Thick morphisms and homotopy bracket structures

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"Microformal geometry" in brief

Key points: thick morphisms and nonlinear pullbacks

- There is a notion of *thick* (or *microformal*) *morphisms* of (super)manifolds generalizing ordinary smooth maps;
- Key difference: the pullback Φ^* by a thick morphism $\Phi\colon\ M_1 \Rrightarrow M_2$,

$$\Phi^*\colon\ \mathbf{C}^{\infty}(M_2) \to \mathbf{C}^{\infty}(M_1)\,,$$

  is, in general, **nonlinear** (actually, formal) map of infinite-dimensional manifolds of even functions.

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"Why we care"; in particular:

- Motivation: L_{∞}-morphisms of homotopy Poisson brackets;
- Applications and development: homotopy structures; duality of vector spaces and bundles; fermionic and quantum versions;
- Hints at a "nonlinear extension" of algebra-geometry duality.

Motivation

L_∞-algebras (or SHLAs) as Q-manifolds

Recall that the following structures are equivalent:

- L_∞-algebra (in antisymmetric version) L
- L_∞-algebra (in symmetric version) $\Pi L = V$
- (Formal) Q-structure on V, i.e., $Q \in \mathrm{Vect}(V)$, $\tilde{Q} = 1$, $Q^2 = 0$.

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L_∞-morphisms of L_∞-algebras as Q-maps

L_∞-morphism $L \rightsquigarrow K$ $\quad\Leftrightarrow\quad$ (nonlinear) Q-morphism $\Pi L \to \Pi K$

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Problem: what for functions?!

If we have L_∞-brackets on $C^\infty(M_1)$ and $C^\infty(M_2)$ (e.g. homotopy Poisson or homotopy Schouten), what is a "natural" construction for L_∞-morphisms? (Should be NONLINEAR maps! Pullbacks will not work!)

Example: higher Koszul brackets

Classical fact: for a Poisson M, there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{A}^k(M) & \xrightarrow{d_P} & \mathfrak{A}^{k+1}(M) \\ \uparrow & & \uparrow \\ \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M)\,, \end{array}$$

and vertical arrows map Koszul bracket to the Schouten bracket.

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Homotopy case: for a homotopy Poisson M, one can still construct

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but vertical arrows cannot map many 'higher Koszul brackets' into one Schouten bracket. An L_∞-morphism?

SOLUTION: pullback by a **thick morphism**!

Definition of a microformal (thick) morphism

Let M_1, M_2 be supermanifolds with local coordinates x^a, y^i.
Let p_a and q_i be conjugate momenta (fiber coordinates in T^*M_1, T^*M_2) and $\omega_1 = dp_a dx^a$, $\omega_2 = dq_i dy^i$ be the symplectic forms.

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Definition

A *microformal* (aka *thick*) *morphism* $\Phi\colon\; M_1 \Rrightarrow M_2$ is a formal Lagrangian submanifold $\Phi \subset T^*M_2 \times T^*M_1$ w.r.t. $\omega_2 - \omega_1$ specified locally by a *generating function* of the form $S(x,q)$:

$$q_i dy^i - p_a dx^a = d(y^i q_i - S) \quad \text{on } \Phi\,,$$

where $S(x,q)$, regarded as a part of the structure, is a formal power series in momenta

$$S(x,q) = S_0(x) + S^i(x)q_i + \frac{1}{2}\,S^{ij}(x)q_jq_i + \frac{1}{3!}\,S^{ijk}(x)q_kq_jq_i + \ldots$$

Pullback by a microformal morphism

Construction of pullback

Let $\Phi\colon\ M_1 \Rrightarrow M_2$ be a thick morphism with the generating function S. The *pullback* Φ^* is a formal mapping $\Phi^*\colon\ \mathbf{C}^{\infty}(M_2) \to \mathbf{C}^{\infty}(M_1)$ of functional supermanifolds (of 'bosonic' functions) defined by

$$\boxed{\Phi^*[g](x) = g(y) + S(x,q) - y^i q_i\,,}$$

for $g \in \mathbf{C}^{\infty}(M_2)$, where q_i and y^i are determined from the equations

$$q_i = \frac{\partial g}{\partial y^i}\,(y)\,, \quad y^i = (-1)^{\tilde{\imath}}\,\frac{\partial S}{\partial q_i}(x,q)$$

(giving $y^i = (-1)^{\tilde{\imath}}\frac{\partial S}{\partial q_i}(x, \frac{\partial g}{\partial y}(y))$ solvable by iterations).

Heuristically, if $f = \Phi^*[g]$, then $\Lambda_f = \Lambda_g \circ \Phi$, where $\Lambda_f = \mathrm{gr}(df)\,.$

General form of pullback

Example

Let $S(x,q) = S^0(x) + \varphi^i(x)q_i$. Then: $\Phi^*[g] = S^0 + \varphi^* g$.
(NB: ordinary maps have generating functions $S = \varphi^i(x)q_i$.)

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For a general $S(x,q) = S^0(x) + \varphi^i(x)q_i + \dots$, the equation $y^i = (-1)^{\tilde{\imath}}\frac{\partial S}{\partial q_i}(x, \frac{\partial g}{\partial y}(y))$ defines a map $\varphi_g \colon M_1 \to M_2$ as a formal perturbation of $\varphi = \varphi_0 \colon M_1 \to M_2$:

$$\varphi^i_g(x) = \varphi^i(x) + S^{ij}(x)\partial_j g(\varphi(x)) + \dots ,$$

and $\Phi^*[g](x) = \bigl(g(y) + S(x,q) - y^i q_i\bigr)\,|_{y=\varphi_g(x), q=\partial g/\partial y(\varphi_g(x))}$, which gives Φ^* as a **formal nonlinear differential operator** :

General form of $\Phi^* \colon \mathbf{C}^\infty(M_2) \to \mathbf{C}^\infty(M_1)$

$$\Phi^*[g](x) = S^0(x) + g(\varphi(x)) + \frac{1}{2}\, S^{ij}(x)\partial_i g(\varphi(x))\partial_j g(\varphi(x)) + \dots$$

Coordinate invariance

Transformation law for generating functions of thick morphisms

A generating function $S(x,q)$ as a geometric object on $M_1 \times M_2$ transforms by

$$S'(x',q') = S(x,q) - y^i q_i + y^{i'} q_{i'}\,.$$

Here $S(x,q)$ is the expression for S in 'old' coordinates and $S'(x',q')$ is the expression for S in 'new' coordinates. At the r.h.s., the variables x^a and $y^{i'}$ are given by substitutions: $x^a = x^a(x')$ and $y^{i'} = y^{i'}(y)$, while q_i and y^i are determined from

$$q_i = \frac{\partial y^{i'}}{\partial y^i}(y)\, q_{i'}\,, \quad y^i = (-1)^{\tilde{\imath}} \frac{\partial S}{\partial q_i}(x,q)\,.$$

The transformation law satisfies the cocycle condition. The canonical relation $\Phi \subset T^*M_2 \times (-T^*M_1)$ specified by S is well-defined. Pullbacks do not depend on a choice of coordinates.

Key fact: derivative of pullback

Theorem

Let $\Phi\colon M_1 \Longrightarrow M_2$ *be a thick morphism. Consider the pullback*

$$\Phi^*\colon \mathbf{C}^{\infty}(M_2) \to \mathbf{C}^{\infty}(M_1)\,.$$

Then for every $g \in \mathbf{C}^{\infty}(M_2)$*, the derivative* $T\Phi^*[g]$ *is given by*

$$T\Phi^*[g] = \varphi_g^*\,,$$

where $\varphi_g^*\colon C^{\infty}(M_2) \to C^{\infty}(M_1)$ *is the usual pullback with respect to the map* $\varphi_g\colon M_1 \to M_2$ *defined by* $y^i = (-1)^{\tilde{\imath}}\frac{\partial S}{\partial q_i}(x, \frac{\partial g}{\partial y}(y))$ *(depending perturbatively on* g*,* $\varphi_g = \varphi_0 + \varphi_1 + \varphi_2 + \ldots$*).*

Corollary

For every g*, the derivative* $T\Phi^*[g]$ *of* Φ^* *is an algebra homomorphism* $C^{\infty}(M_2) \to C^{\infty}(M_1)$*.*

Composition law

Consider thick morphisms $\Phi_{21}\colon M_1 \Rrightarrow M_2$ and $\Phi_{31}\colon M_2 \Rrightarrow M_3$ with generating functions $S_{21} = S_{21}(x, q)$ and $S_{32} = S_{32}(y, r)$.

Theorem

The composition $\Phi_{32} \circ \Phi_{21}$ is well-defined as a thick morphism $\Phi_{31}\colon M_1 \Rrightarrow M_3$ with the generating function $S_{31} = S_{31}(x, r)$, where

$$S_{31}(x, r) = S_{32}(y, r) + S_{21}(x, q) - y^i q_i$$

and y^i and q_i are expressed through (x^a, r_μ) from the system

$$q_i = \frac{\partial S_{32}}{\partial y^i}(y, r)\,, \quad y^i = (-1)^{\tilde{\imath}}\, \frac{\partial S_{21}}{\partial q_i}\,(x, q)\,,$$

which has a unique solution as a power series in r_μ and a functional power series in S_{32}.

Further facts

Formal category

Composition of thick morphisms is associative and and $(\Phi_{32} \circ \Phi_{21})^* = \Phi_{21}^* \circ \Phi_{32}^*$. In the lowest order, the composition is as in $\mathcal{SM}\mathrm{an} \rtimes \mathbf{C}^{\infty}$, whose arrows are pairs (φ_{21}, f_{21}) with the composition $(\varphi_{32}, f_{32}) \circ (\varphi_{21}, f_{21}) = (\varphi_{32} \circ \varphi_{21}, \varphi_{21}^* f_{32} + f_{21})$. Thick morphisms form a *formal category* ("formal thickening" of $\mathcal{SM}\mathrm{an} \rtimes \mathbf{C}^{\infty}$). Denote it $\mathcal{ET}\mathrm{hick}$.

"Fermionic version"

There is a fermionic version based on anticotangent bundles ΠT^*M and odd generating functions $S(x, y^*)$: "odd thick morphisms" $\Psi\colon\; M_1 \Rrightarrow M_2$ induce nonlinear pullbacks $\Psi^*\colon\; \mathbf{\Pi C}^{\infty}(M_2) \to \mathbf{\Pi C}^{\infty}(M_1)$ of **odd** functions ("fermionic fields") and their composition gives another formal category, $\mathcal{OT}\mathrm{hick}$, which contains $\mathcal{SM}\mathrm{an} \rtimes \mathbf{\Pi C}^{\infty}$.

Recollection: L_∞-algebras and L_∞-morphisms – 1

We consider $\mathbb{Z}_2$-graded version. (One can include a $\mathbb{Z}$-grading.)

There are two parallel notions: “symmetric” and “antisymmetric”.

Definition (*L_∞-algebra: antisymmetric version*)

A vector space $L = L_0 \oplus L_1$ with a collection of multilinear operations

$$[-,\dots,-]\colon \underbrace{L\times\dots\times L}_{k\text{ times}} \to L \quad (\text{for } k=0,1,2,\dots)$$

such that

- the parity of the kth bracket is $k \mod 2$;
- all brackets are antisymmetric (in $\mathbb{Z}_2$-graded sense);
- $\sum_{r+s=n}\sum_{\text{shuffles}}(-1)^{\alpha+\sigma}[[x_{\sigma(1)},\dots,x_{\sigma(r)}],\dots,x_{\sigma(r+s)}]=0$, for all $n=0,1,2,3,\dots$

(here $(-1)^\alpha$ comes from parities and $(-1)^\sigma=\operatorname{sign}\sigma$).

Recollection: L_∞-algebras and L_∞-morphisms – 2

A parallel notion is as follows.

Definition (*L_∞-algebra: symmetric version*)

A vector space $V = V_0 \oplus V_1$ with a collection of multilinear operations

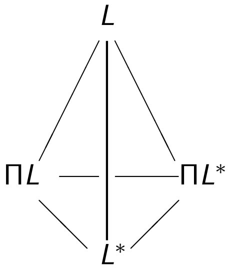
$$\{-,\ldots,-\}\colon \underbrace{V\times\ldots\times V}_{k\ \text{times}} \to V \quad (\text{for } k=0,1,2,\ldots)$$

such that

- all brackets are odd;
- all brackets are symmetric (in $\mathbb{Z}_2$-graded sense);
- $\sum_{r+s=n}\sum_{\text{shuffles}}(-1)^\alpha\{\{v_{\sigma(1)},\ldots,v_{\sigma(r)}\},\ldots,v_{\sigma(r+s)}\}=0$, for all $n = 0, 1, 2, 3, \ldots$

(here $(-1)^\alpha$ comes from parities only).

Recollection: L_∞-algebras and L_∞-morphisms – 3



Equivalent structures:

- Antisymmetric L_∞-algebra structure on L
- Symmetric L_∞-algebra structure on ΠL
- Homological vector field $Q \in \mathrm{Vect}(\Pi L)$, i.e., $\tilde{Q} = 1$, $Q^2 = 0$

(Also: P_∞- on L^* and S_∞- on ΠL^*, to be discussed later.)

NB: we identify super vector spaces with supermanifolds.

Recollection: L_∞-algebras and L_∞-morphisms – 4

Relation between brackets in L and ΠL

$$\{\Pi x_1, \ldots, \Pi x_k\} = (-1)^{(k-1)\tilde{x}_1 + \ldots \tilde{x}_{k-1}} \Pi [x_1, \ldots, x_k] \,.$$

Relation with Q

- $Q(\xi) = \sum \frac{1}{n!} \{\underbrace{\xi, \ldots, \xi}_{n}\}$, where $\xi \in V = \Pi L$
- Higher derived bracket formula: $\iota([x_1, \ldots, xk]) = \pm [\ldots [Q, \iota(x_1)], \ldots, \iota(x_k)](0)$, for $x = x^i e_i \in L$, and $\iota(x) := (-1)^{\tilde{x}} x^i \partial/\partial \xi^i \in \mathrm{Vect}(\Pi L)$ (sign fixed by linearity condition).

Description of L_∞-morphisms

An *L_∞-morphism* $L_1 \rightsquigarrow L_2$ is given by a sequence $\Lambda^n L_1 \to L_2$ or $S^n(\Pi L_1) \to \Pi L_2$ satisfying a sequences of identities ("higher homotopies"). It is equivalent to a Q-map $\Pi L_1 \to \Pi L_2$.

Digression: P_∞- and S_∞-structures

Let M be a (super)manifold. Then a P_∞- (S_∞-) structure on M is an antisymmetric (resp., symmetric) L_∞-structure on $C^\infty(M)$ such that the brackets are multiderivations.

- A P_∞-structure on M is specified by an even $P \in C^\infty(\Pi T^*M)$ satisfying $[P, P] = 0$, by the formula:

$$\{f_1, \ldots, f_k\}_P = [\ldots[P, f_1], \ldots, f_k]|_M \,.$$

- An S_∞-structure on M is specified by an odd $H \in C^\infty(T^*M)$ satisfying $(H, H) = 0$, by the formula:

$$\{f_1, \ldots, f_k\}_H = (\ldots(H, f_1), \ldots, f_k)|_M \,.$$

Homological vector fields ("Hamilton–Jacobi"):

- $Q_P = \int_M Dx\, P(x, \frac{\partial\psi}{\partial x})\frac{\delta}{\delta\psi(x)} \in \mathrm{Vect}(\mathbf{\Pi C}^\infty(M))$
- $Q_H = \int_M Dx\, H(x, \frac{\partial f}{\partial x})\frac{\delta}{\delta f(x)} \in \mathrm{Vect}(\mathbf{C}^\infty(M))$

Key theorem: pullback as an L_∞-morphism

Let M_1 and M_2 be S_∞-manifolds, with $H_i \in C^\infty(T^*M_i)$, $i = 1, 2$.

Definition of an S_∞ ("homotopy Schouten") thick morphism

A thick morphism $\Phi\colon\ M_1 \Rrightarrow M_2$ is S_∞ if $\pi_1^* H_1 = \pi_2^* H_2$ on Φ.

Note: this is expressed by the Hamilton–Jacobi equation

$$H_1\left(x, \frac{\partial S}{\partial x}\right) = H_2\left(\frac{\partial S}{\partial q}, q\right).$$

Theorem

If a thick morphism of S_∞-manifolds $\Phi\colon\ M_1 \Rrightarrow M_2$ is S_∞, then the pullback

$$\Phi^*\colon\ \mathbf{C}^\infty(M_2) \to \mathbf{C}^\infty(M_1)$$

is an L_∞-morphism of the homotopy Schouten brackets.
In greater detail: Φ^* intertwines the homological vector fields $Q_{H_2} \in \mathrm{Vect}(\mathbf{C}^\infty(M_2))$ and $Q_{H_1} \in \mathrm{Vect}(\mathbf{C}^\infty(M_1))$.

Another application: adjoint for a nonlinear transformation

Theorem

1. For a fiberwise map of vector bundles $\Phi\colon\; E_1 \to E_2$, there is a fiberwise thick morphism

$$\Phi^*\colon\; E_2^* \Longrightarrow E_1^*\,,$$

with the same properties as the usual adjoint and coinciding with it if Φ is fiberwise-linear. Construction:
$\Phi^* := \bigl(\boldsymbol{\kappa}\times\boldsymbol{\kappa})(\Phi)\bigr)^{op} \subset T^*E_1^* \times (-T^*E_2^*)$, *where*
$\boldsymbol{\kappa}\colon\; T^*E \to T^*E^*$ *is the Mackenzie–Xu diffeomorphism.*
2. The obtained pushforward of functions on the dual bundles

$$\Phi_* := (\Phi^*)^*\colon\; \mathbf{C}^{\infty}(E_1^*) \to \mathbf{C}^{\infty}(E_2^*)$$

if restricted on the space of sections $\mathbf{C}^{\infty}(M,E_1)$ *takes it to* $\mathbf{C}^{\infty}(M,E_2)$ *and coincides on sections with* $\Phi_*(\boldsymbol{v}) = \Phi\circ\boldsymbol{v}$.

Recollection: L_∞-algebroids

- An L_∞-algebroid is a vector bundle $E \to M$ with an antisymmetric L_∞-algebra structure on sections and a sequence of n-ary anchors $E \times_M \ldots \times_M E \to TM$ so that the Leibniz identities hold:

  $$[u_1, \ldots, u_{n-1}, fu_n] = a(u_1, \ldots, u_{n-1})(f)\, u_n + (-1)^\alpha f\, [u_1, \ldots, u_n]\,,$$

  where $(-1)^\alpha = (-1)^{(\tilde{u}_1 + \ldots + \tilde{u}_{n-1} + n)\tilde{f}}$.
- An L_∞-algebroid structure on $E \to M$ is equivalent to a (formal) homological vector field on the supermanifold ΠE.
- An L_∞-morphism of L_∞-algebroids $\Phi\colon\ E_1 \rightsquigarrow E_2$ is specified by a map (in general, nonlinear) $\Phi\colon\ \Pi E_1 \to \Pi E_2$ such that the corresponding homological vector fields are Φ-related.
- Example: all anchors assemble into an L_∞-morphism $\Pi E \to \Pi TM$.

L_∞-morphisms of Lie-Poisson and Lie-Schouten brackets

Theorem

An L_∞-morphism of L_∞-algebroids over a base M induces L_∞-morphisms of the homotopy Poisson and homotopy Schouten algebras of functions on the dual and antidual bundles respectively.

Corollary

The anchor for an L_∞-algebroid $E \to M$ induces L_∞-morphisms

$$\mathbf{C}^\infty(\Pi E^*) \to \mathbf{C}^\infty(\Pi T^* M)$$

for the homotopy Schouten brackets, and

$$\mathbf{\Pi C}^\infty(E^*) \to \mathbf{\Pi C}^\infty(T^* M)\,.$$

for the homotopy Poisson brackets.

Application to a homotopy Poisson manifold

In particular, we have the following:

Corollary

On a homotopy Poisson manifold M, there is an L_∞-morphism

$$\boldsymbol{\Omega}(M) = \mathbf{C}^\infty(\Pi TM) \to \mathbf{C}^\infty(\Pi T^*M) = \mathfrak{A}(M)\,,$$

between the higher Koszul brackets on forms (induced by a homotopy Poisson structure) and the canonical Schouten bracket on multivector fields.

(This was our initial problem discussed in the beginning.)

Quantum pullbacks and quantum thick morphisms

Definition

A *quantum pullback* $\hat{\Phi}^*\colon\; OC^{\infty}_{\hbar}(M_2) \to OC^{\infty}_{\hbar}(M_1)$ is defined by

$$(\hat{\Phi}^*[w])(x) = \int_{T^*M_2} Dy\bar{D}q\; e^{\frac{i}{\hbar}(S_{\hbar}(x,q)-y^i q_i)}\, w(y)\,.$$

A *quantum thick* (or *microformal*) *morphism* $\hat{\Phi}\colon\; M_1 \to_{\hbar} M_2$ is the corresponding arrow in the dual category.

Here $S_{\hbar}(x,q)$ is a *quantum generating function*:

$$S_{\hbar}(x,q) = S^0_{\hbar}(x) + \varphi^i_{\hbar}(x)q_i + \frac{1}{2}\,S^{ij}_{\hbar}(x)q_jq_i + \frac{1}{3!}\,S^{ijk}_{\hbar}(x)q_kq_jq_i + \dots$$

$OC^{\infty}_{\hbar}(M)$ is the algebra of *oscillatory wave functions*, i.e. sums of formal expressions $w(x) = a_{\hbar}(x)e^{\frac{i}{\hbar}b_{\hbar}(x)}$, where $a_{\hbar}(x)$ and $b_{\hbar}(x)$ are formal power series in $\hbar$.
($\bar{D}q := (2\pi\hbar)^{-n}(i\hbar)^m Dq$ in dimension $n|m$.)

Classical limit

Theorem

Let $\hat{\Phi}\colon M_1 \twoheadrightarrow_{\hbar} M_2$ be a quantum thick morphism with a quantum generating function $S_{\hbar}$. Consider $S_0(x,q) := \lim\limits_{\hbar\to 0} S_{\hbar}(x,q)$ as the (classical) generating function of a (classical) thick morphism $\Phi\colon M_1 \twoheadrightarrow M_2$. Then for any oscillatory wave function of the form $w(y) = e^{\frac{i}{\hbar}g(y)}$ on M_2, the quantum pullback given by

$$\hat{\Phi}^*\left[e^{\frac{i}{\hbar}g}\right] = e^{\frac{i}{\hbar}f_{\hbar}(x)}\,,$$

where $f_{\hbar} = \Phi^*[g] + O(\hbar)$, and Φ^* is the pullback by the classical microformal morphism $\Phi\colon M_1 \twoheadrightarrow M_2$ defined by $S_0(x,q)$.

We say that $\Phi = \lim\limits_{\hbar\to 0} \hat{\Phi}$.

Explicit formula for quantum pullback

Suppose

$$S_\hbar(x,q) = S^0_\hbar(x) + \varphi^i_\hbar(x)q_i + S^+_\hbar(x,q)\,,$$

where $S^+_\hbar(x,q)$ is the sum of all terms of order $\geqslant 2$ in q_i.

Theorem

The action of $\hat{\Phi}^*$ defined by $S_\hbar(x,q)$ can be expressed as follows:

$$\bigl(\hat{\Phi}^* w\bigr)(x) = e^{\frac{i}{\hbar}S^0_\hbar(x)} \left(e^{\frac{i}{\hbar}S^+_\hbar\left(x, \frac{\hbar}{i}\frac{\partial}{\partial y}\right)} w(y) \right)\Big|_{\,y^i=\varphi^i_\hbar(x)}\,.$$

Hence the quantum pullback $\hat{\Phi}^*$ is a special type formal linear differential operator over a 'quantum-perturbed' map $\varphi_\hbar\colon\ M_1 \to M_2$. Here $S^0_\hbar(x)$ gives the phase factor, $\varphi^i_\hbar(x)q_i$ gives the map, and $S^+_\hbar(x,q)$ is responsible for "quantum corrections".

Digression: brackets generated by an operator

Let A be a commutative algebra with 1 over $\mathbb{C}[[\hbar]]$. Let Δ be a linear operator on A. Consider two sequences of multilinear operations (of parity $\tilde{\Delta}$ and symmetric in the supersense):

Definition (a modification of Koszul's)

Quantum brackets generated by Δ:

$$\{a_1,\ldots,a_k\}_{\Delta,\hbar} := (-i\hbar)^{-k}[\ldots[\Delta,a_1],\ldots,a_k](1)\,;$$

classical brackets generated by Δ:

$$\{a_1,\ldots,a_k\}_{\Delta,0} := \lim_{\hbar\to 0}\,(-i\hbar)^{-k}[\ldots[\Delta,a_1],\ldots,a_k](1)$$

- Δ is a *formal $\hbar$-differential operator* if all quantum brackets are defined;
- Δ is an *$\hbar$-differential operator of order $\leqslant n$* if all quantum brackets vanish for $k>n$.

More on brackets generated by Δ

Remark (Explicit formulas)

- For $k=0$, $\{\varnothing\}_{\Delta,\hbar} = \Delta(1)$;
- for $k=1$, $\{a\}_{\Delta,\hbar} = (-i\hbar)^{-1}\big(\Delta(a) - \Delta(1)a\big)$;
- for $k=2$, $\{a,b\}_{\Delta,\hbar} =$
  $(-i\hbar)^{-2}\big(\Delta(ab) - \Delta(a)b - (-1)^{\tilde{a}\tilde{b}}\Delta(b)a + \Delta(1)ab\big)$;
- for general k, $\{a_1,\dots,a_k\}_{\Delta,\hbar} = (-i\hbar)^{-k}$
  $$\sum_{s=0}^{k}(-1)^s \sum_{(k-s,s)\text{-shuffles}} (-1)^{\alpha}\,\Delta\big(a_{\tau(1)}\dots a_{\tau(k-s)}\big)\,a_{\tau(k-s+1)}\dots a_{\tau(k)}\,,$$
  where $(-1)^{\alpha} = (-1)^{\alpha(\tau;\tilde{a}_1,\dots,\tilde{a}_k)}$ is the Koszul sign.

$\hbar$-differential operators

$\operatorname{ord}_\hbar \Delta \leqslant k$ iff for all $a\in A$, $[\Delta, a] = i\hbar B$ where $\operatorname{ord}_\hbar B \leqslant k-1$.

$S_{\infty,\hbar}$-algebras

Let Δ on A be odd. *If $\Delta^2 = 0$, then the quantum brackets define an L_∞-algebra (in the odd symmetric version).* They additionally satisfy the modified Leibniz identity

$$\begin{aligned}\{a_1, \dots, a_{k-1}, ab\}_{\Delta,\hbar} = \{a_1, \dots, a_{k-1}, a\}_{\Delta,\hbar} b \pm \\ a\{a_1, \dots, a_{k-1}, b\}_{\Delta,\hbar} + \underbrace{(-i\hbar)\{a_1, \dots, a_{k-1}, a, b\}_{\Delta,\hbar}}_{\text{extra term}} .\end{aligned}$$

We call such an algebraic structure an $S_{\infty,\hbar}$-*algebra*.
Note: the operator Δ and the whole $S_{\infty,\hbar}$-structure are fully defined by 0- and 1-brackets.

Lemma

The quantum brackets generated by Δ correspond to a "Batalin-Vilkovisky homological vector field" on A (regarded as a supermanifold)

$$Q = e^{-\frac{i}{\hbar}a}\Delta\left(e^{\frac{i}{\hbar}a}\right)\frac{\delta}{\delta a} .$$

BV-manifolds and BV quantum morphisms

Definition

(1) A *BV-manifold*: a supermanifold M equipped with an odd formal $\hbar$-differential operator Δ, $\Delta^2 = 0$. The operator Δ is the *BV-operator*.

(2) A *(quantum) BV-morphism* of BV-manifolds (M_1, Δ_1) and (M_2, Δ_2): a quantum thick morphism $\hat{\Phi}\colon\ M_1 \to_{\hbar} M_2$ such that

$$\Delta_1 \circ \hat{\Phi}^* = \hat{\Phi}^* \circ \Delta_2\,.$$

Since Δ induces a sequence of quantum brackets, and is defined by the 0- and 1-brackets, a BV-structure and an $S_{\infty,\hbar}$-structure on M are equivalent.

Question

How to obtain an L_∞-morphism of quantum brackets generated by BV-operators? (Note: the operator $\hat{\Phi}^*$ is linear, so cannot be the answer.)

L_∞-morphism of quantum brackets induced by a quantum BV-morphism

Define $\hat{\Phi}^!\colon\ \mathbf{C}^{\infty}_{\hbar}(M_2) \to \mathbf{C}^{\infty}_{\hbar}(M_1)$ by

$$\hat{\Phi}^! := \frac{\hbar}{i}\,\ln \circ \hat{\Phi}^* \circ \exp \frac{i}{\hbar}\,,$$

or $\hat{\Phi}^!(g) = \frac{\hbar}{i}\,\ln\,\hat{\Phi}^*\left(e^{\frac{i}{\hbar}g}\right)$, for a $g \in \mathbf{C}^{\infty}_{\hbar}(M_2)$.

Theorem

If $\hat{\Phi}\colon\ M_1 \longrightarrow_{\hbar} M_2$ is a BV quantum morphism, then $\hat{\Phi}^!$ is an L_∞-morphism of the $S_{\infty,\hbar}$-algebras of functions.
Or, in greater detail: $\hat{\Phi}^!$ is a morphism of infinite-dimensional Q-manifolds $\mathbf{C}^{\infty}_{\hbar}(M_2) \to \mathbf{C}^{\infty}_{\hbar}(M_1)$, where

$$Q_\Delta = \int Dx\; e^{-\frac{i}{\hbar}f}\,\Delta\left(e^{\frac{i}{\hbar}f}\right)\frac{\delta}{\delta f(x)}\,.$$

From a quantum BV morphism to a classical S_∞ thick morphism

Let M be a BV-manifold with a BV-operator Δ. In the limit $\hbar \to 0$, Δ gives an S_∞-structure. Its "master Hamiltonian" is

$$H(x,p) = \lim_{\hbar \to 0} e^{-\frac{i}{\hbar}x^a p_a} \Delta\big(e^{\frac{i}{\hbar}x^a p_a}\big)\,.$$

Theorem ("analog of Egorov's theorem")

Let M_1 and M_2 be BV-manifolds and let $\hat{\Phi}\colon\; M_1 \mathrel{\longrightarrow\!\!\!\!\!\!\!\to}_{\hbar} M_2$ be a BV quantum thick morphism. Then its classical limit $\Phi\colon\; M_1 \mathrel{\longrightarrow\!\!\!\!\!\!\!\to} M_2$ is a homotopy Schouten morphism for the induced S_∞-structures.

Explicitly: the intertwining relation $\Delta_1 \circ \hat{\Phi}^* = \hat{\Phi}^* \circ \Delta_2$ implies the Hamilton-Jacobi equation for the classical thick morphism $\Phi = \lim\limits_{\hbar \to 0} \hat{\Phi}$:

$$H_1\Big(x, \frac{\partial S}{\partial x}\Big) = H_2\Big(\frac{\partial S}{\partial q}, q\Big)\,.$$

Some open questions

"Non-linear algebra-geometry duality"

- Define a *non-linear homomorphism* of algebras to be a map $A_1 \to A_2$ such that its derivative at every element $a \in A_1$ is an algebra homomorphism. (Variant: a formal map $A_1 \to A_2$.) Question: how to describe such maps?
- In particular, is it true that all such non-linear homomorphisms between algebras $C^\infty(M)$ are pullbacks by thick morphisms?

Other

- "Thick manifolds": if we have thick diffeomorphisms, what can be obtained by gluing?
- Action of thick morphisms on forms, cohomology, etc. ...

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FINIS

Thank you for attention!