

16 August 2018
London Mathematical Society – EPSRC Durham Symposium
Higher structures in M-theory

Contents

Motivation from symplectic/Poisson geometry

One-parameter groups of symplectomorphisms of a symplectic manifold (M, ω) are determined by *hamiltonian vector fields*:

- For $f \in \mathcal{C}^\infty(M)$, the hamiltonian vector field is X_f with

$$\iota_{X_f}\omega = df \ . \tag{1}$$

- On functions, an inf. symplectomorphism acts via Poisson brackets:

$$X_f(g) = \{f, g\} = \Big[[-\pi, f], g\Big] \ . \tag{2}$$

  "*The Poisson bracket is a derived bracket by the Poisson tensor* π"

- More modern: Take the graded Poisson manifold $T^*[1]M$ with homological function $\mathcal{Q}$:

$$X_f(g) = \Big\{\{\mathcal{Q}, f\}, g\Big\} \ . \tag{3}$$

So the action of an inf. symplectomorphism is given by a derived bracket.

More examples

More examples of actions of infinitesimal (spacetime) symmetries:

- M differentiable, inf. diffeomorphisms determinded by vector fields $X \in \Gamma(TM)$:

$$X(f) = L_X(f) , \quad X(Y) = L_X(Y) = [X, Y] , \tag{4}$$

  more general for **Lie algebroids** $(E, [\cdot, \cdot]_E, \rho)$, a section in $\Gamma(E)$ acts with the anchor on functions on M:

$$s(f) = \rho(s)(f) , \quad s(t) = [s, t]_E . \quad s, t \in \Gamma(E) . \tag{5}$$

- For **generalized geometry** (Hitchin/Gualtieri), inf. elements of $\mathrm{Diff}(M) \ltimes \Omega^2_{\mathrm{cl}}(M)$ (locally) are sections

$$\mathbb{X} = X + \omega, \quad \mathbb{Y} = Y + \beta \quad \in \Gamma(TM \oplus T^*M)$$

  and act via the Dorfman derivative:

$$\mathbb{X}(f) = X(f) , \quad \mathbb{X}(\mathbb{Y}) = [\![\mathbb{X}, \mathbb{Y}]\!] = [X, Y] + L_X\beta - \iota_Y d\omega . \tag{6}$$

More examples

- In **extended field theories**(Hull, Zwiebach, Hohm, Berman, Vaisman) like double or exceptional field theory we have actions of the form

$$S = \int_{M\times\tilde{M}} d\mu\, e^{-\phi}\, \mathfrak{R}(\mathcal{H})\ , \quad \mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}\ . \quad (7)$$

  They have infinitesimal spacetime transformations parameterized by fundamentals of $O(d,d)$, $\mathcal{X}^M = (X^\mu(x,\tilde{x}), X_\mu(x,\tilde{x}))$ and acting on functions and vectors as:

$$\begin{aligned} \mathcal{X}(f) &= \mathcal{X}^M \partial_M(f)\ , \\ \mathcal{X}(\mathcal{Y})^M &= \mathcal{X}^N\partial_N\mathcal{Y}^M - \mathcal{Y}^N\partial_N\mathcal{X}^M + \mathcal{Y}^N\partial^M\mathcal{X}_N\ . \end{aligned} \quad (8)$$

We will show that *all of them are of derived bracket form*. Especially in the last item, this gives insight into the algebraic structure of symmetries of DFT/EFT.

Definition.

A **symplectic pre-NQ-manifold of $\mathbb{N}$-degree** n *is an $\mathbb{N}$-graded manifold $\mathcal{M}$, together with symplectic form ω of degree n and a vector field Q of degree* 1*, satisfying $L_Q\omega = 0$.*

Examples

An important class where in addition $Q^2 = 0$, are the **Vinogradov Lie** n**-algebroids**:

$$\mathcal{V}_n(M) := T^*[n]T[1]M\ .$$

They have the following properties:

- Local coordinates $(x^\mu, \xi^\mu, \zeta_\mu, p_\mu)$ of degrees $0, 1, n-1, n$.
- Symplectic form $\omega = dx^\mu \wedge dp_\mu + d\xi^\mu \wedge d\zeta_\mu$
- Nilpotent vector field Q with Hamiltonian $\mathcal{Q} = \xi^\mu p_\mu$, i.e. $\{\mathcal{Q}, \mathcal{Q}\} = 0$.

Degree 2 and Lie 2-algebra

Getzler, Fiorenza, Manetti, Voronov

Let $\mathcal{M}$ be a symplectic pre-NQ-manifold of degree 2. On $\mathcal{C}_1^\infty(\mathcal{M}) \oplus \mathcal{C}_0^\infty(\mathcal{M})$ consider the maps:

$$\begin{aligned}
\mu_1 : &\quad \mathcal{C}_0^\infty(\mathcal{M}) \to \mathcal{C}_1^\infty(\mathcal{M}) \\
\mu_2 : &\quad \mathcal{C}_i^\infty(\mathcal{M}) \times \mathcal{C}_j^\infty(\mathcal{M}) \to \mathcal{C}_{i+j-1}^\infty(\mathcal{M}) \quad i+j>0 \\
\mu_3 : &\quad \mathcal{C}_i^\infty(\mathcal{M}) \times \mathcal{C}_1^\infty(\mathcal{M}) \times \mathcal{C}_1^\infty(\mathcal{M}) \to \mathcal{C}_0^\infty(\mathcal{M}) \quad i=0,1
\end{aligned} \tag{9}$$

With $\delta f := \{\mathcal{Q}, f\}$ for $f \in \mathcal{C}_1^\infty(\mathcal{M})$ and zero otherwise, they are

$$\begin{aligned}
\mu_1(f) =&\ \{\mathcal{Q}, f\} \\
\mu_2(f,g) =&\ \tfrac{1}{2}\Big(\{\delta f, g\} \pm \{\delta g, f\}\Big) \\
\mu_3(f,g,h) =&\ -\tfrac{1}{12}\Big(\Big\{\{\delta f, g\}, h\Big\} \pm \dots\Big) .
\end{aligned} \tag{10}$$

Degree 2 and Lie 2-algebra

Baez, Crans, Getzler, Fiorenza, Manetti, Voronov

If $\{\mathcal{Q},\mathcal{Q}\}=0$, i.e. in the NQ-case, one can prove the relations:

$$f, f_1, f_2, f_3 \in \mathcal{C}_1^\infty(\mathcal{M})\ , \quad g, g_1, g_2 \in \mathcal{C}_0^\infty(\mathcal{M})$$

$$\begin{aligned} \mu_1(\mu_2(f,g)) &= \mu_2(f,\mu_1(g)) \\ \mu_2(\mu_1(g_1),g_2) &= \mu_2(g_1,\mu_1(g_2)) \\ \mu_1(\mu_3(f_1,f_2,f_3)) &= \mu_2(f_1,\mu_2(f_2,f_3))+\mu_2(f_2,\mu_2(f_3,f_1)) \\ &\quad +\mu_2(f_3,\mu_2(f_1,f_2))\ , \end{aligned} \tag{11}$$

i.e. $\mathcal{C}_1^\infty(\mathcal{M})\oplus\mathcal{C}_0^\infty(\mathcal{M})$ with the μ_i is a **2-term L_∞ algebra** which is equivalent to a **Lie 2-algebra**.

Conditions for L_∞-structure

A.D., Sämann

If $\{\mathcal{Q},\mathcal{Q}\} \neq 0$, one has to choose an appropriate subset of functions to ensure to still have the Lie 2-agebra relations:

Theorem.
Consider a subset of $\mathcal{C}^\infty_1(\mathcal{M}) \oplus \mathcal{C}^\infty_0(\mathcal{M})$, such that the Poisson brackets and the maps μ_i close on this subset. Then the latter is a 2-term L_∞-algebra if and only if

$$\begin{aligned} \{Q^2 f, g\} + \{Q^2 g, f\} &= 0\,,\\ \{Q^2 X, f\} + \{Q^2 f, X\} &= 0\,,\\ \{\{Q^2 X, Y\}, Z\}_{[X,Y,Z]} &= 0\,, \end{aligned} \tag{12}$$

for all functions $f, g \in \mathcal{C}^\infty_0(\mathcal{M})$ and $X, Y, Z \in \mathcal{C}^\infty_1(\mathcal{M})$ and the subscript $[X, Y, Z]$ means the alternating sum over X, Y, Z.

The Lie/Dorfman bracket as a derived bracket

Roytenberg, Weinstein

- M manifold, take $\mathcal{V}_1(M)$. Then

$$\begin{aligned} \mu_1(f) &= df \ , \\ \mu_2(X,Y) &= [X,Y] = \{\{\mathcal{Q},X\},Y\} \ , \end{aligned} \tag{13}$$

  I.e. the Lie bracket is a derived bracket.

- Take $\mathcal{V}_2(M)$. Then objects in $\mathcal{C}_1^\infty(\mathcal{M})$ correspond to generalized vectors and

$$\begin{aligned} \mu_1(f) &= df \ , \\ \mu_2(X+\alpha, Y+\beta) &= [X,Y] + L_X\beta - L_Y\alpha - \tfrac{1}{2}d(\iota_X\beta - \iota_Y\alpha) \ , \\ \mu_3(X,Y,Z) &= \langle [X,Y],Z\rangle + \mathrm{cycl.} \ . \end{aligned} \tag{14}$$

  I.e. the Dorfman bracket is a derived bracket.

The C-bracket as derived bracket

A.D., Sämann, Stasheff

We take the same setting as before, but instead of M as base, we take T^*M, i.e. we take $\mathcal{V}_2(T^*M)$. Locally we have $(x^M, \xi^M, \zeta_M, p_M)$ of degree $(0, 1, 1, 2)$, i.e. $2d + 2d$ vectors. To get $2d$ we project:

$$\theta^M = \frac{1}{\sqrt{2}}\left(\xi^M + \eta^{MN}\zeta_N\right) , \quad \beta^M = \frac{1}{\sqrt{2}}\left(\xi^M - \eta^{MN}\zeta_N\right) . \qquad (15)$$

Results in Poisson structure and hamilton function:

$$\omega = dx^M \wedge dp_M + \tfrac{1}{2}\eta_{MN}\, d\theta^M \wedge d\theta^N , \quad \mathcal{Q} = \theta^M p_M . \qquad (16)$$

We still have $L_{X_{\mathcal{Q}}}\omega = 0$ but $\{\mathcal{Q}, \mathcal{Q}\} \neq 0$, so we get a **pre-NQ manifold**.

The C-bracket as derived bracket

A.D., Sämann, Stasheff

With local coordinates (x^M, θ^M, p_M) of degree $(0, 1, 2)$, and most general degree one objects $\mathcal{X} = \mathcal{X}_M \theta^M$ (using η_{MN} to raise and lower indices) we get:

$$\begin{aligned} \mu_1(f) &= \partial_M f\, \theta^M\;, \\ \mu_2(\mathcal{X}, \mathcal{Y}) &= \tfrac{1}{2}(\mathcal{X}(\mathcal{Y}) - \mathcal{Y}(\mathcal{X}))_M\, \theta^M\;, \\ \mu_3(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) &= \mathcal{X}^M \mathcal{Z}^N \partial_M \mathcal{Y}_N - \mathcal{Y}^M \mathcal{Z}^N \partial_M \mathcal{X}_N + \text{cycl.} \end{aligned} \tag{17}$$

I.e. the C-bracket is a derived bracket.

Recall that $\mathcal{X}(\mathcal{Y})^M = \mathcal{X}^N \partial_N \mathcal{Y}^M - \mathcal{Y}^N \partial_N \mathcal{X}^M + \mathcal{Y}^N \partial^M \mathcal{X}_N$.

The C-bracket as derived bracket

A.D., Sämann, Stasheff

Evaluating the theorem on the pre-NQ-manifold locally in a patch $U \subset T^*M$ with coordinates (x^M, θ^M, p_M) with $f \in \mathcal{C}^\infty(U)$ and $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathcal{C}_1^\infty(T^*[2]T[1]U)$, we get

- $\{Q^2 f, g\} + \{Q^2 g, f\} = 2\partial_M f\, \eta^{MN} \partial_N g = 0$
  This is the strong constraint.
- $\{Q^2 \mathcal{X}, f\} + \{Q^2 f, \mathcal{X}\} = 2(\partial_M \mathcal{X}_K \theta^K)\eta^{MN}\partial_N f = 0$
  This is the strong constraint for vectors and functions.
- $\{\{Q^2 \mathcal{X}, \mathcal{Y}\}, \mathcal{Z}\}_{[\mathcal{X},\mathcal{Y},\mathcal{Z}]} = 2\theta^K((\partial^M \mathcal{X}_K)(\partial_M \mathcal{Y}^N)\mathcal{Z}_N)_{[\mathcal{X},\mathcal{Y},\mathcal{Z}]} = 0$
  Additional constraint for vectors? ...Shows up in properties of the Riemann tensor of double field theory

Infinitesimal symmetries as derived brackets

A.D., Sämann

We have seen that for geometry, generalized geometry and local double field theory infinitesimal spacetime or configuration space transformations are of derived bracket form. Same can be shown for **heterotic generalized geometry** and **heterotic double field theory**. This suggests to use it as a **guiding principle**:

> Infinitesimal spacetime/configuration space transformations are of derived bracket form.

To be further investigated, many other examples:

- Exceptional generalized geometries, exceptional field theories
- Noncommutative field and gravity theories
- Counter examples?

A look on the roadmap

- The same constructions done for $\mathcal{V}_n(M)$ give the Vinogradov brackets on

$$X = X^\mu \zeta_\mu + X_{\mu_1 \dots \mu_{n-1}} \xi^1 \cdots \xi^{n-1} \in \Gamma(TM \oplus \wedge^{n-1} T^* M) \,.$$

  In particular $SL(5)$ exceptional generalized geometry in case of $n = 3$.
- In case of E_5, E_6 exceptional generalized geometry the NQ manifolds are more complicated (Arvanitakis). No NQ manifolds for E_7 or higher so far.
- AKSZ constructions with such NQ-manfiolds and deformation quantization (Chatzistavrakidis, Jonke, Khoo, Kokenyesi, Mylonas, Schupp, Sinkovics, Szabo).
- Exceptional field theories (Berman, Cederwall, Godazgar, Kleinschmidt, Palmkvist, Perry, Samtleben, Thompson), I guess their symmetries are of derived form.
- Everything local in the talk so far. What about global issues, examples?

A look on the roadmap

- The same constructions done for $\mathcal{V}_n(M)$ give the Vinogradov brackets on

$$X = X^\mu \zeta_\mu + X_{\mu_1 \dots \mu_{n-1}} \xi^1 \cdots \xi^{n-1} \in \Gamma(TM \oplus \wedge^{n-1} T^* M) \, .$$

  In particular $SL(5)$ exceptional generalized geometry in case of $n = 3$.
- In case of E_5, E_6 exceptional generalized geometry the NQ manifolds are more complicated (Arvanitakis). No NQ manifolds for E_7 or higher so far.
- AKSZ constructions with such NQ-manfiolds and deformation quantization (Chatzistavrakidis, Jonke, Khoo, Kokenyesi, Mylonas, Schupp, Sinkovics, Szabo).
- Exceptional field theories (Berman, Cederwall, Godazgar, Kleinschmidt, Palmkvist, Perry, Samtleben, Thompson), I guess their symmetries are of derived form.
- Everything local in the talk so far. What about global issues, examples?

Heisenberg nilmanifold: Peridodicity

$$\begin{tikzcd} S^1 \arrow[r, hook] & N_{j,k} \arrow[d, two heads] \\ & T^2 \end{tikzcd}$$

$$(x^1, x^2, x^3) \sim (x^1, x^2 + 1, x^3) \sim (x^1, x^2, x^3 + 1)$$
$$\sim (x^1 + 1, x^2, x^3 - jx^2)$$

- Invariant one-forms: $dx^1 \ , \quad dx^2 \ , \quad dx^3 + jx^1\, dx^2 \ .$
- Metric: $g = (dx^1)^2 + (dx^2)^2 + (dx^3 + jx^1\, dx^2)^2 \ .$
- First chern class:

$$c_1(N_{j,k}) = j\, dx^1 \wedge dx^2 \ , \quad \int_{T^2} c_1(N_j) = \ j \ . \qquad (18)$$

- H-flux representing Dixmier Douady class of $U(1)$-gerbe:

$$H = \ k\, dx^1 \wedge dx^2 \wedge dx^3, \quad \int_{N_{j,k}} H = k \ . \qquad (19)$$

U(1)-gerbe and periodicity

Generalized geometry on $N_{j,k}$:

$$0 \longrightarrow T^* N_{j,k} \longrightarrow \mathscr{C}_{j,k} \longrightarrow TN_{j,k} \longrightarrow 0$$

$$X_{(\beta)} + \gamma_{(\beta)} = X_{(\alpha)} + \gamma_{(\alpha)} + \iota_{X_{(\alpha)}}(B_{(\beta)} - B_{(\alpha)}) \,, \quad \text{on } U_\beta \cap U_\alpha \subset N_{j,k} \,.$$

In terms of periodicity on $N_{j,k}$, e.g.:

$$X + \gamma \,|_{(x^1+1, x^2, x^3 - jx^2)} = X + \gamma \,|_{(x^1, x^2, x^3)} + i_X(k\, dx^2 \wedge dx^3) \,, \qquad (20)$$

for B-field $B = kx^1\, dx^2 \wedge dx^3$. Invariant sections are $e^{-B}(X + \gamma)$, with inverse B-transform e^{-B}.

In terms of NQ-manifolds: $T^*[2]T[1]N_{j,k}$

- Degree-1 coordinates corresponding to the invariant forms:

$$\bar{\xi}^1 \sim dx^1 \ , \quad \bar{\xi}^2 \sim dx^2 \ , \quad \bar{\xi}^3 \sim dx^3 + jx^1\, dx^2 \ , \tag{21}$$

and the corresponding vector fields $(\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3)$.

- The corresponding coordinates invariant under patchings of $\mathscr{C}_{j,k}$ are

$$\begin{aligned} &\xi^i = \bar{\xi}^i \quad i = 1, 2, 3 \ , \\ &\zeta_1 = \bar{\zeta}_1 \ , \quad \zeta_2 = \bar{\zeta}_2 - kx^1\, \bar{\xi}^3 \ , \quad \zeta_3 = \bar{\zeta}_3 + kx^1\, \bar{\xi}^2 \ . \end{aligned} \tag{22}$$

- Degree-2 momenta: Periodicity due to Roytenberg

$$(x^1, x^2, x^3, p_1, p_2, p_3) \sim (x^1 + 1, x^2, x^3 - jx^2, p_1, p_2 + jp_3, p_3) \ . \tag{23}$$

- Periodicity-invariant symplectic form, homological function :

$$\omega = dx^i \wedge dp_i + d\xi^1 \wedge d\zeta_i \ , \quad \mathcal{Q} = \xi^i a_i^j p_j \ , \tag{24}$$

with anchor a_i^j relating ζ_i to the standard basis on $TN_{j,k}$.

T-duality between subalgebras

$$\begin{tikzcd} & N_{j,k} \times_{T^2} N_{k,j} \arrow[dl, "\mathrm{p}"'] \arrow[dr, "\tilde{\mathrm{p}}"] & \\ N_{j,k} \arrow[dr, "\pi"'] & & N_{k,j} \arrow[dl, "\tilde{\pi}"] \\ & T^2 & \end{tikzcd}$$

- $\tilde{\mathrm{p}}^* \tilde{H} - \mathrm{p}^* H = d(\mathrm{p}^* A \wedge \tilde{\mathrm{p}}^* \tilde{A})$ (Bouwknegt, Evslin, Mathai, Hannabuss, Rosenberg)
- Interchanging j and k (Buscher)
  $g = g_0 + (dx^3 + jx^1 dx^2)^2$, $B = kx^1 dx^{23}$
  $\tilde{g} = g_0 + (dx^4 + kx^1 dx^2)^2$, $\tilde{B} = jx^1 dx^{24}$

- Construct pre-NQ manifold $\mathscr{E}_{j,k} = \mathscr{E}_{k,j}$ using periodicity invariant (x, ξ, ζ, p) by reducing a Courant algebroid on $N_{j,k} \times_{T^2} N_{k,j}$.
- The conditions of theorem give embeddings:

$$\begin{tikzcd} & \mathcal{C}^\infty(\mathscr{E}_{j,k}) & \\ \mathcal{C}^\infty(T^*[2]T[1]N_{j,k}) \arrow[ur, hook, "\iota"] & & \mathcal{C}^\infty(T^*[2]T[1]N_{k,j}) \arrow[ul, hook', "\tilde{\iota}"'] \end{tikzcd}$$

T-duality between subalgebras

A.D., Sämann, work in progress

Studying T-dual subalgebras means solving the following problem:

Find embeddings ι, $\tilde{\iota}$ such that $Im(\iota), Im(\tilde{\iota}) \subset \mathcal{C}^\infty(\mathscr{E}_{j,k})$ satisfy the conditions:

$$\begin{aligned} \{Q^2 f, g\} + \{Q^2 g, f\} &= 0\,, \\ \{Q^2 X, f\} + \{Q^2 f, X\} &= 0\,, \\ \{\{Q^2 X, Y\}, Z\}_{[X,Y,Z]} &= 0\,, \end{aligned} \tag{25}$$

for f of degree-0 and X, Y, Z of degree-1 in the image of $\iota, \tilde{\iota}$.

Interesting: Q^2 now differs from the flat case:

$$Q^2 = (p_3 - k\xi^1\xi^2)(p_4 - j\xi^1\xi^2)\,. \tag{26}$$

Compare to "solving the section condition" in DFT?

T-duality between subalgebras

A.D., Sämann, work in progress

Is there a more geometric interpretation of Q^2?

- Recall the local version $p_3 p_4$ and $\{p_i, f\} = \partial_i f$.
- Here e.g. $\{D, \cdot\} = \{p_3 - k\xi^1\xi^2, \cdot\}$
- Meaning e.g. on degree-1:

$$\{D, X\} = \partial_3 X + k(X^1\xi^2 - X^2\xi^1)$$

- NOT the connective structure (would be $\xi^2\xi^3$)
- Courant algebroid connection? (Alekseev, Gualtieri, Xu), see Jan's talk.

Next steps and outlook

- Find and classify embeddings ι, study properties (Lagrangian etc.).
- Can we compare to DFT? Solving the section condition $\leftrightarrow$ finding dual subalgebras. If so, interpret the additional terms containing the Dixmier Douady classes $k\xi^1\xi^2$ and $j\xi^1\xi^2$.
- For NQ-manifolds, we studied aspects of Riemannian geometry. Apply here.
- What about the exotic cases where there are Q- and R-fluxes?
- The formalism extends readily to (so far some) exceptional generalized geometries.
- Important operations are given in terms of Poisson brackets. What happens if we replace $\{\cdot,\cdot\}$ by $[\cdot,\cdot]_\star$?

Appendix: Construction of $\mathscr{E}_{j,k}$

As we have coordinates with periodicity, we imitate the local case:

- First construct a Courant algebroid on $N_{j,k} \times_{T^2} N_{k,j}$, containing $\mathrm{p}^* H \pm \tilde{\mathrm{p}}^* \tilde{H}$ as twist. In particular, the degree-2 momenta have periodicity:

  $$\begin{aligned}&(x^1, x^2, x^3, x^4, \bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4)\\ &\sim (x^1 + 1, x^2, x^3 - jx^2, x^4 - kx^2, \bar{p}_1, \bar{p}_2 + j\bar{p}_3 + k\bar{p}_4, \bar{p}_3, \bar{p}_4)\end{aligned}$$

  Possible to choose periodicity invariant coordinates $(x^i, \xi^i, \zeta_i, p_i)$.

- Reduce to the pre-NQ-case by setting

  $$\theta_{\pm} := \frac{1}{2}\left(\xi^3 \pm \zeta_4\right) , \quad \theta^2_{\pm} := \frac{1}{2}\left(\xi^4 \pm \zeta_3\right) .$$

  and only keeping θ^1_+ and θ^2_+.

- Reduced invariant symplectic form and hamiltonian:

  $$\begin{aligned}\omega &= dx^i \wedge dp_i + d\xi^1 \wedge d\zeta_1 + d\xi^2 \wedge d\zeta_2 + d\theta^1_+ \wedge d\theta^2_+ ,\\ \mathcal{Q} &= \xi^1 p_1 + \xi^2 p_2 + \theta^1_+ p_3 + \theta^2_+ p_4 - jx^1\xi^2 p_3 - kx^1\xi^2 p_4 .\end{aligned}$$