Large-scale behavior of mean curvature flow with periodic forcing

Inwon Kim (joint with Hongwei Gao), work in progress [1]

[1]Dept. of Mathematics, UCLA

August 20, 2018

Let $g : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function which is periodic with respect to translations by $\mathbb{Z}^d$. We consider the evolution of sets $(\Omega_t)_{t>0}$ in $\mathbb{R}^d$ by the flow

$$V = -\varepsilon\kappa + g(\frac{x}{\varepsilon}) \text{ on } \partial\Omega_t.$$

Here $\kappa = \kappa_{x,t}$ denotes the mean curvature of $\partial\Omega_t$ at given boundary point, positive if convex with respect to Ω_t.

Note that zooming in by the coordinate change $(x, t) \to (x/\varepsilon, t/\varepsilon)$ we have

$$V = -\kappa + g(x) \text{ on } \partial\tilde{\Omega}_t.$$

We are interested in the behavior of Ω_t or $\tilde{\Omega}_t$ as $\varepsilon \to 0$.

Literature

Lions-Souganidis (2005) (periodic media) and Armstrong-Cardaliaguet (2015) (Random) shows homogenization in the setting of viscous Hamilton-Jacobi equations, for positive g, under a smallness assumption on $g^2/|Dg|$. Here the cell problem has Lipschitz continuous solutions, and the homogenized speed is Lipschitz continuous.

Caffarelli-Monneau (2014) shows homogenization for positive and Lipschitz g, periodic media, in two dimensions. Here the cell problem solution is only bounded and the homogenized speed is only continuous.

They also provide a counterexample in three dimensions, in laminar setting.

Literature, Cont'd

Dirr-Karali-Yip (2008) shows, when g is sufficiently small (allowing sign change), existence and uniqueness of pulsating traveling wave solution, and homogenization. The speed is Lipschitz continuous with respect to the normal. Dirr-Dondl-Scheuzow (Random media).

Cardaliaguet-Lions-Sougnaidis (2009) discusses examples of pinning and failure of homogenization as well as homogenization, with sign changing g.

Cesaroni - Novaga (2013) considers laminar setting $g(x) = g(x', x_n)$ and graph solutions $\{x_n = u(x', t)\}$ which is periodic with respect to x'. They show existence of (generalized) traveling wave solution with maximal speed $\bar{c}$ and convergence of solutions to the wave. In particular they have

$$\max \frac{u(t,x) - \bar{c}t - \frac{\ln(1+t)}{\bar{c}}}{t} \leq C.$$

We employ the level set PDE

$$u_t = F(D^2u, Du, x) = |Du|(\varepsilon\nabla\cdot(Du/|Du|) + g(\frac{x}{\varepsilon})) \text{ in } \mathbb{R}^n \times (0,\infty), \quad (1)$$

which is a singular and nonlinear parabolic equation. Here the initial data is given as $u(x,0) = \chi_{\Omega_0} - \chi_{\Omega_0^C}$, and we will study the zero level set of $u(x,t)$ instead of Ω_t. Viscosity solutions will be used as notions of solutions.

Lions-Souganidis (2005) shows homogenization of (1) in periodic media, assuming that g satisfies

$$\inf_x (g^2(x) - |Dg|) > 0.$$

This is to provide equicontinuity of solutions of the approximate cell problem $v_\lambda + F(\lambda D^2 v, Dv, x/\lambda) = 0$, to obtain a bounded solution of the cell problem. The above assumption on g provides uniform Lipschitz continuity for v_λ, to yield a unique constant $\bar{F}(p, x)$ such that

$$F(D^2 v, Dv + p, x) = \bar{F}(p) = \bar{g}(\frac{p}{|p|})|p|,$$

which provides the homogenized velocity $\bar{g}(p)$. In particular for $\varepsilon = 1$ and $\nu \in \mathcal{S}^n$, we have a plane-like solution of the form

$$w(x,t) := \bar{g}(\nu)t - \nu \cdot x - v(x).$$

The regularity of $\bar{g}$ is related to the regularity of the cell problem solution v, continuity of $\bar{g}$ comes from boundedness of v.

We are interested in the description of maximal and minimal asymptotic speed ("head" and "tail" speed) for u_ε in general setting, where we may not have plane-like solutions.

Theorem

There exist two upper- and lower-semicontinuous functions $\bar{s}, \underline{s} : \mathcal{S}^d \to \mathbb{R}$, $\underline{s} \le \bar{s}$ *with the following properties:*

- $\bar{s}$ *and* $\underline{s}$ *are continuous in* $\mathcal{S}^d \setminus \mathbb{R}\mathbb{Z}^d$ *("irrational directions").*
- *Let* u^ε *solve* (1) *with initial data* $u^\varepsilon(x,0) = x \cdot \nu$. *Then for all* $t > 0$

$$\bar{s}(\nu) = \lim_{\varepsilon \to 0} \frac{\sup\{x \cdot \nu : u^\varepsilon(x,t) = 0\}}{t},$$

$$\underline{s}(\nu) = \lim_{\varepsilon \to 0} \frac{\inf\{x \cdot \nu : u^\varepsilon(x,t) = 0\}}{t}.$$

Theorem (Cont'd)

In particular, when $\bar{s}(\nu) \neq \underline{s}(\nu)$, the interface $\{u^\varepsilon = 0\}$ oscillates by unit size as $\varepsilon \to 0$ and thus homogenization fails.

For general initial data, we can say that

$$\limsup{}^* u^\varepsilon \text{ is a subsolution of } u_t \leq \bar{s}\left(\frac{Du}{|Du|}\right)|Du|$$

and

$$\liminf{}_* u^\varepsilon \text{ is a supersolution of } u_t \geq \underline{s}\left(\frac{Du}{|Du|}\right)|Du|.$$

It is not clear to us at the moment whether $\bar{s}$ and $\underline{s}$ are continuous at rational directions.

In the laminar case, when $g(x) = g(x', x_n)$, stronger continuity results are available.

Theorem

$\bar{s}$ and $\underline{s}$ are continuous at irrational directions and also when $\nu \cdot e_n \neq 0$.

We will also discuss the existence of generalized travelling wave solutions in this setting.

Head and Tail speed: Definition

Set $\varepsilon = 1$. For given $s \in \mathbb{R}$ and $\nu \mathcal{S}^n$, we define $\bar{u}_{s,\nu}$ by the maximal subsolution of (1) which lies below the obstacle $O_{s,\nu}(x,t) := st - \nu \cdot x$. Then we define the head speed by

$$\bar{s}(\nu) := \inf\{s : \bar{u}_{s,\nu} < O_{s,\nu} \text{ after some time } t > T\}.$$

Similarly $\underline{s}(\nu)$ can be defined using $\underline{u}_{s,\nu}$: the minimal supersolution which lies above the obstacle $O_{s,\nu}$.

It is possible to define above solutions in the whole domain $\mathbb{R}^n \times [0,\infty)$. However we often need localized solutions in finite domains to discuss solutions at irrational directions ν, or to compare solutions over varying ν. When defined in finite domain, we impose boundary conditions to coincide with the obstacle.

Characterizing homogenized operators by obstacle problem, in the lack of cell problem, has been introduced by Caffarelli-Souganidis-Wang '05.

Birkhoff properties

The maximal and minimal obstacle solutions feature space-time monotonicity (Birkhoff) properties. For instance $\bar{u}_{s,\nu}$ satisfy

$$\bar{u}_{s,\nu}(x+\xi, t+\tau) \leq \bar{u}_{s,\nu}(x,t)$$

if $\xi \in \mathbb{Z}^d$ and if $s\tau - \xi \cdot \nu \leq 0$. In particular if ν is irrational then for any small $\tau > 0$ one can find $\xi \in \mathbb{Z}^d$ such that $s\tau \leq \xi \cdot \nu \leq C\tau$.
This property is central in the proof of the following proposition.

Let u and v satisfy

$$u(x,t) \leq u(x-\xi, t-\tau_1) \text{ and } v(x,t) \leq v(x+\xi, t+\tau_2),$$

where $|\xi \cdot \nu| \leq \tau_1, \tau_2$.

Proposition (Local comparison)

Let u, v be a subsolution and supersolution with above property. Suppose $\tau_1, \tau_2 \leq e^{-LT}$, where L is the Lipschitz constant for g. Then the following holds: If $u(\cdot, 0) < v(\cdot, 0)$ in $|(x - x_0)| \leq e^{2LT}$, then

$$u(\cdot, t) \leq v(\cdot, t) \text{ in } \{|x - x_0| \leq R(t)\} \text{ for } 0 \leq t \leq T,$$

with $R(T) = C(|\xi|)e^{2L(T-t)}$.

This property localizes the flow from far-away regions.

To prove the local comparison, take an inf-convolution $\tilde{v}(x,t;h) := \inf_{|x-y|\leq h(t)} v(y,t)$, where $h(t) > 0$ decreases fast enough over time to preserve the supersolution property.

Suppose $\tilde{v}$ crosses u from above at (x_0, t_0) in $\{|x| \leq R(t)\} \times \{0 \leq t \leq t_0\}$ for the first time. Note that for $t \leq t_0$ and for $|x| \leq R(t-\tau_1)$,

$$u(x,t) \leq u(x-\xi, t-\tau_1) \leq \tilde{v}(x-\xi, t-\tau_1) \leq \tilde{v}(x, t+\Delta t).$$

In particular we have, for $\Delta t = \tau_2 - \tau_1$,

$$u(\cdot, t-\Delta t) \leq \tilde{v}(\cdot, t) \text{ in } \Sigma(t) := |x - x_0| \leq R(t-t_2) - R(t).$$

Due to the finite propagation property, if Δt is small enough compared to $h^2(t)$, then it follows that

$$u(\cdot, t) \leq \tilde{v}(\cdot, t; \hat{h}) \text{ in } \Sigma(t) \text{ for } \hat{h} = h/2.$$

So now we have u crossing $\tilde{v}(\cdot; h)$ at $t = t_0$ or before in domain $\Sigma(t)$ that contains $|x - x_0| \leq t_2 R(T)$, whereas u stays below $\tilde{v}(\cdot, t; h/2)$ in the same domain before $t \leq t_0$. This situation leads to a contradiction if $t_2 R(t) \geq C$, by applying a space-time inf-convolution.

If u is a supersolution, then so is

$$\tilde{u}(x,t) := \inf_{|x-y| \leq h(t)\varphi(x)} u(y,t),$$

If

$$\varphi(x) = 1 - c|(x - x_0)|^2 \text{ where } c << 1.$$

This space-time convolution is a variation from the one constructed by Athanasoupolos-Caffrarelli-Salsa for the Stefan problem.

Let us discuss several consequences of the above localized comparison now.

Lemma

Let $\nu \in \mathcal{S}^n \setminus \mathbb{RZ}^d$, and let $s = \bar{s}(\nu) + \delta$. If $u = \bar{u}_{s,\nu}$ is defined in a finite domain of size $C(\nu)e^{2LT}$, then the zero set of $\bar{u}_{s,\nu}$ is at least $\frac{\delta}{2}t$ away from that of the obstacle $O_{s,\nu}$ in $|x| \leq C(\nu)e^{2LT}/2$ for $t_0(\delta,\nu) \leq t \leq T$.

This creates a curved solution u^ε which has lateral boundary data $O_{s,\nu}$ on the lateral boundary of the domain $\{|x| \leq Ce^{2LT}\}$ but grows apart from it in the middle region. This type of function is used in the perturbed test function argument, also to show the continuity of $\bar{s}$ and $\underline{s}$.

Theorem

$\bar{s}(\nu)$ and $\underline{s}(\nu)$ are continuous at irrational directions.

We use the fact that for ν_n converging to an irrational direction ν, the constant dependence $C(\nu_n)$ can be chosen uniformly with respect to n.

Theorem

Let $u^\varepsilon(\cdot,0) = \chi_{\Omega_0} - \chi_{\Omega_0^C}$, *where* Ω_0 *is a domain in* $\mathbb{R}^n$. *Then* $U_1 := \limsup^* u^\varepsilon$ *is a subsolution of* $V = \bar{s}(\nu)$ *and* $U_2 = \liminf u^\varepsilon$ *is a supersolution of* $V = \underline{s}(\nu)$.

Without knowing continuity of $\bar{s}$ and $\underline{s}$ for all directions, this theorem is not very useful, except for Ω_0 being a hyperspace or bounding the evolution by convex sets.

In the laminar case, the translation along the direction e_n preserves the equation, this allows Birkhoff property with small vertical shifts for directions with nonzero e_n components. Thus local comparison holds for obstacle solutions for such directions. This leads to continuity for $\bar{s}$ and $\underline{s}$ for ν with nonzero e_n components, besides irrational directions. So here the above theorem is more meaningful.

Sign-changing g

For sign-changing g, our arguments still hold for nonzero values of $\bar{s}$ and $\underline{s}$. We expect parallel results to hold for this case, however it remains to be investigated.

Laminar case

Here we assume $g(x) = g(x', x_n)$ and $\nu = e_n$. In this case one can study the flow in the graph setting, where the interface Γ_t can be written as a graph $x_n = u(x,t)$, and u solves the graph PDE

$$\frac{u_t}{\sqrt{(1+|Du|^2)}} = \nabla \cdot (\frac{Du}{\sqrt{1+|Du|^2}}) + g(x) \text{ in } \mathbb{R}^{n-1} \times (0,\infty).$$

Here Cesaroni-Novaga shows that, if $\nabla \cdot (\frac{Du}{\sqrt{1+|Du|^2}})$ is uniformly bounded, then u is locally $C^{1,\alpha}$ in space, uniformly in time. We are in this situation with our obstacle solutions, since $\min g \leq u_t \leq \max g$.

Based on this regularity we consider the future time sequences of the obstacle subsolutions at $s = s_n = \bar{s} + \frac{1}{n^2}$ and supersolutions at $s = s_n = \underline{s} - \frac{1}{n^2}$ to obtain the following:

Consider the flow $V = -\kappa + g(x)$, $g(x', x_n) = g(x_n)$, $\nu = e_n$.

Theorem

Suppose $\bar{s}(e_n) > \underline{s}(e_n)$. Then there exists open, disjoint sets E_1 and E_2 in $\mathbb{R}^{n-1}$ and $U_i : E_i \to \mathbb{R}$ such that

(a) *$E_i \times (-\infty.\infty)$ are stationary solutions, and the graphs $\Gamma_i := \{x_n = U_i(x,t)\}$ are solutions of the flow.*

(b) *$U_1 \to -\infty$ as $x \to \partial E_1$ and $U_2 \to +\infty$ as $x \to \partial E_2$.*

(c) *Γ_i's are travelling wave solutions respectively with speeds $s_1 = \bar{s}$ and $s_2 = \underline{s}$, i.e.,*

$$U_i(x,t) = U_i(x) + \bar{s}t \text{ in } E_i.$$

• Cessaroni-Novaga previously showed the existence maximal TW solution U_1, as well as estimating $u^\varepsilon - U_1$, for more general setting than positive g.

For general settings, *assuming* that some regularity results are available (what we need is uniform local equi-continuity of the zero level set in $\varepsilon = 1$ scale), we expect to be able to obtain pulsating travelling waves

$$u(x+\xi, t) = u(x, t + s^{-1}\xi \cdot \nu) \text{ for } \xi \in \mathbb{Z}^n,$$

with $s = \bar{s}$ and $s = \underline{s}$.

Thank you for your attention!