Twisted string algebras and fake flatness

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Self-dual string and fake flatness

- self-dual string equation: $\mathcal{H} = *\mathrm{d}\Phi$
- consider 2 models for string Lie 2-algebra:

$$\mathfrak{string}_{\mathrm{sk}} : \mathbb{R} \xrightarrow{0} \mathfrak{g} \ ,$$
$$\mathfrak{string}_{\Omega} : \widehat{\Omega}\mathfrak{g} \xrightarrow{\mu_1} P_0\mathfrak{g} \ .$$

$$\rightsquigarrow$$

- Infinitesimal gauge transformation: $\delta\mathcal{H} = \mu_2(\mathcal{F}, \Lambda)$
- Under categorical equivalence: $\mathcal{H} \to \Phi_0(\mathcal{H}) + \Phi_2(\mathcal{F}, A)$

Fake-flatness condition, $\mathcal{F} = 0$, needed! $\leftarrow$ problematic

Higher Gauge Theory as morphisms of dgas

- (Higher) Gauge Theory can be encoded in morphisms between differential graded algebras
- Morphisms from $\mathsf{CE}(\mathfrak{g})$ to $\Omega^\bullet(X)$ yield potentials and flat curvatures
- Gauge transformations are given by homotopies between these

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Example:

$$A : \mathsf{CE}(\mathfrak{g}) \longrightarrow \Omega^\bullet(X)$$
$$t^\alpha \longmapsto A^\alpha$$
$$Qt^\alpha = -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta t^\gamma \longmapsto 0 = \mathrm{d}A + \tfrac{1}{2}[A, A]$$

Weil algebra $\mathsf{W}(\mathfrak{g})$:

- dga: $(\wedge^\bullet(\mathfrak{g}^*[1] \oplus \mathfrak{g}^*[2]), Q_{\mathsf{W}})$
- $Q_{\mathsf{W}}|_{\mathfrak{g}^*[1]} = Q_{\mathsf{CE}} + \sigma$ and $Q_{\mathsf{W}}|_{\mathfrak{g}^*[2]} = -\sigma Q_{\mathsf{CE}} \sigma^{-1}$
- $\sigma : \mathfrak{g}^*[1] \to \mathfrak{g}^*[2]$ is the shift isomorphism

Higher Gauge Theory: Weil algebra

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Example:

- ordinary Lie algebra $\mathfrak{g}$, generators t^α and $r^\alpha = \sigma t^\alpha$
- $Qt^\alpha = -\frac{1}{2} f^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma + r^\alpha$
- $Qr^\alpha = -f^\alpha_{\beta\gamma} t^\beta \wedge r^\gamma$

Higher Gauge Theory as morphisms of dgas

- (Higher) Gauge Theory can be encoded in morphisms between differential graded algebras, i.e. morphisms that commute with Q
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Example:

$$(A, \mathcal{F}) : \mathsf{W}(\mathfrak{g}) \longrightarrow \Omega^\bullet(X)$$

$$t^\alpha, r^\alpha = \sigma t^\alpha \longmapsto A^\alpha, \mathcal{F}^\alpha$$

$$Qt^\alpha = -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta t^\gamma + r^\alpha \longmapsto \mathcal{F}^\alpha = \mathrm{d}A + \tfrac{1}{2}[A, A]$$

$$Qr^\alpha = -f^\alpha_{\beta\gamma} t^\beta r^\gamma \longmapsto (\nabla \mathcal{F})^\alpha = 0$$

Higher Gauge Theory: Invariant polynomials

Invariant polynomials $\mathrm{inv}(\mathfrak{g})$:

- inv. pol.: $P \in \mathsf{W}(\mathfrak{g})|_{\wedge^\bullet(\mathfrak{g}^*[2])}$ s.t. $Q_\mathsf{W} P \in \mathsf{W}(\mathfrak{g})|_{\wedge^\bullet(\mathfrak{g}^*[2])}$
- $P_1 \sim P_2$ if $P_1 - P_2 = Q_\mathsf{W}\tau$ for $\tau \in \mathsf{W}(\mathfrak{g})|_{\wedge^\bullet(\mathfrak{g}^*[2])}$
- $\mathrm{inv}(\mathfrak{g})$: invariant polynomials mod horizontal equiv.

→

- identified with characteristic classes in Chern-Weil theory
- sits in short exact sequence:

$$\mathsf{CE}(\mathfrak{g}) \twoheadleftarrow \mathsf{W}(\mathfrak{g}) \hookleftarrow \mathrm{inv}(\mathfrak{g})$$

Higher Gauge Theory: $\mathfrak{g}$-connection object

Let $P \to X$ be a principal bundle and define $\mathfrak{g}$-connection object:

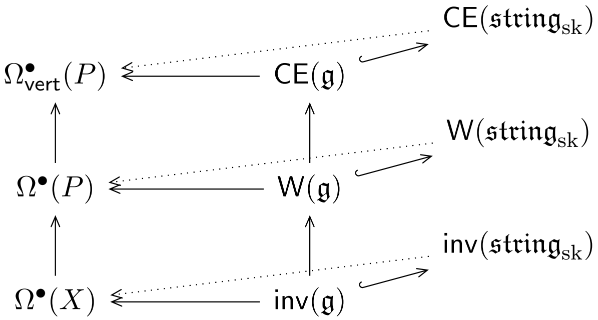
$$
\begin{array}{ccc}
\Omega^\bullet_{\mathsf{vert}}(P) & \xleftarrow{\;A\;} & \mathsf{CE}(\mathfrak{g}) \\
\uparrow & & \uparrow \\
\Omega^\bullet(P) & \xleftarrow{\;(A,\mathcal{F})\;} & \mathsf{W}(\mathfrak{g}) \\
\uparrow & & \uparrow \\
\Omega^\bullet(X) & \xleftarrow{\;<\mathcal{F}>\;} & \mathsf{inv}(\mathfrak{g})
\end{array}
$$

$\Longleftarrow$ 1^{st} Ehresmann condition (upper square)

$\Longleftarrow$ 2^{nd} Ehresmann condition (lower square)

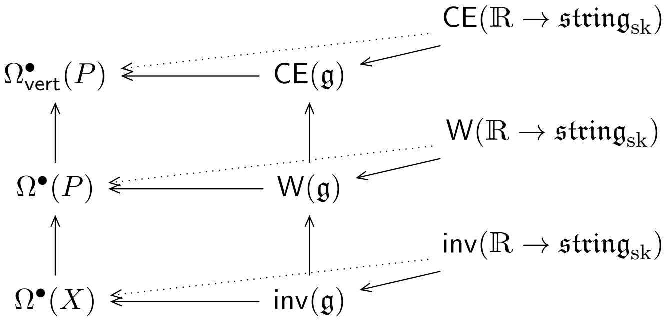
Higher Gauge Theory: Lifting problem

Try to lift to $\mathfrak{string}_{\mathrm{sk}}$:



Higher Gauge Theory: Lifting problem

Try to lift to $\mathbb{R} \to \mathfrak{string}_{\mathrm{sk}}$:



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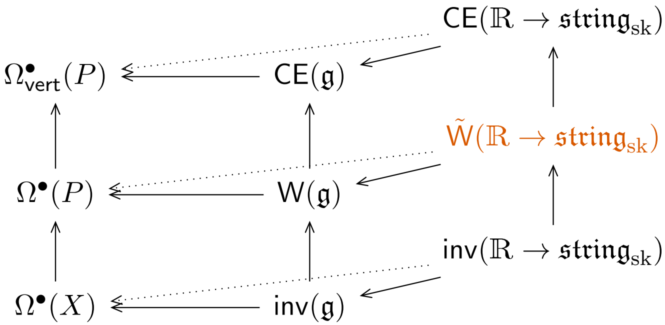

Modified string algebra: Skeletal model

This yields:

$$\widetilde{\mathfrak{string}}_{\rm sk} = \mathbb{R}[2] \hookrightarrow \mathbb{R}[1] \xrightarrow{0} \mathfrak{g} ,$$

together with:

$$\begin{aligned}
\mathcal{F} &= \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) ,\\
\mathcal{H} &= \mathrm{d}B + \tfrac{1}{6}\mu_3(A,A,A) - \kappa(A,\mathcal{F}) + \mu_1(C) ,\\
\mathcal{G} &= \mathrm{d}C ,\\
\mathrm{d}\mathcal{F} &= -\mu_2(A,\mathcal{F}) ,\\
\mathrm{d}\mathcal{H} &= -\kappa(\mathcal{F},\mathcal{F}) + \mu_1(\mathcal{G}) ,\\
\mathrm{d}\mathcal{G} &= 0 .
\end{aligned}$$

$\Longrightarrow$ SDS equation gauge covariant even for non-vanishing $\mathcal{F}$.

Modified string algebra: Loop model

Analogous story yields:

$$\widetilde{\mathfrak{string}}_\Omega = \mathbb{R}[2] \hookrightarrow \hat{\Omega}(\mathfrak{g})[1] \xrightarrow{\mu_1} P_0\mathfrak{g} \ ,$$

together with:

$$\begin{aligned}
\mathcal{F} &= \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) + \mu_1(B) \ , \\
\mathcal{H} &= \mathrm{d}B + \mu_2(A,B) - \kappa(A,\mathcal{F}) + \mu_1(C) \ , \\
\mathcal{G} &= \mathrm{d}C \ , \\
\mathrm{d}\mathcal{F} &= -\mu_2(A,\mathcal{F}) + \mu_1(\mathcal{H}) \ , \\
\mathrm{d}\mathcal{H} &= -\kappa(\mathcal{F},\mathcal{F}) + \mu_1(\mathcal{G}) \ , \\
\mathrm{d}\mathcal{G} &= 0 \ .
\end{aligned}$$

$\Longrightarrow$ SDS equation gauge covariant even for non-vanishing $\mathcal{F}$.

$\Longrightarrow$ Categorical equivalence okay even for non-vanishing $\mathcal{F}$.

Further application

Consider:

$$
\begin{array}{ccccccc}
\mathfrak{g}^*[3] & \hookrightarrow & \mathfrak{g}^*[2] & & \mathbb{R}^*[1] & \longrightarrow & \mathbb{R}^* \\
 & & \oplus & & \oplus & & \oplus \\
 & & \mathbb{R}[2] & \longrightarrow & \mathbb{R}[1] & & \mathfrak{g}
\end{array}
$$

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