Branching random walks in random environment

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Joint work with **Matt Roberts (Bath)**

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Branching random walks in a random potential

Branching random walks

- **Motion:** Start with single particle at the origin that performs a simple random walk on $\mathbb{Z}^d$ (in continuous time).
- **Branching:** After an exponential waiting time, the particle splits into two new particles.
- The new particles behave independently (**no interaction**).

in a random potential:

- the potential $\{\xi(z), z \in \mathbb{Z}^d\}$ is a collection of i.i.d. non-negative random variables.
- **Modification:** when at site z, particles branch at rate $\xi(z)$.

Note: Other models introduce a random offspring distribution instead of changing the rates, e.g. space i.i.d., time i.i.d. or space-time i.i.d.

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Typical questions:

Start with one particle at the origin, then we can ask:

- How far do particles spread by time t?
- Equivalently: when do faraway sites z get hit?
- What does the height profile look like, i.e. how many particles $N(t, z)$ are there at site z at time t?

More specifically:

- We are interested in large scale behaviour $\rightsquigarrow$ scaling limit?
- Can we describe the site with the maximal number of particles?

Need to understand:

1. The role of **averaging**:
   - over the environment.
   - over the branching/migration mechanism.
2. The competition between the benefit of **high peaks** vs. **cost of getting there**.

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Branching random walk with constant branching rate

No migration: Consider a branching process, where particles split at rate r, but there is no migration. The expected number of particles u_t satisfies

$$\frac{d}{dt} u_t = r\, u_t.$$

I.e. if we start with one particle, $u_t = e^{rt}$.

Branching random walk with homogeneous branching rate. Suppose $\xi(x) \equiv r$ for all $x \in \mathbb{Z}^d$. A first moment calculation shows that:

Particle growth in constant environment

Particles spread in a ball of radius growing linearly in t.

More interesting questions: corrections to linear growth term.

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Averaging: The parabolic Anderson model

Fix the (inhomogeneous) potential ξ, let

$$u(t,x) = E^{\xi}[\#\{\text{particles at site } x \text{ at time t }\}]$$

for $t \geq 0, x \in \mathbb{Z}^d$. Then u solves the following equation that defines the **parabolic Anderson model**

$$\begin{aligned} \frac{\partial}{\partial t} u(t,z) &= \Delta u(t,z) + \xi(z) u(t,z)\,, \\ u(0,z) &= \mathbb{1}_0(z)\,, \end{aligned}$$

where Δ is the discrete Laplacian, defined as

$$\Delta f(x) = \sum_{y \in \mathbb{Z}^d : y \sim x} (f(y) - f(x))\,,$$

and $y \sim x$ if y is a neighbour of x.

Lots of research activity during the last 20 years in particular by [Donsker, Varadhan, **Gärtner, Molchanov**, Sznitman, Antal, Carmona, den Hollander, Biskup, König, van der Hofstad, Mörters, Sidorova, Lacoin, O., Schnitzler, Twarowski, Fiodorov, Muirhead, Chouk, Gairing, Perkowski, ...]

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Intermittency for the parabolic Anderson model

The main idea is to understand

Intermittency

The solution u is concentrated in a **small** number of **remote** islands, where the potential ξ is particularly large.

- The behaviour of the model depends crucially on the decay of the tail probability $\mathrm{Prob}\{\xi(0) > x\} \sim ?$ for $x \to \infty$.

For this talk, we will focus on these:

Example A: ξ has a Pareto distrbution, for some $\alpha > 0$:

$$\mathrm{Prob}\{\xi(0) > x\} = x^{-\alpha}.$$

Example B: ξ has a Weibull distribution, for some $\gamma > 0$:

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Previous work on parabolic Anderson model

Theorem 1

For either Pareto potential ($\alpha > d$) or Weibull potential (any $\gamma > 0$), there exists a process Z_t such that as $t \to \infty$,

$$\frac{u(t, Z_t)}{\sum_z u(t,z)} \to 1, \quad \textit{in probability.}$$

- Proved by [König, Lacoin, Mörters, Sidorova '09] – Pareto, [N. Sidorova, A. Twarowski '14] [Fiodorov, Muirhead '14] – Weibull.
- For lighter tails (double exponential), need a island of finite size that supports solution, [K'onig, Biskup, dos Santos '16].

Earlier results mostly concern asymptotics of exptected total mass.

Question

Do these results help to understand the actual number of particles in the branching random walk?

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Back to BRW: Controlling the environment

Main question: If a BRW manages to cover a ball of radius r – what is the largest potential it has seen along the way?

$$\text{How does } \max_{x \in B(0,r) \cap \mathbb{Z}^d} \xi(x) \text{ grow ?}$$

More precise question: What is the **geometry of the peaks** of the potential on large scales?

Extreme value theory tells us:

- Fix a large scaling parameter T.
- Assume ξ is Pareto distributed, i.e. $\mathrm{Prob}\{\xi(z) > x\} \sim x^{-\alpha}$, $\alpha > d$.
- For $q = \frac{d}{\alpha - d}$, introduce scaling for potential and space

$$a_T = \left(\tfrac{T}{\log T}\right)^q, \quad r_T = \left(\tfrac{T}{\log T}\right)^{q+1}.$$

Then, the rescaled environment converges:

$$\Pi_T := \sum_{z \in \mathbb{Z}^d} \delta_{\left(\frac{z}{r_T}, \frac{\xi(z)}{a_T}\right)} \Rightarrow \Pi,$$

where Π is a Poisson point process with intensity $\frac{\alpha}{y^{\alpha+1}} dz \otimes dy$.

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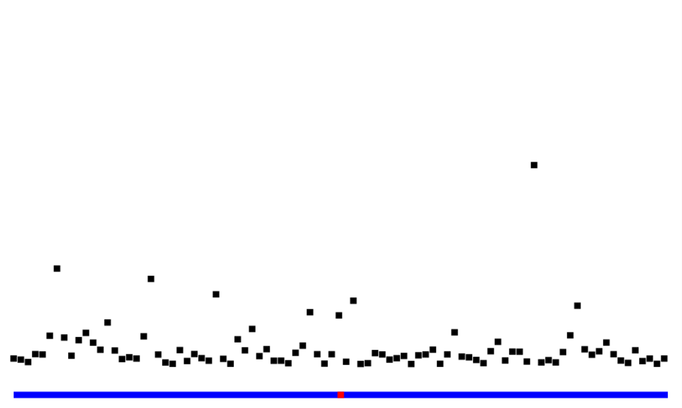

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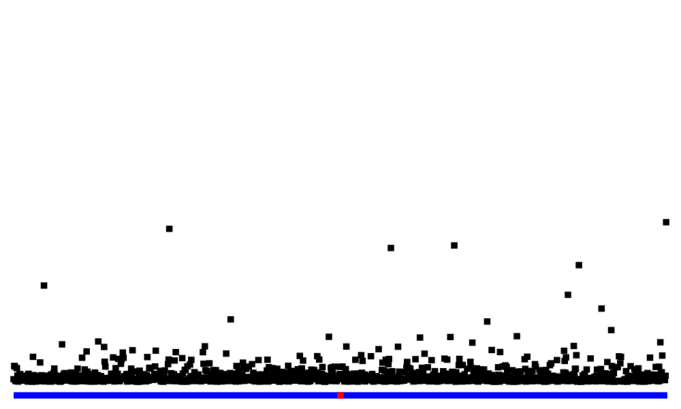

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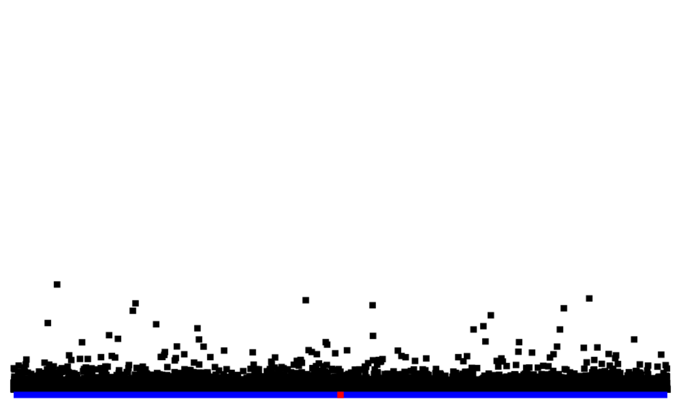

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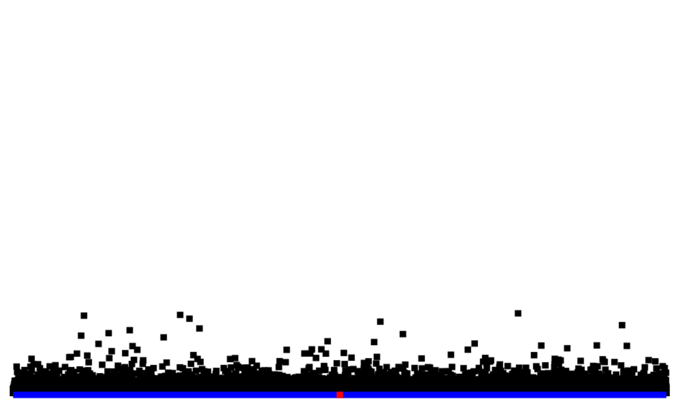

Main result: a scaling limit

Consider for $z \in r_T^{-1}\mathbb{Z}^d, t \geq 0$:

Hitting times: $H_T(z) = \inf\{t \geq 0 : N(tT, r_T z) \geq 1\}$,

Support: $S_T(t) = \{z \in \mathbb{R}^d \;:\; H_T(z) \leq t\}$

Rescaled number of particles: $M_T(t,z) = \frac{1}{a_T} \log_+ N(tT, r_T z)$

with interpolation for $z \notin r_T^{-1}\mathbb{Z}^d$.

Theorem 2 (O., Roberts '16, '18)

The triple

$$\Big((H_T(z))_{z\in\mathbb{R}^d}, (S_T(t))_{t\geq 0}, (M_T(t,z))_{t\geq 0, z\in\mathbb{R}^d}\Big),$$

converges in distribution (in a suitable topology) to

$$(h_\Pi, s_\Pi, m_\Pi) = ((h_\Pi(z))_{z\in\mathbb{R}^d}, (s_\Pi(t))_{t\geq 0}, (m_\Pi(t,z))_{t\geq 0, z\in\mathbb{R}^d}),$$

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$$(h_\Pi, s_\Pi, m_\Pi) = ((h_\Pi(z))_{z\in\mathbb{R}^d}, (s_\Pi(t))_{t\geq 0}, (m_\Pi(t,z))_{t\geq 0, z\in\mathbb{R}^d}),$$

where the limiting object is a functional of the Poisson point process Π.

Main result: a scaling limit

Consider for $z \in r_T^{-1}\mathbb{Z}^d, t \geq 0$:

Hitting times: $H_T(z) = \inf\{t \geq 0 : N(tT, r_T z) \geq 1\}$,

Support: $S_T(t) = \{z \in \mathbb{R}^d \ : \ H_T(z) \leq t\}$

Rescaled number of particles: $M_T(t,z) = \frac{1}{a_T} \log_+ N(tT, r_T z)$

with interpolation for $z \notin r_T^{-1}\mathbb{Z}^d$.

Theorem 2 (O., Roberts '16, '18)

The triple

$$\Big((H_T(z))_{z\in\mathbb{R}^d}, (S_T(t))_{t\geq 0}, (M_T(t,z))_{t\geq 0, z\in\mathbb{R}^d}\Big),$$

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Starting in a point z with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at z and branch at rate $\xi(z)$
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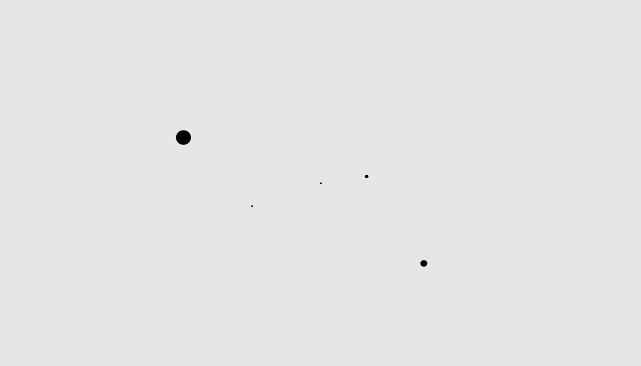

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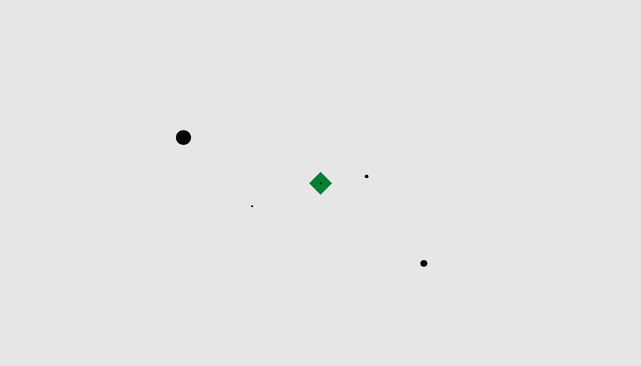

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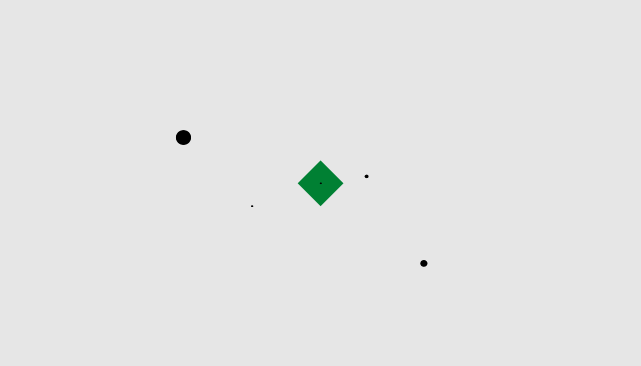

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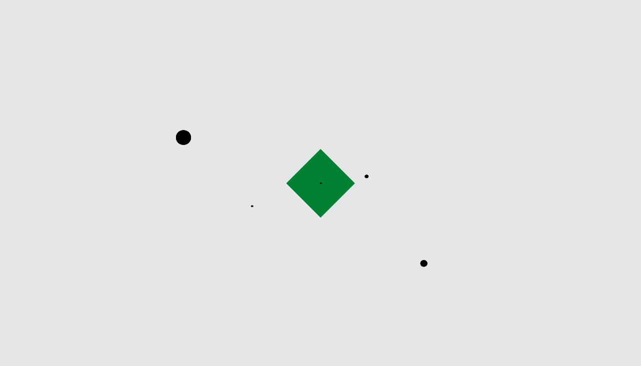

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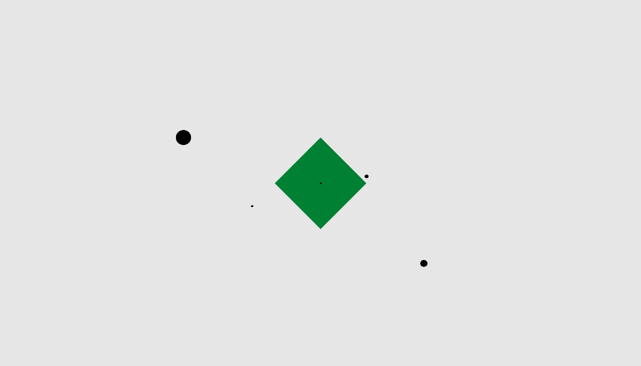

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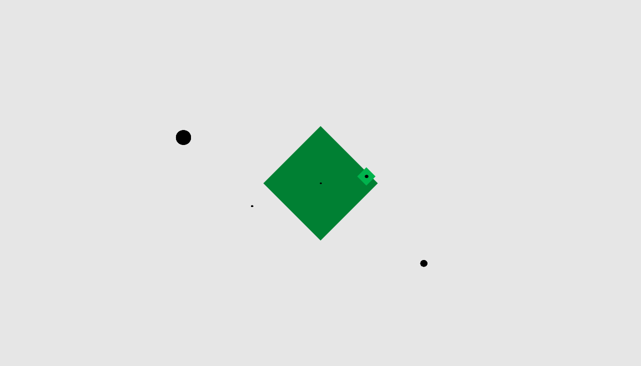

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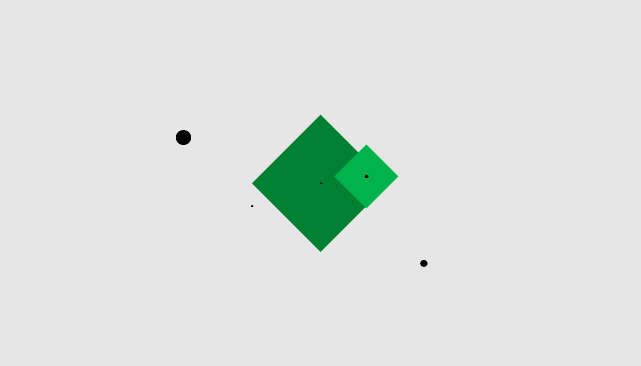

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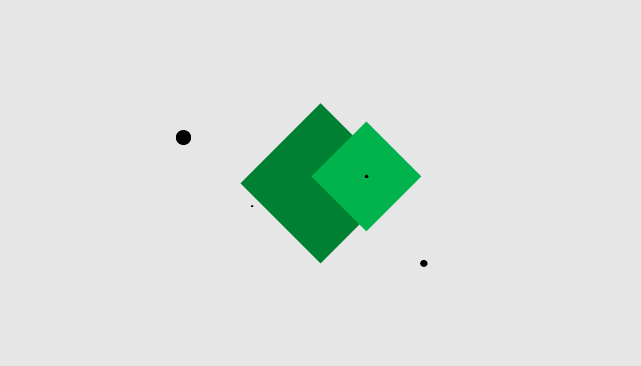

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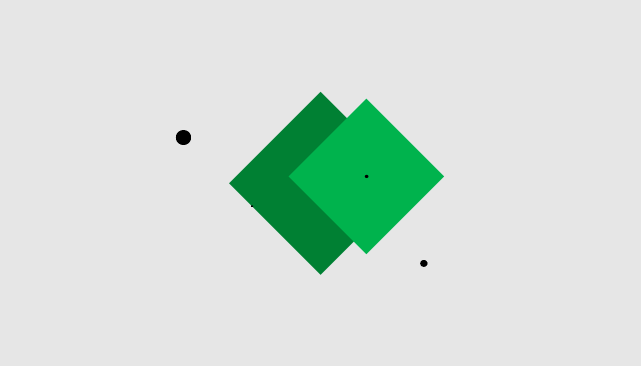

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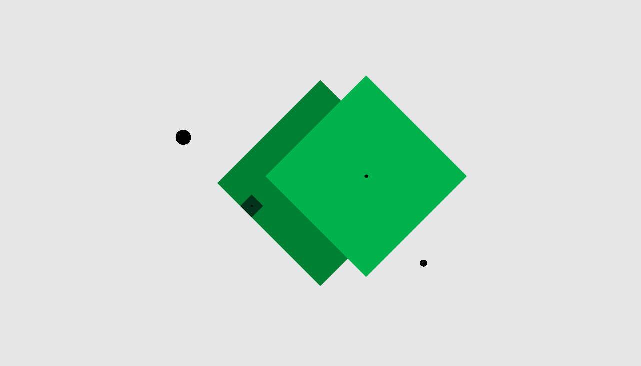

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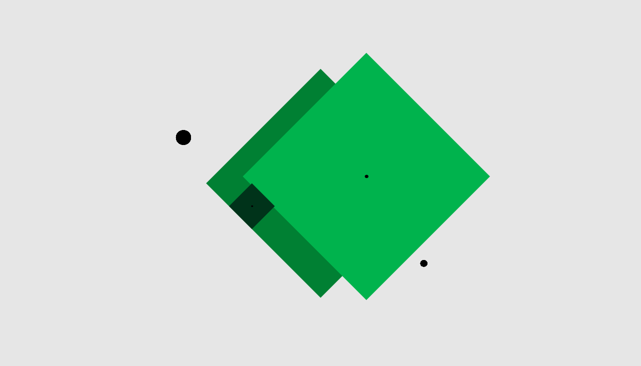

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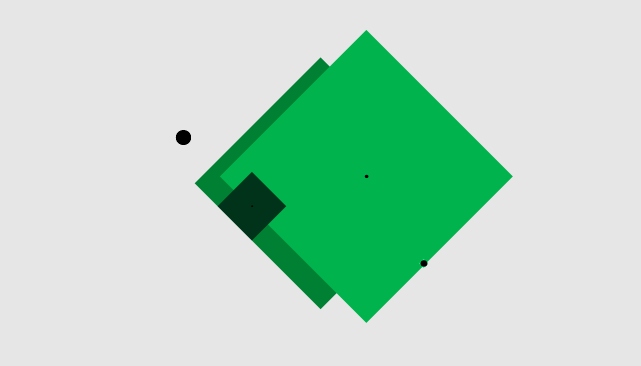

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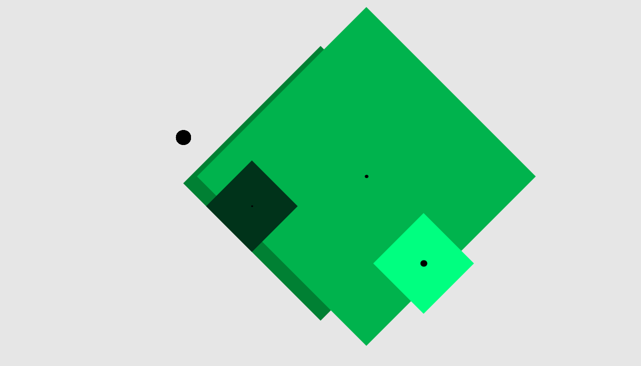

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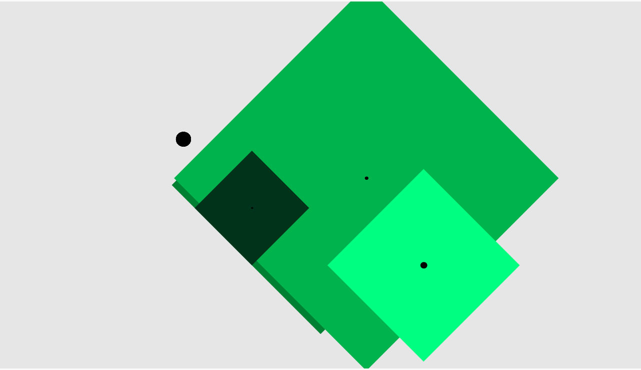

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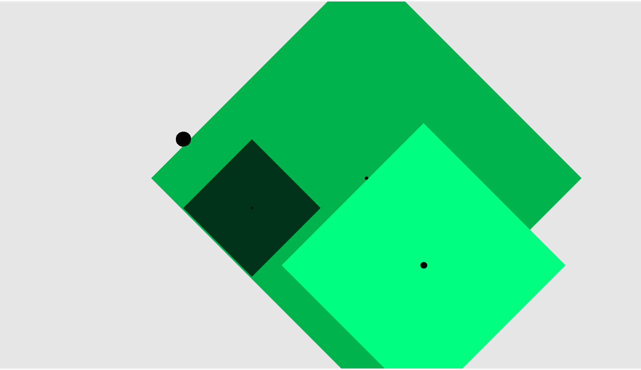

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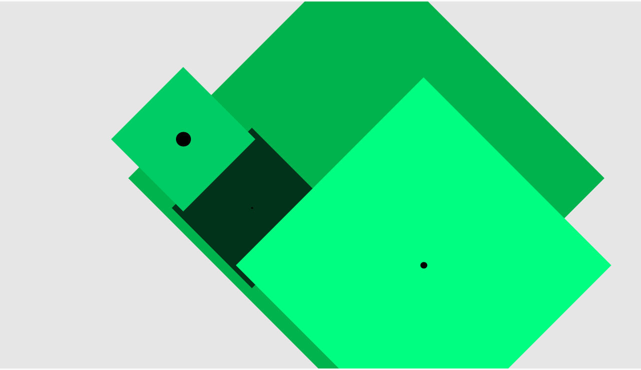

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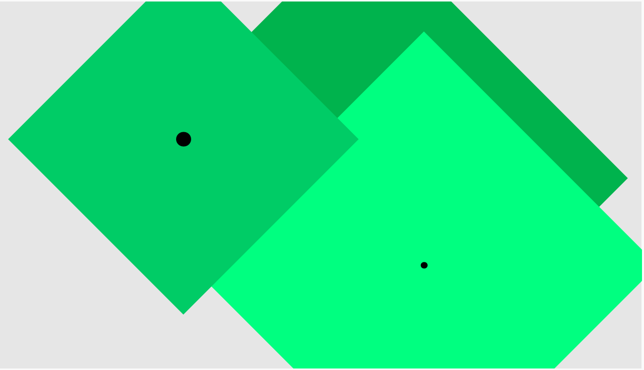

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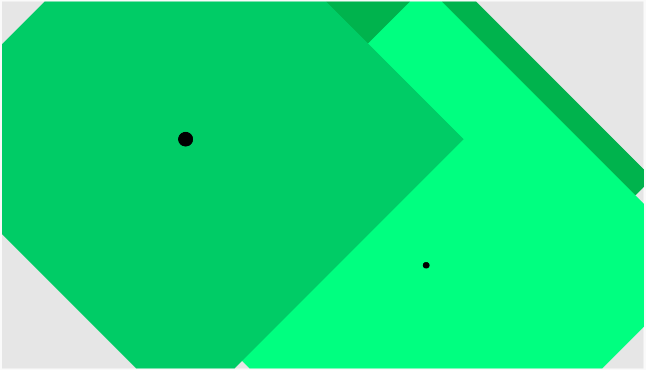

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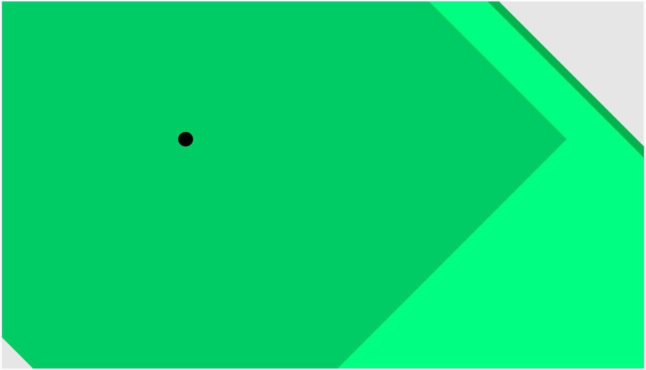

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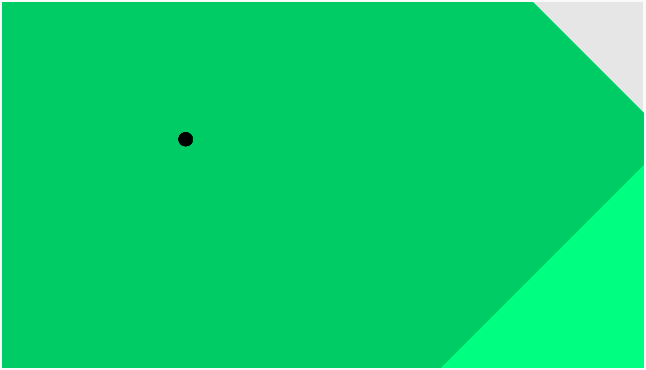

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Let $h(0) = 0$, and we define the hitting time of $z \in \mathbb{R}^d$ by the **lilypad model** as

$$h(z) = \inf\Big(\sum_{j=0}^{\infty} q\,\frac{|y_{j+1} - y_j|}{\xi(y_{j+1})}\Big),$$

where $|\cdot|$ is the ℓ^1-norm and the inf is over all sequences (y_i) with $y_0 = z$ and $(y_i, \xi(y_i)) \in \Pi, i \geq 1$ such that $|y_n| \to 0$.

- Need to show this is well-defined.
- Support and number of particles are corollaries.

Balance between spatial and temporal scale

Claim

'Lilypad' of particles spreads out at speed proportional to $\xi(z)$.

Recall that we rescale our systems

$$\text{space } r_T = \left(\frac{T}{\log T}\right)^{q+1} \quad \text{potential } a_T = \left(\frac{T}{\log T}\right)^{q}.$$

We start in a point $r_T x$ with potential of size $\xi_T(x) = \xi(r_T x)/a_T \asymp 1$ and assume there are no further good points nearby. When do we reach a point $r_T z$?

$$\begin{aligned}\mathbb{E}_{r_T x}[N(tT, r_T z)] &\approx e^{\xi_T(x) a_T t T} \mathbb{P}_{r_T x}\{ \text{ reach } r_T z \text{ in time } o(tT)\} \\ &\approx e^{\xi_T(x) a_T t T} e^{-q|z-x| r_T \log T} \\ &= e^{a(T) T (\xi_T(x) t - q|z-x|)}.\end{aligned}$$

We reach the point z when this expectation is ≈ 1, i.e. at time

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In particular, this shows that r_T is the right spatial scaling.

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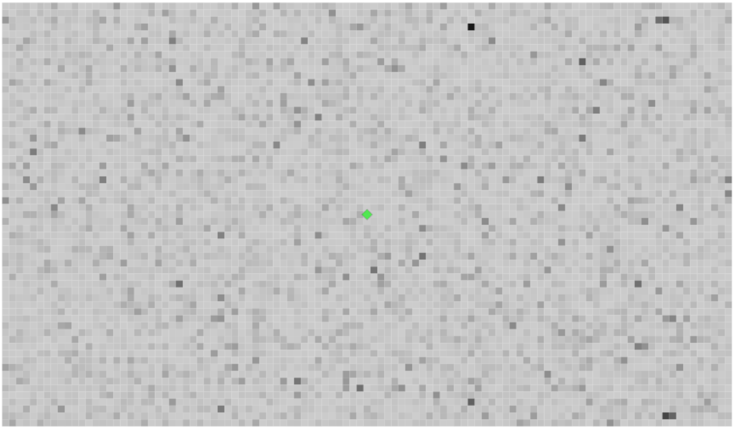
$$s(t) := \{z \in \mathbb{R}^d \; : \; h(z) \leq t\}.$$



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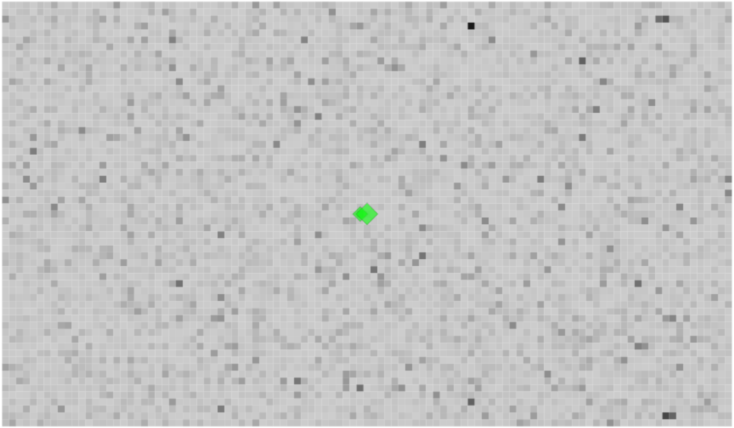

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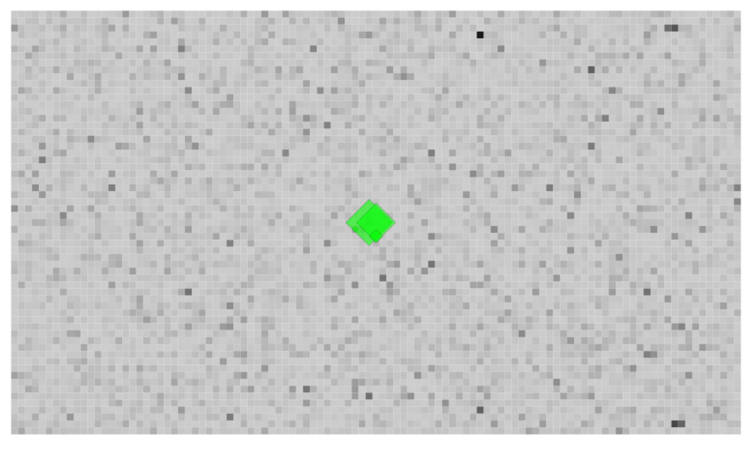

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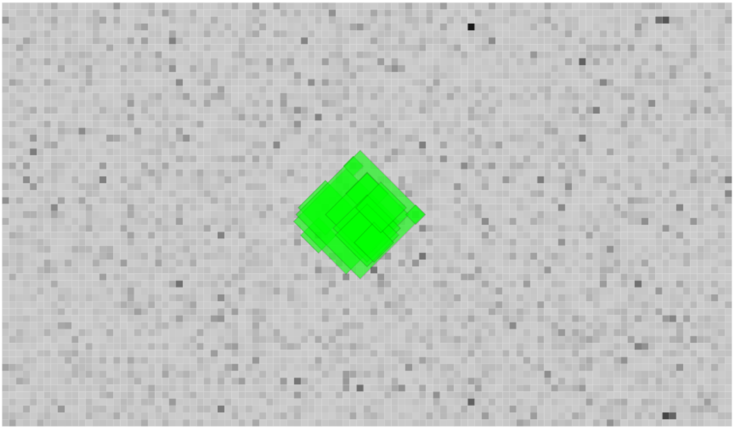

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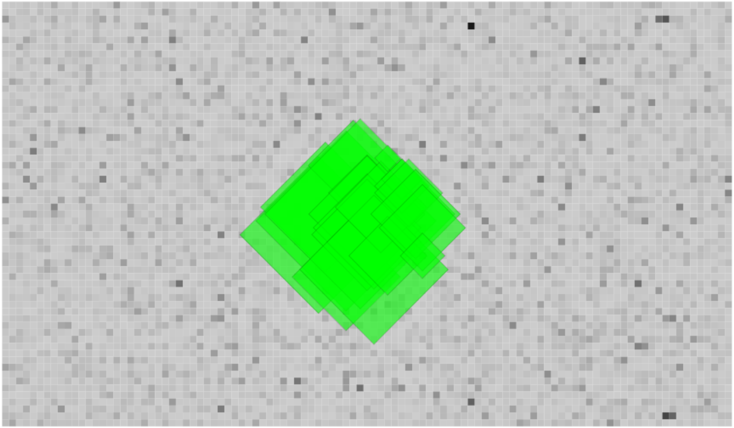

Pictures: The support in $d = 2$

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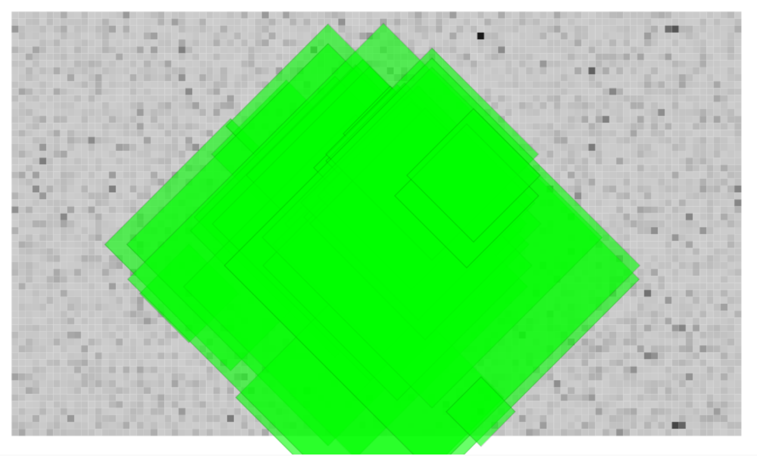

Pictures: The number of particles

The (log-)number of particles $m(t, z)$ follows two rules:

- If z is a site with high potential, number of particles start growing at rate $\xi(z)$ as soon as z is hit.
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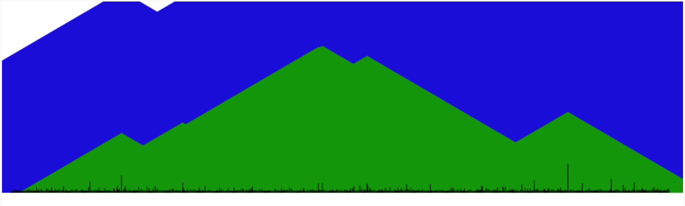
Thus,

$$m(t, z) = \xi(z)(t - h(z)).$$



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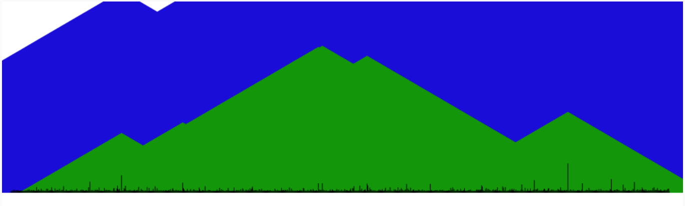

Comments on scaling limit

- Limit is random in contrast to earlier work on BRWRE [Comets, Popov '07], but also not of SDE/SPDE-type.
- Corollary: Log of number of particles at site is random in leading order!
- We call the limit process the **lilypad process**.
- Lilypads grow like ℓ^1-balls:
  - Reason is that the front is driven by extreme large deviation events (underlying RW talkes $\gg T$ steps in time T).
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Proof of scaling limit

Step 1: Decoupling the randomness:

- Define a discrete lilypad process in terms of the point process

$$\Pi_T = \sum_{z \in \mathbb{Z}^d} \delta_{\left(\frac{z}{r_T}, \frac{\xi(z)}{a_T}\right)}.$$

We show in [O. and Roberts '16] that the branching random walks hitting times are well approximated by the hitting times in the discrete lilypad process (which only depend on the environment!)
  - Use moments, but starting from a good point!
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- It remains to show that the discrete lilypad model converges.

Step 2: Continuous mapping theorem:

- Since $\Pi_T \Rightarrow \Pi$, any continuous functional of the point process will also converge.
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$$\frac{u(t, Z_t^{\mathrm{PAM}})}{\sum_{z\in\mathbb{Z}^d} u(t,z)} \to 1 \text{ in probability as } t \to \infty.$$

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Recall that we write $N(t,z)$ for the number of particles at site z at time t.

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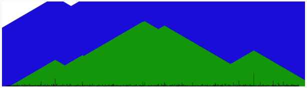

This implies that there is localisation in the rescaled picture, i.e. there exists $\varepsilon > 0$ and a process Z_t such that

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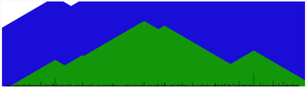

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  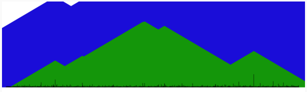

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- Recall: The solution $u(t,x)$ of the parabolic Anderson model describes the **expected number** of particles in the branching random walk (when averaging over branching/migration).

Our methods also give a scaling limit for

$$\Lambda_T(t,z) = \frac{1}{a_T T} \log u(tT, r_T z), \quad z \in \mathbb{R}^d$$

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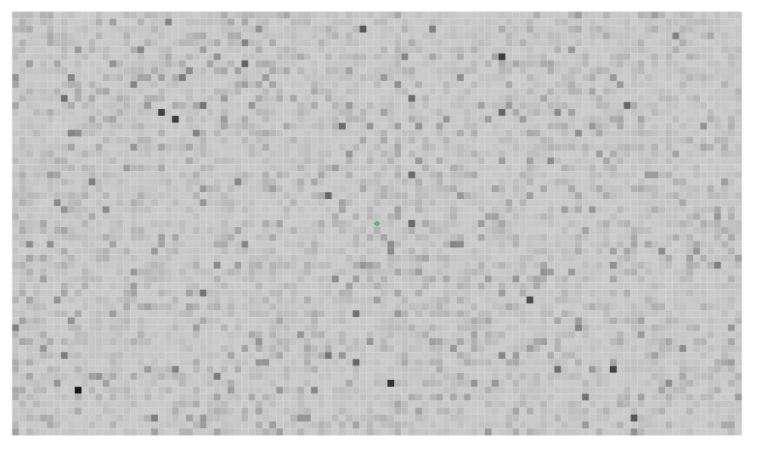

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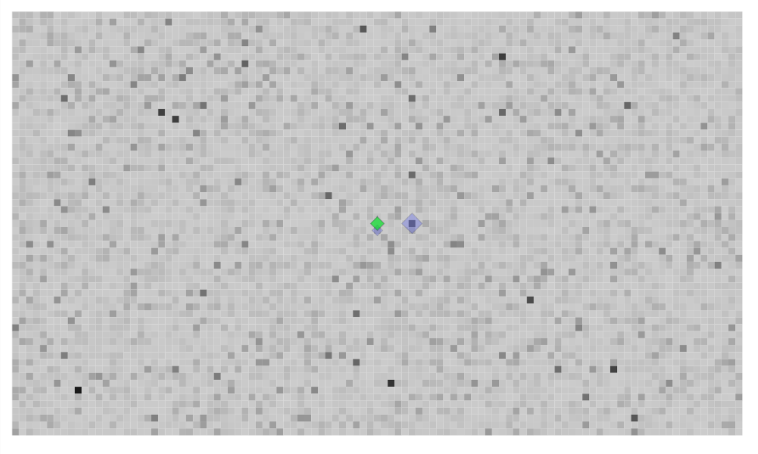

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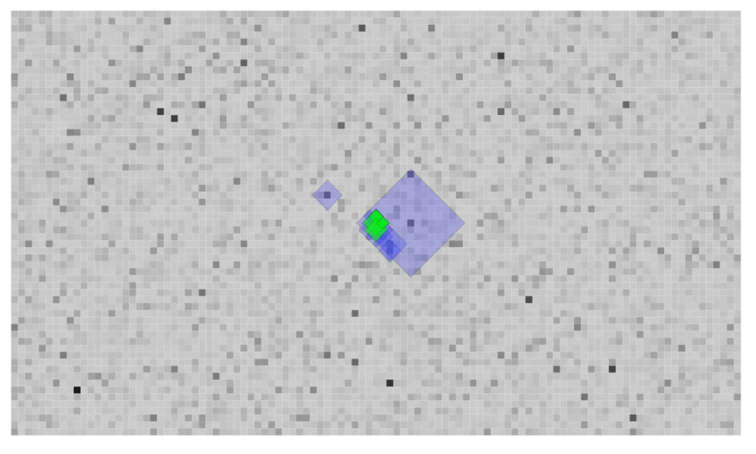

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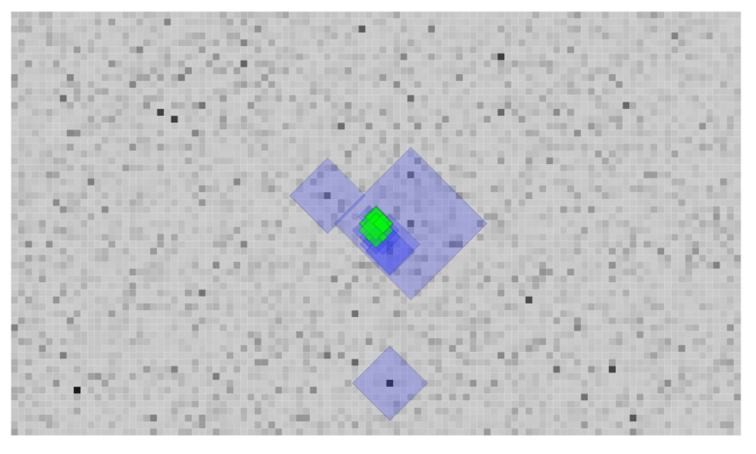

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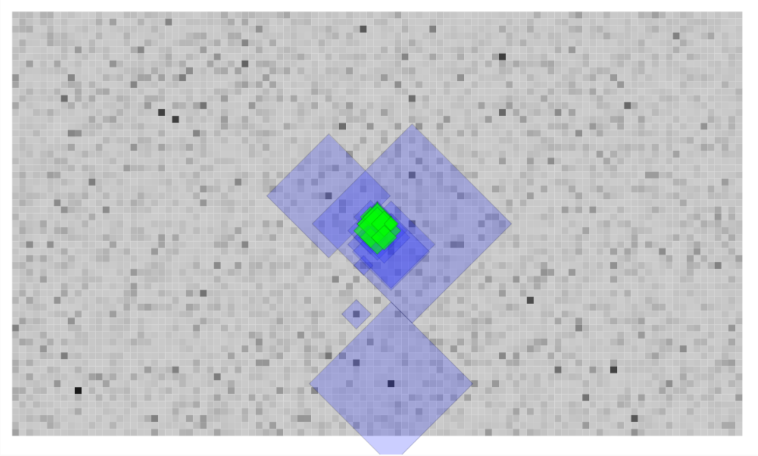

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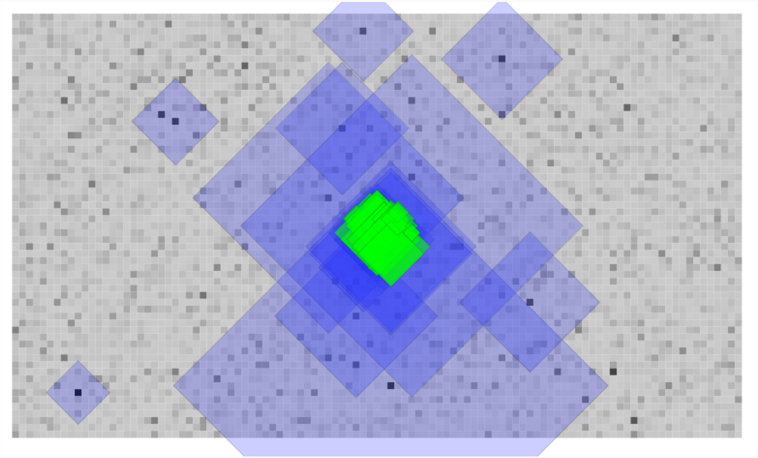

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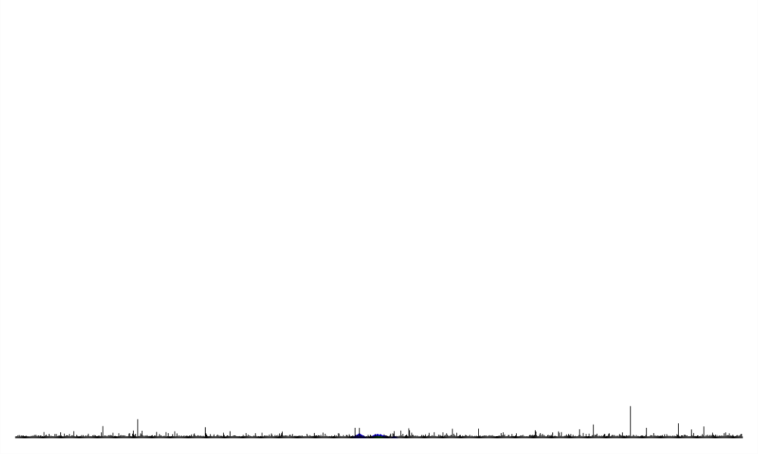

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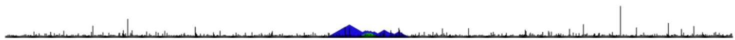

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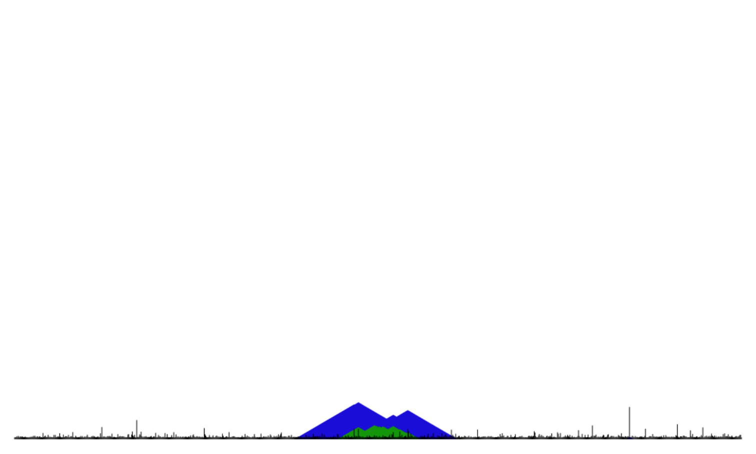

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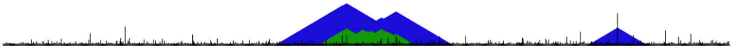

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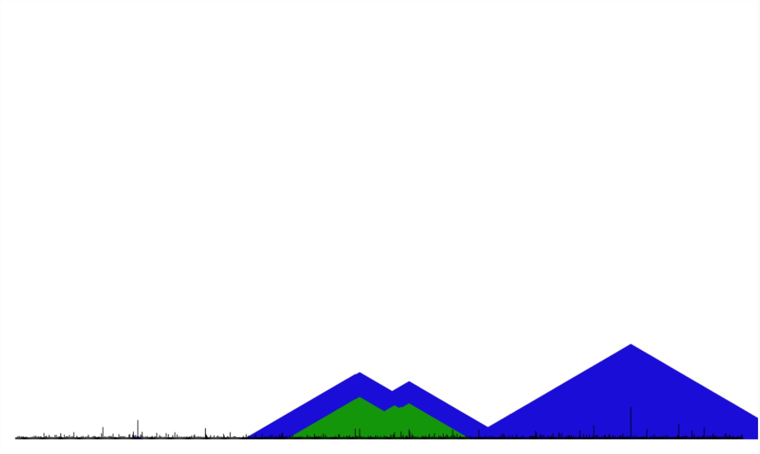

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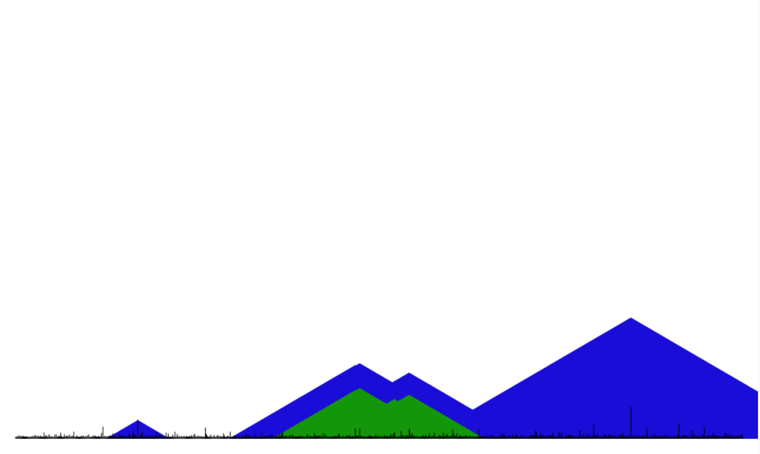

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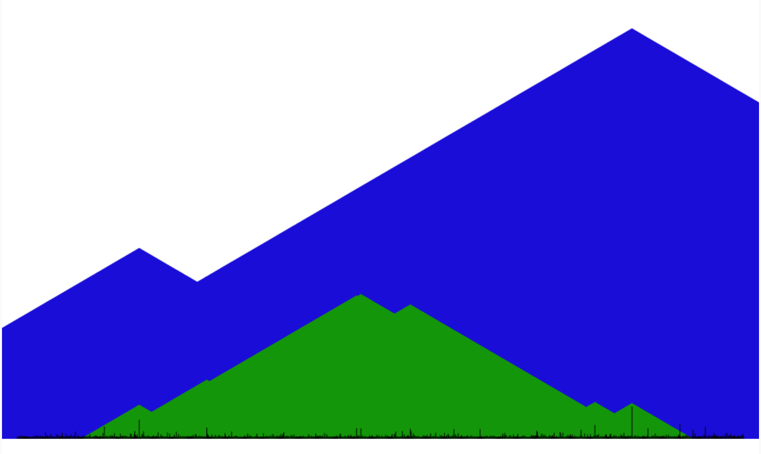

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Localisation and asymptotics of total mass of the parabolic Anderson model well understood:

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Extreme value theory tells us to rescale differently this time:

Spatial rescaling:

$$r_T = \frac{T(\log T)^{\frac{1}{\gamma}-1}}{\log\log T}.$$

For the potential we need:

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converges to a Poisson point process on $\mathbb{R}^d \times \mathbb{R}$.

Note the leading order of maximal value of Π_T on a compact set is **deterministic**!

Also it is known that there exists Z_T^1:

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Note the leading order of maximal value of Π_T on a compact set is **deterministic**!

Also it is known that there exists Z^1_T:

$$\frac{1}{T}\log\sum_z u(T,z) \sim \frac{1}{T}\log u(T,Z^1_T) \sim a_T + b_T \text{ random term.}$$

Rescaling the environment

Extreme value theory tells us to rescale differently this time:

Spatial rescaling:

$$r_T = \frac{T(\log T)^{\frac{1}{\gamma}-1}}{\log\log T}.$$

For the potential we need:

$$a_T = (d\log r_T)^{\frac{1}{\gamma}}, \quad b_T = (d\log r_T)^{\frac{1}{\gamma}-1}.$$

Then, the rescaled point process

$$\Pi_T = \sum_{z\in\mathbb{Z}^d} \delta_{\left(\frac{z}{r_T}, \frac{\xi(z)-a_T}{b_T}\right)},$$

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Our work in progress

Q: Are BRW and PAM still different?

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For Weibull potential with γ small, we have that

$$\frac{1}{Tb_T}\Big(\log\sum_z u(T,z) - \log\sum_z N(T,z)\Big) \to 0,$$

in probability. I.e. PAM and BRW agree to first orders (including the random term).

Moreover, there exists $\varepsilon > 0$ and a site X_T with

$$|X_T| \geq r_T \log\log(T)^{\varepsilon}.$$

such that $N(T, X_T) \geq 1$.

- Recall for the maximizer in the PAM $|Z_T^1|/r_T$ converges.
- So the support of the BRW grows on different scale from maximizer.
- Claim: On the scale of the maximizer, there are particles everywhere.

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Proof idea for Weibull case

Identify the optimal strategy for BRW:

- Try to get to a good site z with $z_T := z/r_T$ and $\xi_T(z) = \frac{\xi(z)-a_T}{b_T}$ of order one.
- Taking the route via a decent site w near the origin, we can show that the first particle arrives at z no later than

$$\frac{|z_T|}{\gamma d^{1/\gamma}} \frac{T}{\log T}.$$

- Then, by time T, we have at least the following number of particles:

$$\exp\Big\{\xi(z)\Big(T - \frac{|z_T|}{\gamma d^{1/\gamma}} \frac{T}{\log T}\Big)\Big\}$$
$$= \exp\Big\{a_T T + b_T T\Big(\xi_T(z) - \frac{|z_T|}{\gamma d^{1/\gamma - 1}}\Big) + o(b_T T)\Big\}$$

- This gives the same optimization problem as for the PAM.

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Conjecture:

For the parabolic Anderson model / branching random walks:

$$\log u(tT, r_T x) \sim tTa_T + Tb_T\Lambda_T(t,x),$$

where Λ_T converges to the following functional of a Poisson point process (taking a supremum at each spatial position):

$$\Lambda(t,x) = \sup_{z\in\Pi}\left\{t\xi(z) - \frac{|z-x|}{\gamma d^{1/\gamma-1}}\right\}.$$

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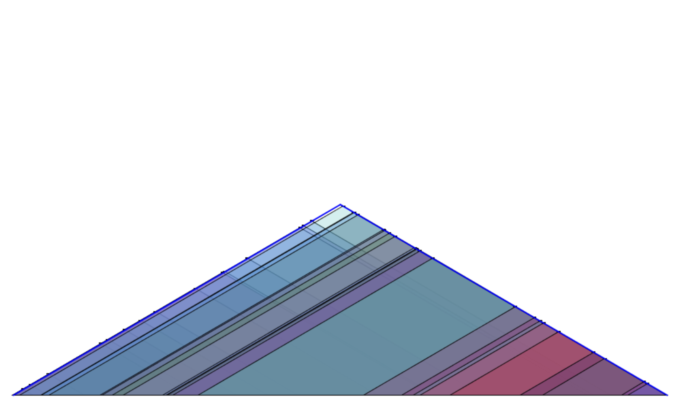

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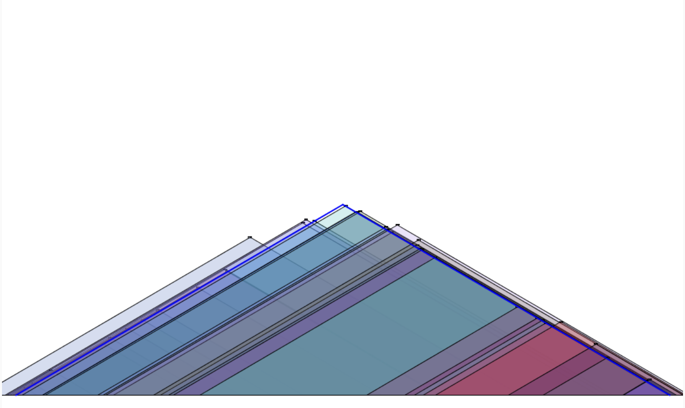

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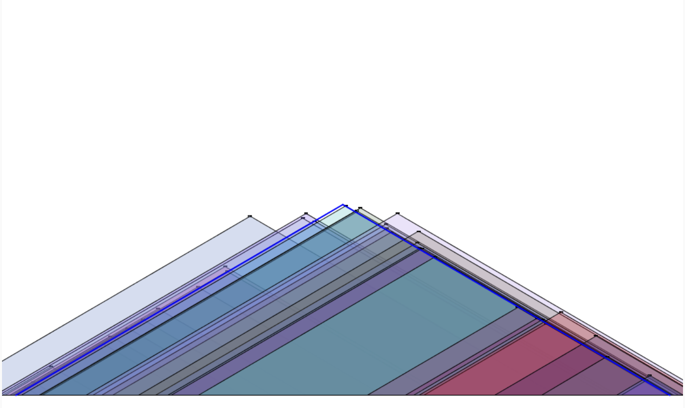

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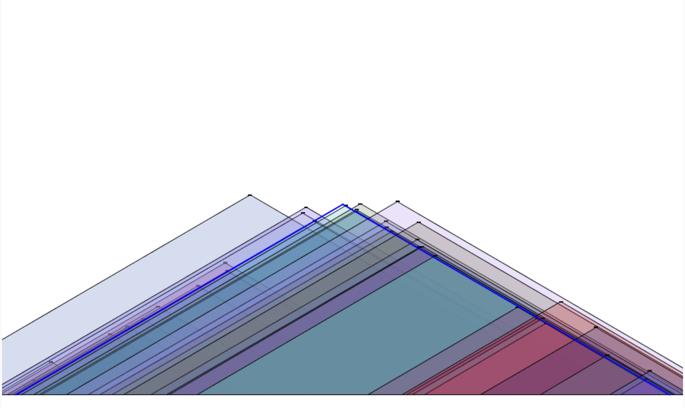

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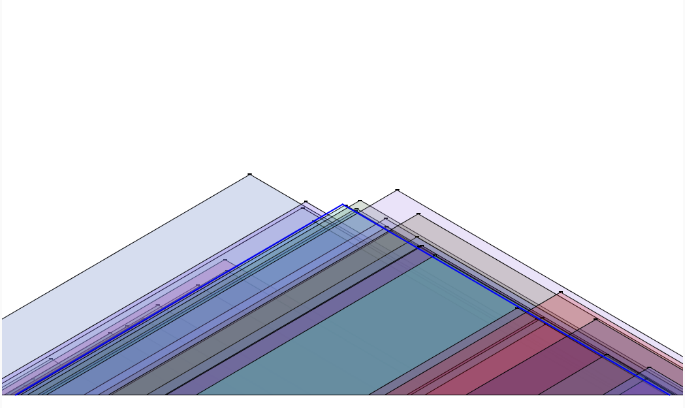

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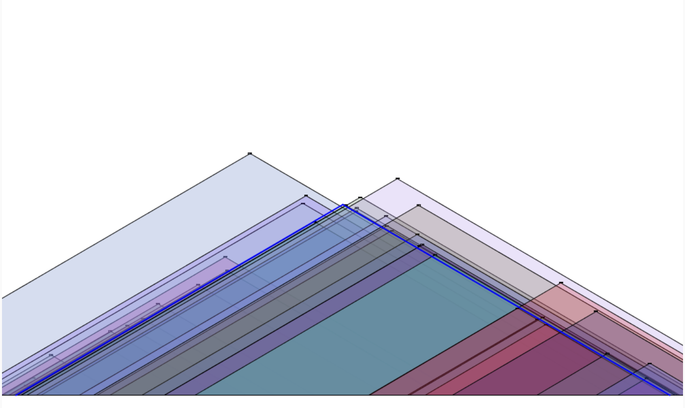

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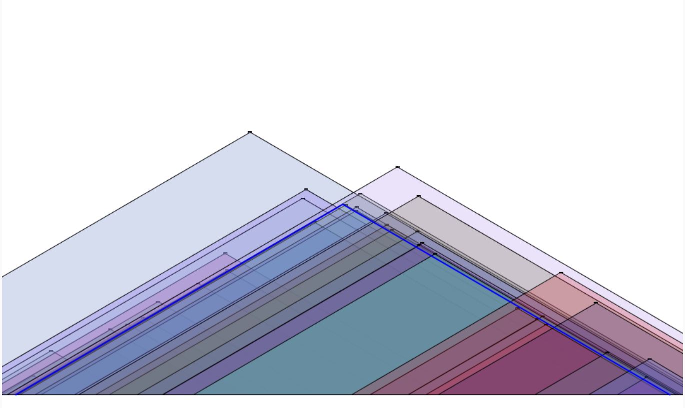

Open problems:

For branching random walks in random environment

- Double exponential potential?
- Branching rate 1 and (soft or hard) killing according to random potential?
  $\rightsquigarrow$ corresponds to parabolic Anderson model with bounded potential. [Engländer 2011, 2015]
- Correlated potentials? $\rightsquigarrow$ any new effects?

Related (more realistic) models of population growth in random environment:

- In Pareto case: the population growth is super-exponential and front of particles is driven by extreme large-deviations events.
- Is there an interesting model with more realistic particle behaviour that shows similar effect as our lilypad model?
- Incorporate local competition to restrain population growth?

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