Branching random walks in random environment

Marcel Ortgiese

Joint work with Matt Roberts (Bath)

LMS Durham Symposium
20 August 2018
Branching random walks in a random potential

Branching random walks

- **Motion:** Start with single particle at the origin that performs a simple random walk on $\mathbb{Z}^d$ (in continuous time).
- **Branching:** After an exponential waiting time, the particle splits into two new particles.
- The new particles behave independently (**no interaction**).

in a random potential:

- the potential $\{\xi(z), z \in \mathbb{Z}^d\}$ is a collection of i.i.d. non-negative random variables.
- **Modification:** when at site $z$, particles branch at rate $\xi(z)$.

Note: Other models introduce a random offspring distribution instead of changing the rates, e.g. space i.i.d., time i.i.d. or space-time i.i.d.
Branching random walks

- **Motion**: Start with a single particle at the origin that performs a simple random walk on $\mathbb{Z}^d$ (in continuous time).
- **Branching**: After an exponential waiting time, the particle splits into two new particles.
- The new particles behave independently (**no interaction**).

**in a random potential:**

- The potential $\{\xi(z), z \in \mathbb{Z}^d\}$ is a collection of i.i.d. non-negative random variables.
- **Modification**: When at site $z$, particles branch at rate $\xi(z)$.

Note: Other models introduce a random offspring distribution instead of changing the rates, e.g. space i.i.d., time i.i.d. or space-time i.i.d.
Branching random walks

- **Motion**: Start with a single particle at the origin that performs a simple random walk on $\mathbb{Z}^d$ (in continuous time).

- **Branching**: After an exponential waiting time, the particle splits into two new particles.

- The new particles behave independently (no interaction).

**in a random potential**: 

- the potential $\{\xi(z), z \in \mathbb{Z}^d\}$ is a collection of i.i.d. non-negative random variables.

- **Modification**: when at site $z$, particles branch at rate $\xi(z)$.

Note: Other models introduce a random offspring distribution instead of changing the rates, e.g. space i.i.d., time i.i.d. or space-time i.i.d.
Typical questions:

Start with one particle at the origin, then we can ask:

- How far do particles spread by time $t$?
- Equivalently: when do faraway sites $z$ get hit?
- What does the height profile look like, i.e. how many particles $N(t, z)$ are there at site $z$ at time $t$?

More specifically:

- We are interested in large scale behaviour $\sim$ scaling limit?
- Can we describe the site with the maximal number of particles?

Need to understand:

1. The role of **averaging**:
   - over the environment.
   - over the branching/migration mechanism.
2. The competition between the benefit of **high peaks** vs. **cost of getting there**.
Typical questions:

Start with one particle at the origin, then we can ask:

- How far do particles spread by time $t$?
- Equivalently: when do faraway sites $z$ get hit?
- What does the height profile look like, i.e. how many particles $N(t, z)$ are there at site $z$ at time $t$?

More specifically:

- We are interested in large scale behaviour $\sim$ scaling limit?
- Can we describe the site with the maximal number of particles?

Need to understand:

1. The role of **averaging**:
   - over the environment.
   - over the branching/migration mechanism.

2. The competition between the benefit of **high peaks** vs. **cost of getting there**.
Typical questions:

Start with one particle at the origin, then we can ask:

- How far do particles spread by time $t$?
- Equivalently: when do faraway sites $z$ get hit?
- What does the height profile look like, i.e. how many particles $N(t, z)$ are there at site $z$ at time $t$?

More specifically:

- We are interested in large scale behaviour $\sim$ scaling limit?
- Can we describe the site with the maximal number of particles?

Need to understand:

1. The role of **averaging**:
   - over the environment.
   - over the branching/migration mechanism.

2. The competition between the benefit of **high peaks** vs. **cost of getting there**.
Typical questions:

Start with one particle at the origin, then we can ask:

- How far do particles spread by time $t$?
- Equivalently: when do faraway sites $z$ get hit?
- What does the height profile look like, i.e. how many particles $N(t, z)$ are there at site $z$ at time $t$?

More specifically:

- We are interested in large scale behaviour $\sim$ scaling limit?
- Can we describe the site with the maximal number of particles?

Need to understand:

1. The role of **averaging**:
   - over the environment.
   - over the branching/migration mechanism.
2. The competition between the benefit of **high peaks** vs. **cost of getting there**.
No migration: Consider a branching process, where particles split at rate $r$, but there is no migration. The expected number of particles $u_t$ satisfies

$$\frac{d}{dt} u_t = r u_t.$$  

I.e. if we start with one particle, $u_t = e^{rt}$. 

Branching random walk with homogeneous branching rate. Suppose $\xi(x) \equiv r$ for all $x \in \mathbb{Z}^d$. A first moment calculation shows that:

Particle growth in constant environment

Particles spread in a ball of radius growing linearly in $t$.

More interesting questions: corrections to linear growth term.
**Branching random walk with constant branching rate**

**No migration:** Consider a branching process, where particles split at rate $r$, but there is no migration. The expected number of particles $u_t$ satisfies

$$ \frac{d}{dt} u_t = r u_t. $$

I.e. if we start with one particle, $u_t = e^{rt}$.

**Branching random walk with homogeneous branching rate.** Suppose $\xi(x) \equiv r$ for all $x \in \mathbb{Z}^d$. A first moment calculation shows that:

**Particle growth in constant environment**

Particles spread in a ball of radius growing linearly in $t$.

More interesting questions: corrections to linear growth term.
No migration: Consider a branching process, where particles split at rate $r$, but there is no migration. The expected number of particles $u_t$ satisfies
\[ \frac{d}{dt} u_t = r u_t. \]
I.e. if we start with one particle, $u_t = e^{rt}$.

Branching random walk with homogeneous branching rate. Suppose $\xi(x) \equiv r$ for all $x \in \mathbb{Z}^d$. A first moment calculation shows that:

Particle growth in constant environment
Particles spread in a ball of radius growing linearly in $t$.

More interesting questions: corrections to linear growth term.
Fix the (inhomogeneous) potential \( \xi \), let

\[
    u(t, x) = E^\xi[\#\{\text{particles at site } x \text{ at time } t \}]
\]

for \( t \geq 0, x \in \mathbb{Z}^d \). Then \( u \) solves the following equation that defines the parabolic Anderson model

\[
    \frac{\partial}{\partial t} u(t, z) = \Delta u(t, z) + \xi(z)u(t, z),
\]

\[
    u(0, z) = 1_{0}(z),
\]

where \( \Delta \) is the discrete Laplacian, defined as

\[
    \Delta f(x) = \sum_{y \in \mathbb{Z}^d : y \sim x} (f(y) - f(x)),
\]

and \( y \sim x \) if \( y \) is a neighbour of \( x \).

Lots of research activity during the last 20 years in particular by

[Donsker, Varadhan, Gärtner, Molchanov, Sznitman, Antal, Carmona, den Hollander, Biskup, König, van der Hofstad, Mörters, Sidorova, Lacoin, O., Schnitzler, Twarowski, Fiodorov, Muirhead, Chouk, Gairing, Perkowski, ...]
Averaging: The parabolic Anderson model

Fix the (inhomogeneous) potential $\xi$, let
\[
u(t, x) = E^{\xi}[\# \{ \text{particles at site } x \text{ at time } t \}]
\]
for $t \geq 0, x \in \mathbb{Z}^d$. Then $u$ solves the following equation that defines the parabolic Anderson model
\[
\frac{\partial}{\partial t} u(t, z) = \Delta u(t, z) + \xi(z) u(t, z),
\]
\[
u(0, z) = 1_{0}(z),
\]
where $\Delta$ is the discrete Laplacian, defined as
\[
\Delta f(x) = \sum_{y \in \mathbb{Z}^d : y \sim x} (f(y) - f(x)),
\]
and $y \sim x$ if $y$ is a neighbour of $x$.

Lots of research activity during the last 20 years in particular by
[Donsker, Varadhan, Gärtner, Molchanov, Sznitman, Antal, Carmona, den Hollander, Biskup, König, van der Hofstad, Mörters, Sidorova, Lacoin, O., Schnitzler, Twarowskii, Fiodorov, Muirhead, Chouk, Gairing, Perkowski, ...]
Averaging: The parabolic Anderson model

Fix the (inhomogeneous) potential $\xi$, let

$$u(t, x) = E^\xi[\# \{ \text{particles at site } x \text{ at time } t \}]$$

for $t \geq 0, x \in \mathbb{Z}^d$. Then $u$ solves the following equation that defines the **parabolic Anderson model**

$$\frac{\partial}{\partial t} u(t, z) = \Delta u(t, z) + \xi(z) u(t, z),$$
$$u(0, z) = 1_{l_0}(z),$$

where $\Delta$ is the discrete Laplacian, defined as

$$\Delta f(x) = \sum_{y \in \mathbb{Z}^d : y \sim x} (f(y) - f(x)),$$

and $y \sim x$ if $y$ is a neighbour of $x$.

Lots of research activity during the last 20 years in particular by

[Donsker, Varadhan, Gärtner, Molchanov, Sznitman, Antal, Carmona, den Hollander, Biskup, König, van der Hofstad, Mörters, Sidorova, Lacoin, O., Schnitzler, Twarowski, Fiodorov, Muirhead, Chouk, Gairing, Perkowski, ...]
Intermittency for the parabolic Anderson model

The main idea is to understand

**Intermittency**

The solution $u$ is concentrated in a **small** number of **remote** islands, where the potential $\xi$ is particularly large.

- The behaviour of the model depends crucially on the decay of the tail probability $\text{Prob}\{\xi(0) > x\} \sim \? \text{ for } x \rightarrow \infty$.

For this talk, we will focus on these:

**Example A:** $\xi$ has a Pareto distribution, for some $\alpha > 0$:

$$\text{Prob}\{\xi(0) > x\} = x^{-\alpha}.$$  

**Example B:** $\xi$ has a Weibull distribution, for some $\gamma > 0$:

$$\text{Prob}\{\xi(0) > x\} = e^{-x^\gamma}.$$
Intermittency for the parabolic Anderson model

The main idea is to understand

**Intermittency**

The solution $u$ is concentrated in a **small** number of **remote** islands, where the potential $\xi$ is particularly large.

- The behaviour of the model depends crucially on the decay of the tail probability $\text{Prob}\{\xi(0) > x\} \sim ?$ for $x \to \infty$.

For this talk, we will focus on these:

**Example A:** $\xi$ has a Pareto distribution, for some $\alpha > 0$:

$$\text{Prob}\{\xi(0) > x\} = x^{-\alpha}.$$

**Example B:** $\xi$ has a Weibull distribution, for some $\gamma > 0$:

$$\text{Prob}\{\xi(0) > x\} = e^{-x^\gamma}.$$
The main idea is to understand

**Intermittency**

The solution $u$ is concentrated in a **small** number of **remote** islands, where the potential $\xi$ is particularly large.

- The behaviour of the model depends crucially on the decay of the tail probability $\operatorname{Prob}\{\xi(0) > x\} \sim ?$ for $x \to \infty$.

For this talk, we will focus on these:

**Example A:** $\xi$ has a Pareto distribution, for some $\alpha > 0$:

$$\operatorname{Prob}\{\xi(0) > x\} = x^{-\alpha}.$$

**Example B:** $\xi$ has a Weibull distribution, for some $\gamma > 0$:

$$\operatorname{Prob}\{\xi(0) > x\} = e^{-x^\gamma}.$$
The main idea is to understand

**Intermittency**

The solution $u$ is concentrated in a **small** number of **remote** islands, where the potential $\xi$ is particularly large.

- The behaviour of the model depends crucially on the decay of the tail probability $\text{Prob}\{\xi(0) > x\} \sim ?$ for $x \to \infty$.

For this talk, we will focus on these:

**Example A:** $\xi$ has a Pareto distribution, for some $\alpha > 0$:

$$\text{Prob}\{\xi(0) > x\} = x^{-\alpha}.$$  

**Example B:** $\xi$ has a Weibull distribution, for some $\gamma > 0$:

$$\text{Prob}\{\xi(0) > x\} = e^{-x^\gamma}.$$
Previous work on parabolic Anderson model

**Theorem 1**

*For either Pareto potential ($\alpha > d$) or Weibull potential (any $\gamma > 0$), there exists a process $Z_t$ such that as $t \to \infty$,*

$$\frac{u(t, Z_t)}{\sum_z u(t, z)} \to 1, \text{ in probability.}$$

- Proved by [König, Lacoin, Mörters, Sidorova '09] – Pareto,
  [N. Sidorova, A. Twarowski '14] [Fiodorov, Muirhead '14] – Weibull.

- For lighter tails (double exponential), need a island of finite size that supports solution, [König, Biskup, dos Santos ’16].

Earlier results mostly concern asymptotics of expected total mass.

**Question**

Do these results help to understand the actual number of particles in the branching random walk?
Previous work on parabolic Anderson model

**Theorem 1**

*For either Pareto potential ($\alpha > d$) or Weibull potential (any $\gamma > 0$), there exists a process $Z_t$ such that as $t \to \infty$,*

$$\frac{u(t, Z_t)}{\sum_z u(t, z)} \to 1, \quad \text{in probability.}$$


  - For lighter tails (double exponential), need a island of finite size that supports solution, [König, Biskup, dos Santos ’16].

Earlier results mostly concern asymptotics of expected total mass.

**Question**

Do these results help to understand the actual number of particles in the branching random walk?
Previous work on parabolic Anderson model

**Theorem 1**

*For either Pareto potential \((\alpha > d)\) or Weibull potential (any \(\gamma > 0\)), there exists a process \(Z_t\) such that as \(t \to \infty\),

\[
\frac{u(t, Z_t)}{\sum_z u(t, z)} \to 1, \quad \text{in probability.}
\]*

- Proved by [König, Lacoin, Mörters, Sidorova ’09] – Pareto,
  [N. Sidorova, A. Twarowski ’14] [Fiodorov, Muirhead ’14] – Weibull.
- For lighter tails (double exponential), need a island of finite size that supports solution, [König, Biskup, dos Santos ’16].

Earlier results mostly concern asymptotics of expected total mass.

**Question**

Do these results help to understand the actual number of particles in the branching random walk?
### Previous work on parabolic Anderson model

#### Theorem 1

For either Pareto potential ($\alpha > d$) or Weibull potential (any $\gamma > 0$), there exists a process $Z_t$ such that as $t \to \infty$,

$$\frac{u(t, Z_t)}{\sum_z u(t, z)} \to 1, \text{ in probability.}$$

- Proved by [König, Lacoin, Mörters, Sidorova ’09] – Pareto,
  [N. Sidorova, A. Twarowski ’14] [Fiodorov, Muirhead ’14] – Weibull.

- For lighter tails (double exponential), need a island of finite size that supports solution, [König, Biskup, dos Santos ’16].

Earlier results mostly concern asymptotics of expected total mass.

### Question

Do these results help to understand the actual number of particles in the branching random walk?
Back to BRW: Controlling the environment

Main question: If a BRW manages to cover a ball of radius $r$ – what is the largest potential it has seen along the way?

How does $\max_{x \in B(0,r) \cap \mathbb{Z}^d} \xi(x)$ grow?

More precise question: What is the geometry of the peaks of the potential on large scales?

Extreme value theory tells us:

- Fix a large scaling parameter $T$.
- Assume $\xi$ is Pareto distributed, i.e. $\text{Prob}\{\xi(z) > x\} \sim x^{-\alpha}$, $\alpha > d$.
- For $q = \frac{d}{\alpha - d}$, introduce scaling for potential and space

$$a_T = \left( \frac{T}{\log T} \right)^q, \quad r_T = \left( \frac{T}{\log T} \right)^{q+1}.$$ 

Then, the rescaled environment converges:

$$\Pi_T := \sum_{z \in \mathbb{Z}^d} \delta\left(\frac{z}{r_T}, \frac{\xi(z)}{a_T}\right) \Rightarrow \Pi,$$

where $\Pi$ is a Poisson point process with intensity $\frac{\alpha}{y^{\alpha+1}} \, dz \otimes dy$. 
Main question: If a BRW manages to cover a ball of radius \( r \) – what is the largest potential it has seen along the way?

How does \( \max_{x \in B(0,r) \cap \mathbb{Z}^d} \xi(x) \) grow?

More precise question: What is the **geometry of the peaks** of the potential on large scales?

Extreme value theory tells us:

- Fix a large scaling parameter \( T \).
- Assume \( \xi \) is Pareto distributed, i.e. \( \text{Prob}\{\xi(z) > x\} \sim x^{-\alpha}, \, \alpha > d \).
- For \( q = \frac{d}{\alpha-d} \), introduce scaling for potential and space

\[
a_T = \left( \frac{T}{\log T} \right)^q, \quad r_T = \left( \frac{T}{\log T} \right)^{q+1}.
\]

Then, the rescaled environment converges:

\[
\Pi_T := \sum_{z \in \mathbb{Z}^d} \delta\left(\frac{z}{a_T}, \frac{\xi(z)}{r_T}\right) \Rightarrow \Pi,
\]

where \( \Pi \) is a Poisson point process with intensity \( \frac{\alpha}{y^{\alpha+1}} \, dz \otimes dy \).
Main question: If a BRW manages to cover a ball of radius $r$ – what is the largest potential it has seen along the way?

How does $\max_{x \in B(0,r) \cap \mathbb{Z}^d} \xi(x)$ grow?

More precise question: What is the geometry of the peaks of the potential on large scales?

Extreme value theory tells us:

- Fix a large scaling parameter $T$.
- Assume $\xi$ is Pareto distributed, i.e. $\text{Prob}\{\xi(z) > x\} \sim x^{-\alpha}$, $\alpha > d$.
- For $q = \frac{d}{\alpha - d}$, introduce scaling for potential and space

$$a_T = \left(\frac{T}{\log T}\right)^q, \quad r_T = \left(\frac{T}{\log T}\right)^{q+1}.$$ 

Then, the rescaled environment converges:

$$\Pi_T := \sum_{z \in \mathbb{Z}^d} \delta\left(\frac{z}{r_T}, \frac{\xi(z)}{a_T}\right) \Rightarrow \Pi,$$

where $\Pi$ is a Poisson point process with intensity $\frac{\alpha}{y^{\alpha+1}} dz \otimes dy$. 


Main question: If a BRW manages to cover a ball of radius $r$ – what is the largest potential it has seen along the way?

How does $\max_{x \in B(0, r) \cap \mathbb{Z}^d} \xi(x)$ grow?

More precise question: What is the geometry of the peaks of the potential on large scales?

Extreme value theory tells us:

- Fix a large scaling parameter $T$.
- Assume $\xi$ is Pareto distributed, i.e. $\text{Prob}\{\xi(z) > x\} \sim x^{-\alpha}$, $\alpha > d$.
- For $q = \frac{d}{\alpha - d}$, introduce scaling for potential and space

\[a_T = \left(\frac{T}{\log T}\right)^q, \quad r_T = \left(\frac{T}{\log T}\right)^{q+1}.\]

Then, the rescaled environment converges:

\[\Pi_T := \sum_{z \in \mathbb{Z}^d} \delta\left(\frac{z}{r_T}, \frac{\xi(z)}{a_T}\right) \Rightarrow \Pi,\]

where $\Pi$ is a Poisson point process with intensity $\frac{\alpha}{y^{\alpha+1}} dz \otimes dy$. 
Main question: If a BRW manages to cover a ball of radius $r$ – what is the largest potential it has seen along the way?

How does $\max_{x \in B(0,r) \cap \mathbb{Z}^d} \xi(x)$ grow?

More precise question: What is the **geometry of the peaks** of the potential on large scales?

Extreme value theory tells us:

- Fix a large scaling parameter $T$.
- Assume $\xi$ is Pareto distributed, i.e. $\text{Prob}\{\xi(z) > x\} \sim x^{-\alpha}$, $\alpha > d$.
- For $q = \frac{d}{\alpha-d}$, introduce scaling for potential and space

$$a_T = \left(\frac{T}{\log T}\right)^q, \quad r_T = \left(\frac{T}{\log T}\right)^{q+1}.$$

Then, the rescaled environment converges:

$$\Pi_T := \sum_{z \in \mathbb{Z}^d} \delta\left(\frac{z}{r_T}, \frac{\xi(z)}{a_T}\right) \Rightarrow \Pi,$$

where $\Pi$ is a Poisson point process with intensity $\frac{\alpha}{y^{\alpha+1}} \, dz \otimes dy$. 
Main question: If a BRW manages to cover a ball of radius $r$ – what is the largest potential it has seen along the way?

How does $\max_{x \in B(0,r) \cap \mathbb{Z}^d} \xi(x)$ grow?

More precise question: What is the geometry of the peaks of the potential on large scales?

Extreme value theory tells us:

- Fix a large scaling parameter $T$.
- Assume $\xi$ is Pareto distributed, i.e. $\text{Prob}\{\xi(z) > x\} \sim x^{-\alpha}, \alpha > d$.
- For $q = \frac{d}{\alpha - d}$, introduce scaling for potential and space
  
  \[ a_T = \left( \frac{T}{\log T} \right)^q, \quad r_T = \left( \frac{T}{\log T} \right)^{q+1}. \]

Then, the rescaled environment converges:

\[ \Pi_T := \sum_{z \in \mathbb{Z}^d} \delta\left( \frac{z}{r_T}, \frac{\xi(z)}{a_T} \right) \Rightarrow \Pi, \]

where $\Pi$ is a Poisson point process with intensity $\frac{\alpha}{y^{\alpha+1}} \, dz \otimes dy$. 
Then, the rescaled environment converges:

\[ \Pi_T := \sum_{z \in \mathbb{Z}^d} \delta_{(\frac{z}{r_T}, \frac{\xi(z)}{a_T})} \Rightarrow \Pi, \]

where \( \Pi \) is a Poisson point process with intensity \( \frac{\alpha}{y^{\alpha+1}} \, dz \otimes dy \).
Then, the rescaled environment converges:

$$\Pi_T := \sum_{z \in \mathbb{Z}^d} \delta\left(\frac{z}{r_T}, \frac{\xi(z)}{a_T}\right) \Rightarrow \Pi,$$

where $\Pi$ is a Poisson point process with intensity $\frac{\alpha}{y^{\alpha + 1}} \, dz \otimes dy$. 

Then, the rescaled environment converges:

$$\Pi_T := \sum_{z \in \mathbb{Z}^d} \delta\left(\frac{z}{r_T}, \frac{\xi(z)}{a_T}\right) \Rightarrow \Pi,$$

where $\Pi$ is a Poisson point process with intensity $\frac{\alpha}{y^{\alpha+1}} \, dz \otimes dy$. 
Then, the rescaled environment converges:

\[ \Pi_T := \sum_{z \in \mathbb{Z}^d} \delta\left(\frac{z}{r_T}, \frac{\xi(z)}{a_T}\right) \Rightarrow \Pi, \]

where \( \Pi \) is a Poisson point process with intensity \( \frac{\alpha}{y^{\alpha+1}} \, dz \otimes dy \).
Main result: a scaling limit

Consider for $z \in r_1^{-1}\mathbb{Z}^d$, $t \geq 0$:

- **Hitting times:** $H_T(z) = \inf \{ t \geq 0 : N(tT, r_Tz) \geq 1 \}$,

- **Support:** $S_T(t) = \{ z \in \mathbb{R}^d : H_T(z) \leq t \}$

- **Rescaled number of particles:** $M_T(t, z) = \frac{1}{a_T} \log_+ N(tT, r_Tz)$

with interpolation for $z \notin r_1^{-1}\mathbb{Z}^d$.

**Theorem 2 (O., Roberts '16, '18)**

The triple

$$\left( (H_T(z))_{z \in \mathbb{R}^d}, (S_T(t))_{t \geq 0}, (M_T(t, z))_{t \geq 0, z \in \mathbb{R}^d} \right)$$

converges in distribution (in a suitable topology) to

$$(h_\Pi, s_\Pi, m_\Pi) = ((h_\Pi(z))_{z \in \mathbb{R}^d}, (s_\Pi(t))_{t \geq 0}, (m_\Pi(t, z))_{t \geq 0, z \in \mathbb{R}^d})$$

where the limiting object is a functional of the Poisson point process $\Pi$. 
Main result: a scaling limit

Consider for \( z \in r_T^{-1}\mathbb{Z}^d, t \geq 0: \)

- **Hitting times:** \( H_T(z) = \inf\{t \geq 0 : N(tT, r_Tz) \geq 1\}, \)
- **Support:** \( S_T(t) = \{z \in \mathbb{R}^d : H_T(z) \leq t\} \)
- **Rescaled number of particles:** \( M_T(t, z) = \frac{1}{a_T} \log_+ N(tT, r_Tz) \)

with interpolation for \( z \notin r_T^{-1}\mathbb{Z}^d \).

**Theorem 2 (O., Roberts ’16, ’18)**

The triple

\[
\left((H_T(z))_{z \in \mathbb{R}^d}, (S_T(t))_{t \geq 0}, (M_T(t, z))_{t \geq 0, z \in \mathbb{R}^d}\right),
\]

converges in distribution (in a suitable topology) to

\[
(h_\Pi, s_\Pi, m_\Pi) = \left(((h_\Pi(z))_{z \in \mathbb{R}^d}, (s_\Pi(t))_{t \geq 0}, (m_\Pi(t, z))_{t \geq 0, z \in \mathbb{R}^d}\right),
\]

where the limiting object is a functional of the Poisson point process \( \Pi \).
Main result: a scaling limit

Consider for \( z \in r_T^{-1}\mathbb{Z}^d, t \geq 0 \):

**Hitting times:** \( H_T(z) = \inf\{t \geq 0 : N(tT, r_Tz) \geq 1\} \),

**Support:** \( S_T(t) = \{z \in \mathbb{R}^d : H_T(z) \leq t\} \)

**Rescaled number of particles:** \( M_T(t, z) = \frac{1}{aT}\log_+ N(tT, r_Tz) \)

with interpolation for \( z \notin r_T^{-1}\mathbb{Z}^d \).

**Theorem 2 (O., Roberts '16, '18)**

The triple

\[
\left( (H_T(z))_{z \in \mathbb{R}^d}, (S_T(t))_{t \geq 0}, (M_T(t, z))_{t \geq 0, z \in \mathbb{R}^d} \right),
\]

converges in distribution (in a suitable topology) to

\[
(h_\Pi, s_\Pi, m_\Pi) = \left( (h_\Pi(z))_{z \in \mathbb{R}^d}, (s_\Pi(t))_{t \geq 0}, (m_\Pi(t, z))_{t \geq 0, z \in \mathbb{R}^d} \right),
\]

where the limiting object is a functional of the Poisson point process \( \Pi \).
Main result: a scaling limit

Consider for \( z \in r_T^{-1}\mathbb{Z}^d, t \geq 0 \):

**Hitting times:**
\[
H_T(z) = \inf \{ t \geq 0 : N(tT, r_T z) \geq 1 \}
\]

**Support:**
\[
S_T(t) = \{ z \in \mathbb{R}^d : H_T(z) \leq t \}
\]

**Rescaled number of particles:**
\[
M_T(t, z) = \frac{1}{a_T} \log_+ N(tT, r_T z)
\]

with interpolation for \( z \notin r_T^{-1}\mathbb{Z}^d \).

**Theorem 2 (O., Roberts ’16, ’18)**

The triple
\[
\left( (H_T(z))_{z \in \mathbb{R}^d}, (S_T(t))_{t \geq 0}, (M_T(t, z))_{t \geq 0, z \in \mathbb{R}^d} \right),
\]
converges in distribution (in a suitable topology) to
\[
(h_\Pi, s_\Pi, m_\Pi) = \left( (h_\Pi(z))_{z \in \mathbb{R}^d}, (s_\Pi(t))_{t \geq 0}, (m_\Pi(t, z))_{t \geq 0, z \in \mathbb{R}^d} \right),
\]
where the limiting object is a functional of the Poisson point process \( \Pi \).
Predicting the hitting times: The lilypad process

Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. (⋆)
- Continue until point with higher potential is found. $\rightsquigarrow$ start of a new ‘lilypad’.

\[
\begin{align*}
\text{Let } h(0) &= 0, \text{ and we define the hitting time of } z \in \mathbb{R}^d \text{ by the lilypad model as } \\
&= \inf \left\{ \sum_{j=0}^{\infty} q_{j+1} - q_j \xi(q_j+1) \right\},
\end{align*}
\]

where $|\cdot|$ is the $\ell_1$-norm and the inf is over all sequences $(y_i)$ with $y_0 = z$ and $(y_i, \xi(y_i)) \in \Pi$, $i \geq 1$ such that $|y_n| \to 0$.

- Need to show this is well-defined.
- Support and number of particles are corollaries.
Predicting the hitting times: The lilypad process

Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. ($\star$)
- Continue until point with higher potential is found. $\leadsto$ start of a new ‘lilypad’.
Predicting the hitting times: The lilypad process

Starting in a point \( z \) with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at \( z \) and branch at rate \( \xi(z) \)
- ‘lilypad’ of particles spreads out at speed proportional to \( \xi(z) \). (⋆)
- Continue until point with higher potential is found. \( \sim \) start of a new ‘lilypad’.

Let \( h(0) = 0 \), and we define the hitting time of \( z \in \mathbb{R}^d \) by the lilypad model as

\[
    h(z) = \inf \left( \sum_{j=0}^{\infty} q|y_{j+1} - y_j| \xi(y_{j+1}) \right),
\]

where \( |·| \) is the \( \ell_1 \)-norm and the inf is over all sequences \((y_i)\) with \( y_0 = z \) and \((y_i, \xi(y_i)) \in \Pi, i \geq 1 \) such that \( |y_n| \to 0 \).

Need to show this is well-defined.

Support and number of particles are corollaries.
Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. ($\star$)
- Continue until point with higher potential is found. $\leadsto$ start of a new ‘lilypad’.

Let $h(0) = 0$, and we define the hitting time of $z \in \mathbb{R}^d$ by the lilypad model as

$$h(z) = \inf \left\{ \sum_{j=0}^{\infty} q |y_{j+1} - y_j| \xi(y_{j+1}) \right\},$$

where $|\cdot|$ is the $\ell^1$-norm and the inf is over all sequences $(y_i)$ with $y_0 = z$ and $(y_i, \xi(y_i)) \in \Pi$, $i \geq 1$ such that $|y_n| \to 0$.

Need to show this is well-defined.

Support and number of particles are corollaries.
Predicting the hitting times: The lilypad process

Starting in a point \( z \) with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at \( z \) and branch at rate \( \xi(z) \)
- ‘lilypad’ of particles spreads out at speed proportional to \( \xi(z) \). (⋆)
- Continue until point with higher potential is found. \( \sim \) start of a new ‘lilypad’.

\[ h(z) = \inf \left( \sum_{j=0}^{\infty} q |y_{j+1} - y_j| \xi(y_{j+1}) \right), \]

where \(|·|\) is the \( \ell_1 \)-norm and the inf is over all sequences \((y_i, \xi(y_i))\) with \( y_0 = z \) and \((y_i, \xi(y_i)) \in \Pi, i \geq 1 \) such that \( |y_n| \rightarrow 0 \).

Need to show this is well-defined.

Support and number of particles are corollaries.
Predicting the hitting times: The lilypad process

Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. ($\star$)
- Continue until point with higher potential is found. $\leadsto$ start of a new ‘lilypad’.

\[
\begin{align*}
\text{Let } h(0) &= 0, \text{ and we define the hitting time of } z \in \mathbb{R}^d \text{ by the lilypad model as } \\
&= \inf \left( \sum_{j=0}^{\infty} q |y_{j+1} - y_j| \xi(y_{j+1}) \right),
\end{align*}
\]

where $|\cdot|$ is the $\ell_1$-norm and the inf is over all sequences $(y_i)$ with $y_0 = z$ and $(y_i, \xi(y_i)) \in \Pi$, $i \geq 1$ such that $|y_n| \to 0$.

- Need to show this is well-defined.
- Support and number of particles are corollaries.
Predicting the hitting times: The lilypad process

Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. (⋆)
- Continue until point with higher potential is found. $\leadsto$ start of a new ‘lilypad’.
Predicting the hitting times: The lilypad process

Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. \(\star\)
- Continue until point with higher potential is found. \(\sim\) start of a new ‘lilypad’.

\[ h(z) = \inf \left( \sum_{j=0}^{\infty} q |y_{j+1} - y_j| \xi(y_{j+1}) \right), \]

where $|\cdot|$ is the $\ell_1$-norm and the inf is over all sequences $(y_i)$ with $y_0 = z$ and $(y_i, \xi(y_i)) \in \Pi$, $i \geq 1$ such that $|y_n| \to 0$. 

- Need to show this is well-defined.
- Support and number of particles are corollaries.
Predicting the hitting times: The lilypad process

Starting in a point \( z \) with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at \( z \) and branch at rate \( \xi(z) \)
- ‘lilypad’ of particles spreads out at speed proportional to \( \xi(z) \). (⋆)
- Continue until point with higher potential is found. \( \leadsto \) start of a new ‘lilypad’.

\[ h(z) = \inf \left( \sum_{j=0}^{\infty} q |y_{j+1} - y_j| \xi(y_{j+1}) \right), \]

where \( |·| \) is the \( \ell_1 \)-norm and the inf is over all sequences \((y_i)\) with \( y_0 = z \) and \((y_i, \xi(y_i)) \in \Pi, i \geq 1 \) such that \( |y_n| \to 0 \).

- Need to show this is well-defined.
- Support and number of particles are corollaries.
Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. (⋆)
- Continue until point with higher potential is found. $\rightsquigarrow$ start of a new ‘lilypad’.

$$h(z) = \inf \left( \sum_{j=0}^{\infty} q \left| y_{j+1} - y_j \right| \xi(y_{j+1}) \right),$$

where $\| \cdot \|$ is the $\ell_1$-norm and the inf is over all sequences $(y_i)$ with $y_0 = z$ and $(y_i, \xi(y_i)) \in \Pi$, $i \geq 1$ such that $\|y_n\| \to 0$.

Need to show this is well-defined.

Support and number of particles are corollaries.
Predicting the hitting times: The lilypad process

Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. (⋆)
- Continue until point with higher potential is found. $\leadsto$ start of a new ‘lilypad’.
Predicting the hitting times: The lilypad process

Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. ($\star$)
- Continue until point with higher potential is found. $\leadsto$ start of a new ‘lilypad’.

\[ h(z) = \inf \left\{ \sum_{j=0}^{\infty} q |y_{j+1} - y_j| \xi(y_{j+1}) \right\}, \]

where $|\cdot|$ is the $\ell_1$-norm and the inf is over all sequences $(y_i)$ with $y_0 = z$ and $(y_i, \xi(y_i)) \in \Pi$, $i \geq 1$ such that $|y_n| \to 0$.
Predicting the hitting times: The lilypad process

Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. ($\star$)
- Continue until point with higher potential is found. $\leadsto$ start of a new ‘lilypad’.
Predicting the hitting times: The lilypad process

Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. (⋆)
- Continue until point with higher potential is found. $\leadsto$ start of a new ‘lilypad’.

Let $h(0) = 0$, and we define the hitting time of $z \in \mathbb{R}^d$ by the lilypad model as $h(z) = \inf \left( \sum_{j=0}^{\infty} q |y_{j+1} - y_j| \xi(y_{j+1}) \right)$, where $|·|$ is the $\ell_1$-norm and the inf is over all sequences $(y_i)$ with $y_0 = z$ and $(y_i, \xi(y_i)) \in \Pi$, $i \geq 1$ such that $|y_n| \to 0$.

Need to show this is well-defined.

Support and number of particles are corollaries.
Predicting the hitting times: The lilypad process

Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. (⋆)
- Continue until point with higher potential is found. $\leadsto$ start of a new ‘lilypad’.
Predicting the hitting times: The lilypad process

Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. (⋆)
- Continue until point with higher potential is found. $\leadsto$ start of a new ‘lilypad’.

Let $h(0) = 0$, and we define the hitting time of $z \in \mathbb{R}^d$ by the lilypad model as

$$h(z) = \inf \left( \sum_{j=0}^{\infty} q |y_{j+1} - y_j| \xi(y_{j+1}) \right),$$

where $|·|$ is the $\ell_1$-norm and the inf is over all sequences $(y_i)$ with $y_0 = z$ and $(y_i, \xi(y_i)) \in \Pi$, $i \geq 1$ such that $|y_n| \to 0$. 

Need to show this is well-defined.

Support and number of particles are corollaries.
Predicting the hitting times: The lilypad process

Starting in a point \( z \) with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at \( z \) and branch at rate \( \xi(z) \)
- ‘lilypad’ of particles spreads out at speed proportional to \( \xi(z) \). (⋆)
- Continue until point with higher potential is found. \( \sim \) start of a new ‘lilypad’.

\[
\begin{align*}
\text{h}(0) &= 0, \\
\text{h}(z) &= \inf \left( \sum_{j=0}^{\infty} q_{y_j+1} - y_j \right) \xi(y_j+1),
\end{align*}
\]

where \( |\cdot| \) is the \( \ell_1 \)-norm and the inf is over all sequences \((y_i)\) with \( y_0 = z \) and \((y_i, \xi(y_i)) \in \Pi\), \( i \geq 1 \) such that \( |y_n| \to 0 \).

Need to show this is well-defined.

Support and number of particles are corollaries.
Predicting the hitting times: The lilypad process

Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$.
- Continue until point with higher potential is found. \(\leadsto\) start of a new ‘lilypad’.

\[ h(z) = \inf \sum_{j=0}^{\infty} q|y_{j+1} - y_j| \xi(y_{j+1}) \]

where $|\cdot|$ is the $\ell_1$-norm and the inf is over all sequences $(y_i)$ with $y_0 = z$ and $(y_i, \xi(y_i)) \in \Pi_i, i \geq 1$ such that $|y_n| \to 0$.

Need to show this is well-defined.

Support and number of particles are corollaries.
Predicting the hitting times: The lilypad process

Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. ($\star$)
- Continue until point with higher potential is found. $\leadsto$ start of a new ‘lilypad’.

Let $h(0) = 0$, and we define the hitting time of $z \in \mathbb{R}^d$ by the lilypad model as

$$h(z) = \inf \infty \sum_{j=0}^{\infty} q |y_{j+1} - y_j| \xi(y_{j+1}),$$

where $|\cdot|$ is the $\ell_1$-norm and the inf is over all sequences $(y_i)$ with $y_0 = z$ and $(y_i, \xi(y_i)) \in \Pi$, $i \geq 1$ such that $|y_n| \to 0$.

Need to show this is well-defined.

Support and number of particles are corollaries.
Predicting the hitting times: The lilypad process

Starting in a point \( z \) with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at \( z \) and branch at rate \( \xi(z) \)
- ‘lilypad’ of particles spreads out at speed proportional to \( \xi(z) \). (⋆)
- Continue until point with higher potential is found. \( \leadsto \) start of a new ‘lilypad’.
Predicting the hitting times: The lilypad process

Starting in a point $z$ with high potential, we observe the following spread of mass (in the rescaled picture):

- particles sit at $z$ and branch at rate $\xi(z)$
- ‘lilypad’ of particles spreads out at speed proportional to $\xi(z)$. (⋆)
- Continue until point with higher potential is found. $\rightsquigarrow$ start of a new ‘lilypad’.

Let $h(0) = 0$, and we define the hitting time of $z \in \mathbb{R}^d$ by the lilypad model as

$$h(z) = \inf \left( \sum_{j=0}^{\infty} q \frac{|y_{j+1} - y_j|}{\xi(y_{j+1})} \right),$$

where $|\cdot|$ is the $\ell^1$-norm and the inf is over all sequences $(y_i)$ with $y_0 = z$ and $(y_i, \xi(y_i)) \in \Pi, i \geq 1$ such that $|y_n| \to 0$.

- Need to show this is well-defined.
- Support and number of particles are corollaries.
Balance between spatial and temporal scale

Claim

‘Lilypad’ of particles spreads out at speed proportional to $\xi(z)$.

Recall that we rescale our systems

$$
\text{space } r_T = \left( \frac{T}{\log T} \right)^{q+1} \quad \text{potential } a_T = \left( \frac{T}{\log T} \right)^q.
$$

We start in a point $r_T x$ with potential of size $\xi_T(x) = \xi(r_T x)/a_T \approx 1$ and assume there are no further good points nearby. When do we reach a point $r_T z$?

$$
E_{r_T x}[N(tT, r_T z)] \approx e^{\xi_T(x) a_T t T} \mathbb{P}_{r_T x}\left\{ \text{reach } r_T z \text{ in time } o(tT) \right\}
\approx e^{\xi_T(x) a_T t T} e^{-q|z-x| r_T \log T}
= e^{a(T) T (\xi_T(x) t - q|z-x|)}.
$$

We reach the point $z$ when this expectation is $\approx 1$, i.e. at time

$$
t = q \frac{|z-x|}{\xi_T(x)}.
$$

In particular, this shows that $r_T$ is the right spatial scaling.
Balance between spatial and temporal scale

<table>
<thead>
<tr>
<th>Claim</th>
</tr>
</thead>
<tbody>
<tr>
<td>‘Lilypad’ of particles spreads out at speed proportional to $\xi(z)$.</td>
</tr>
</tbody>
</table>

Recall that we rescale our systems

$$
space \ r_T = \left( \frac{T}{\log T} \right)^{q+1} \quad \text{potential} \ a_T = \left( \frac{T}{\log T} \right)^q.
$$

We start in a point $r_Tx$ with potential of size $\xi_T(x) = \xi(r_Tx)/a_T \approx 1$ and assume there are no further good points nearby. When do we reach a point $r_Tz$?

$$
\mathbb{E}_{r_Tx}[N(tT, r_Tz)] \approx e^{\xi_T(x)a_TtT} \mathbb{P}_{r_Tx}\{ \text{reach } r_Tz \text{ in time } o(tT) \} \
\approx e^{\xi_T(x)a_TtT} e^{-q|z-x|r_T\log T} \
= e^{a(T)T(\xi_T(x)t-q|z-x|)}.
$$

We reach the point $z$ when this expectation is $\approx 1$, i.e. at time

$$
t = q \frac{|z-x|}{\xi_T(x)}.
$$

In particular, this shows that $r_T$ is the right spatial scaling.
Balance between spatial and temporal scale

**Claim**

‘Lilypad’ of particles spreads out at speed proportional to ξ(z).

Recall that we rescale our systems

\[ r_T = \left( \frac{T}{\log T} \right)^{q+1}, \quad a_T = \left( \frac{T}{\log T} \right)^q. \]

We start in a point \( r_T x \) with potential of size \( \xi_T(x) = \xi(r_T x)/a_T \gtrsim 1 \) and assume there are no further good points nearby. When do we reach a point \( r_T z \)?

\[ \mathbb{E}_{r_T x}[N(tT, r_T z)] \approx e^{\xi_T(x)a_T tT} \mathbb{P}_{r_T x}\{ \text{reach } r_T z \text{ in time } o(tT) \} \]

\[ \approx e^{\xi_T(x)a_T tT} e^{-q|z-x|r_T \log T} \]

\[ = e^{a(T) T (\xi_T(x) t - q|z-x|)}. \]

We reach the point \( z \) when this expectation is \( \approx 1 \), i.e. at time

\[ t = q \frac{|z - x|}{\xi_T(x)}. \]

In particular, this shows that \( r_T \) is the right spatial scaling.
Balance between spatial and temporal scale

**Claim**

‘Lilypad’ of particles spreads out at speed proportional to $\xi(z)$.

Recall that we rescale our systems

$$r_T = \left(\frac{T}{\log T}\right)^{q+1} \quad \text{potential} \quad a_T = \left(\frac{T}{\log T}\right)^q.$$  

We start in a point $r_T x$ with potential of size $\xi_T(x) = \xi(r_T x)/a_T \approx 1$ and assume there are no further good points nearby. When do we reach a point $r_T z$?

$$\mathbb{E}_{r_T x}[N(tT, r_T z)] \approx e^{\xi_T(x) a_T t T} \mathbb{P}_{r_T x}\{ \text{reach } r_T z \text{ in time } o(tT)\}$$  

$$\approx e^{\xi_T(x) a_T t T} e^{-q|z-x|r_T \log T}$$  

$$= e^{a(T) T (\xi_T(x) t - q|z-x|)}.$$  

We reach the point $z$ when this expectation is $\approx 1$, i.e. at time

$$t = q \frac{|z-x|}{\xi_T(x)}.$$  

In particular, this shows that $r_T$ is the right spatial scaling.
Balance between spatial and temporal scale

Claim

‘Lilypad’ of particles spreads out at speed proportional to $\xi(z)$.

Recall that we rescale our systems

$$\text{space } r_T = \left( \frac{T}{\log T} \right)^{q+1} \quad \text{potential } a_T = \left( \frac{T}{\log T} \right)^q.$$

We start in a point $r_T x$ with potential of size $\xi_T(x) = \xi(r_T x) / a_T \asymp 1$ and assume there are no further good points nearby. When do we reach a point $r_T z$?

$$\mathbb{E}_{r_T x}[N(tT, r_T z)] \approx e^{\xi_T(x) a_T t T} \mathbb{P}_{r_T x}\{ \text{reach } r_T z \text{ in time } o(tT) \}$$

$$\approx e^{\xi_T(x) a_T t T} e^{-q |z - x| r_T \log T}$$

$$= e^{a(T) T (\xi_T(x) t - q |z - x|)}.$$

We reach the point $z$ when this expectation is $\approx 1$, i.e. at time

$$t = q \frac{|z - x|}{\xi_T(x)}.$$

In particular, this shows that $r_T$ is the right spatial scaling.
Balance between spatial and temporal scale

**Claim**

‘Lilypad’ of particles spreads out at speed proportional to $\xi(z)$.

Recall that we rescale our systems

$$r_T = \left( \frac{T}{\log T} \right)^{q+1} \quad \text{potential} \quad a_T = \left( \frac{T}{\log T} \right)^q.$$

We start in a point $r_T x$ with potential of size $\xi_T(x) = \xi(r_T x)/a_T \approx 1$ and assume there are no further good points nearby. When do we reach a point $r_T z$?

$$E_{r_T x}[N(t T, r_T z)] \approx e^{\xi_T(x) a_T t T} \mathbb{P}_{r_T x}\{ \text{reach } r_T z \text{ in time } o(t T) \} \approx e^{\xi_T(x) a_T t T} e^{-q|z-x|r_T \log T} \approx e^{a(T) T (\xi_T(x)t - q|z-x|)}.$$

We reach the point $z$ when this expectation is $\approx 1$, i.e. at time

$$t = q \frac{|z-x|}{\xi_T(x)}.$$

In particular, this shows that $r_T$ is the right spatial scaling.
Balance between spatial and temporal scale

Claim

‘Lilypad’ of particles spreads out at speed proportional to $\xi(z)$.

Recall that we rescale our systems

$$r_T = \left( \frac{T}{\log T} \right)^{q+1} \quad \text{potential} \quad a_T = \left( \frac{T}{\log T} \right)^q.$$

We start in a point $r_T x$ with potential of size $\xi_T(x) = \xi(r_T x)/a_T \approx 1$ and assume there are no further good points nearby. When do we reach a point $r_T z$?

$$E_{r_T x}[N(tT, r_T z)] \approx e^{\xi_T(x)a_T tT} \mathbb{P}_{r_T x}\{ \text{reach } r_T z \text{ in time } o(tT) \}$$

$$\approx e^{\xi_T(x)a_T tT} e^{-q|z-x|r_T \log T}$$

$$= e^{a(T) T (\xi_T(x)t - q|z-x|)}.$$

We reach the point $z$ when this expectation is $\approx 1$, i.e. at time

$$t = q \frac{|z-x|}{\xi_T(x)}.$$

In particular, this shows that $r_T$ is the right spatial scaling.
The limiting support is defined as

\[ s(t) := \{ z \in \mathbb{R}^d : h(z) \leq t \}. \]
The limiting support is defined as

\[ s(t) := \{ z \in \mathbb{R}^d : h(z) \leq t \}. \]
The limiting support is defined as

\[ s(t) := \{ z \in \mathbb{R}^d : h(z) \leq t \}. \]
The limiting support is defined as

$$s(t) := \{ z \in \mathbb{R}^d : h(z) \leq t \}.$$
The limiting support is defined as

$$s(t) := \{ z \in \mathbb{R}^d : h(z) \leq t \}.$$
The limiting support is defined as

\[ s(t) := \{ z \in \mathbb{R}^d : h(z) \leq t \}. \]
The (log-)number of particles $m(t, z)$ follows two rules:

- If $z$ is a site with high potential, number of particles start growing at rate $\xi(z)$ as soon as $z$ is hit.
- Costs to go from nearest good site $y$ to $z$ is $q|y - z|$ (on logarithmic scale).

Thus,

$$m(t, z) = \xi(z)(t - h(z)).$$
The (log-)number of particles $m(t, z)$ follows two rules:

- If $z$ is a site with high potential, number of particles start growing at rate $\xi(z)$ as soon as $z$ is hit.
- Costs to go from nearest good site $y$ to $z$ is $q|y - z|$ (on logarithmic scale).

Thus,

$$m(t, z) = \sup_y \{\xi_T(y)(t - h(y)) - q|y - z|\}.$$
• Limit is random in contrast to earlier work on BRWRE [Comets, Popov '07], but also not of SDE/SPDE-type.
• Corollary: Log of number of particles at site is random in leading order!
• We call the limit process the **lilypad process**.
• Lilypads grow like $\ell^1$-balls:
  • Reason is that the front is driven by extreme large deviation events (underlying RW talks $\gg T$ steps in time $T$).
  • Dominating term comes from number of steps taken to get from $x$ to $z \sim \ell^1$ norm.
  • Scaling limit is not universal (e.g. not the same for other lattices).
Limit is random in contrast to earlier work on BRWRE \cite{Comets, Popov ’07}, but also not of SDE/SPDE-type.

Corollary: Log of number of particles at site is random in leading order!

We call the limit process the lilypad process.

Lilypads grow like $\ell^1$-balls:

- Reason is that the front is driven by extreme large deviation events (underlying RW takes $\gg T$ steps in time $T$).
- Dominating term comes from number of steps taken to get from $x$ to $z \sim \ell^1$ norm.
- Scaling limit is not universal (e.g. not the same for other lattices).
Comments on scaling limit

- Limit is random in contrast to earlier work on BRWRE \cite{Comets,Popov'07}, but also not of SDE/SPDE-type.
- Corollary: Log of number of particles at site is random in leading order!
- We call the limit process the \textit{lilypad process}.
- Lilypads grow like $\ell^1$-balls:
  - Reason is that the front is driven by extreme large deviation events (underlying RW takes $\gg T$ steps in time $T$).
  - Dominating term comes from number of steps taken to get from $x$ to $z \sim \ell^1$ norm.
  - Scaling limit is not universal (e.g. not the same for other lattices).
Comments on scaling limit

- Limit is random in contrast to earlier work on BRWRE [Comets, Popov ’07], but also not of SDE/SPDE-type.
- Corollary: Log of number of particles at site is random in leading order!
- We call the limit process the **lilypad process**.
- Lilypads grow like \( \ell^1 \)-balls:
  - Reason is that the front is driven by extreme large deviation events (underlying RW takes \( \gg T \) steps in time \( T \)).
  - Dominating term comes from number of steps taken to get from \( x \) to \( z \sim \ell^1 \) norm.
  - Scaling limit is not universal (e.g. not the same for other lattices).
Comments on scaling limit

- Limit is random in contrast to earlier work on BRWRE [Comets, Popov ’07], but also not of SDE/SPDE-type.
- Corollary: Log of number of particles at site is random in leading order!
- We call the limit process the lilypad process.
- Lilypads grow like $\ell^1$-balls:
  - Reason is that the front is driven by extreme large deviation events (underlying RW talkes $\gg T$ steps in time $T$).
  - Dominating term comes from number of steps taken to get from $x$ to $z \sim \ell^1$ norm.
  - Scaling limit is not universal (e.g. not the same for other lattices).
Comments on scaling limit

• Limit is random in contrast to earlier work on BRWRE [Comets, Popov ’07], but also not of SDE/SPDE-type.

• Corollary: Log of number of particles at site is random in leading order!

• We call the limit process the lilypad process.

• Lilypads grow like $\ell^1$-balls:
  • Reason is that the front is driven by extreme large deviation events (underlying RW takes $\gg T$ steps in time $T$).
  • Dominating term comes from number of steps taken to get from $x$ to $z \sim \ell^1$ norm.

• Scaling limit is not universal (e.g. not the same for other lattices).
Comments on scaling limit

- Limit is random in contrast to earlier work on BRWRE [Comets, Popov ’07], but also not of SDE/SPDE-type.
- Corollary: Log of number of particles at site is random in leading order!
- We call the limit process the lilypad process.
- Lilypads grow like \( \ell^1 \)-balls:
  - Reason is that the front is driven by extreme large deviation events (underlying RW takes \( \gg T \) steps in time \( T \)).
  - Dominating term comes from number of steps taken to get from \( x \) to \( z \sim \ell^1 \) norm.
  - Scaling limit is not universal (e.g. not the same for other lattices).
Proof of scaling limit

Step 1: Decoupling the randomness:

- Define a discrete lilypad process in terms of the point process
  \[ \Pi_T = \sum_{z \in \mathbb{Z}^d} \delta \left( \frac{z}{r_T}, \frac{\xi(z)}{a_T} \right) . \]
  We show in [O. and Roberts ’16] that the branching random walks hitting times are well approximated by the hitting times in the discrete lilypad process (which only depend on the environment!)
  - Use moments, but starting from a good point!
  - plus elaborate induction arguments.
- It remains to show that the discrete lilypad model converges.

Step 2: Continuous mapping theorem:

- Since \( \Pi_T \Rightarrow \Pi \), any continuous functional of the point process will also converge.
- Our functionals are only continuous if they depend on finitely many points: thus need to ‘cut off’ points with small potential or that are too far out.
Proof of scaling limit

Step 1: Decoupling the randomness:

- Define a discrete lilypad process in terms of the point process

\[ \Pi_T = \sum_{z \in \mathbb{Z}^d} \delta\left( \frac{z}{r_T}, \frac{\xi(z)}{a_T} \right). \]

We show in [O. and Roberts ’16] that the branching random walks hitting times are well approximated by the hitting times in the discrete lilypad process (which only depend on the environment!)

- Use moments, but starting from a good point!
- plus elaborate induction arguments.
- It remains to show that the discrete lilypad model converges.

Step 2: Continuous mapping theorem:

- Since \( \Pi_T \rightarrow \Pi \), any continuous functional of the point process will also converge.
- Our functionals are only continuous if they depend on finitely many points: thus need to ‘cut off’ points with small potential or that are too far out.
Proof of scaling limit

Step 1: Decoupling the randomness:

- Define a discrete lilypad process in terms of the point process
\[ \Pi_T = \sum_{z \in \mathbb{Z}^d} \delta_{(\frac{z}{r_T}, \xi(z) \frac{a_T}{a_T})}. \]

We show in [O. and Roberts '16] that the branching random walks hitting times are well approximated by the hitting times in the discrete lilypad process (which only depend on the environment!)

- Use moments, but starting from a good point!
- plus elaborate induction arguments.

- It remains to show that the discrete lilypad model converges.

Step 2: Continuous mapping theorem:

- Since \( \Pi_T \Rightarrow \Pi \), any continuous functional of the point process will also converge.
- Our functionals are only continuous if they depend on finitely many points: thus need to ‘cut off’ points with small potential or that are too far out.
Proof of scaling limit

Step 1: Decoupling the randomness:

- Define a discrete lilypad process in terms of the point process

\[ \Pi_T = \sum_{z \in \mathbb{Z}^d} \delta\left( \frac{z}{r_T}, \frac{\xi(z)}{a_T} \right). \]

We show in [O. and Roberts ’16] that the branching random walks hitting times are well approximated by the hitting times in the discrete lilypad process (which only depend on the environment!)

- Use moments, but starting from a good point!
- plus elaborate induction arguments.

- It remains to show that the discrete lilypad model converges.

Step 2: Continuous mapping theorem:

- Since \( \Pi_T \Rightarrow \Pi \), any continuous functional of the point process will also converge.

- Our functionals are only continuous if they depend on finitely many points: thus need to ‘cut off’ points with small potential or that are too far out.
Proof of scaling limit

Step 1: Decoupling the randomness:

- Define a discrete lilypad process in terms of the point process
\[
\Pi_T = \sum_{z \in \mathbb{Z}^d} \delta\left(\frac{z}{r_T}, \frac{\xi(z)}{a_T}\right).
\]

We show in [O. and Roberts '16] that the branching random walks hitting times are well approximated by the hitting times in the discrete lilypad process (which only depend on the environment!)

- Use moments, but starting from a good point!
- plus elaborate induction arguments.

- It remains to show that the discrete lilypad model converges.

Step 2: Continuous mapping theorem:

- Since \( \Pi_T \Rightarrow \Pi \), any continuous functional of the point process will also converge.

- Our functionals are only continuous if they depend on finitely many points: thus need to ‘cut off’ points with small potential or that are too far out.
Proof of scaling limit

Step 1: Decoupling the randomness:

- Define a discrete lilypad process in terms of the point process

\[ \Pi_T = \sum_{z \in \mathbb{Z}^d} \delta\left( \frac{z}{t_T}, \frac{\xi(z)}{a_T} \right). \]

We show in \([O. \text{ and Roberts } '16]\) that the branching random walks hitting times are well approximated by the hitting times in the discrete lilypad process (which only depend on the environment!)

- Use moments, but starting from a good point!
- plus elaborate induction arguments.

- It remains to show that the discrete lilypad model converges.

Step 2: Continuous mapping theorem:

- Since \( \Pi_T \Rightarrow \Pi \), any continuous functional of the point process will also converge.
- Our functionals are only continuous if they depend on finitely many points: thus need to ‘cut off’ points with small potential or that are too far out.
One-point localisation

For $u(x, t)$ the solution of the parabolic Anderson model (i.e. the expected number of particles) it is known from [König et.al '09] that there exists a process $Z_{t}^{\text{PAM}}$ such that

$$\frac{u(t, Z_{t}^{\text{PAM}})}{\sum_{z \in \mathbb{Z}^d} u(t, z)} \to 1 \text{ in probability as } t \to \infty.$$ 

Q: Does the same hold for the branching random walk?

Recall that we write $N(t, z)$ for the number of particles at site $z$ at time $t$.

Theorem 1 (O. and Roberts '17)

There exists a process $Z_{t}^{(1)}$ such that

$$\frac{N(t, Z_{t}^{(1)})}{\sum_{z \in \mathbb{Z}^d} N(t, z)} \to 1 \text{ in probability as } t \to \infty.$$ 

- Convergence cannot hold almost surely, otherwise we need two points for transition times (conjecture).
One-point localisation

For $u(x, t)$ the solution of the parabolic Anderson model (i.e. the expected number of particles) it is known from [König et.al '09] that there exists a process $Z_{t}^{\text{PAM}}$ such that

$$\frac{u(t, Z_{t}^{\text{PAM}})}{\sum_{z\in\mathbb{Z}^{d}} u(t, z)} \rightarrow 1 \text{ in probability as } t \to \infty.$$ 

Q: Does the same hold for the branching random walk?

Recall that we write $N(t, z)$ for the number of particles at site $z$ at time $t$.

**Theorem 1 (O. and Roberts '17)**

There exists a process $Z_{t}^{(1)}$ such that

$$\frac{N(t, Z_{t}^{(1)})}{\sum_{z\in\mathbb{Z}^{d}} N(t, z)} \rightarrow 1 \text{ in probability as } t \to \infty.$$ 

- Convergence cannot hold almost surely, otherwise we need two points for transition times (conjecture).
Proof of the one-point localisation

• From scaling limit theorem, we now that at a typical large time $t$, we have

$$\frac{1}{a_t} \log N(t, r_t^{-1} z) \approx \log N(t, z)$$

This implies that there is localisation in the rescaled picture, i.e. there exists $\varepsilon > 0$ and a process $Z_t$ such that

$$\frac{\sum_{z \in B(Z_t, \varepsilon r_t)} N(t, z)}{\sum_{w \in \mathbb{Z}^d} N(t, w)} \to 1 \text{ in prob.}$$

Here $Z_t$ is defined as the maximizer of the corresponding lilypad process, see [O. and Roberts '16].

• Remains to worry about particles in a ‘small’ ball around $Z_t$.

Strategy:

• Need to control when exactly the good point $Z_t$ is hit for the first time $\rightsquigarrow$ Stopping lines.

• Then show that it is too expensive to leave the good point! (here we very much rely on the extreme growth of potential!)
Proof of the one-point localisation

- From scaling limit theorem, we now that at a typical large time $t$, we have

$$\frac{1}{a_t} \log N(t, r_t^{-1} z) \approx$$

This implies that there is localisation in the rescaled picture, i.e. there exists $\varepsilon > 0$ and a process $Z_t$ such that

$$\frac{\sum_{z \in B(Z_t, \varepsilon r_t)} N(t, z)}{\sum_{w \in \mathbb{Z}^d} N(t, w)} \rightarrow 1 \text{ in prob.}$$

Here $Z_t$ is defined as the maximizer of the corresponding lilypad process, see [O. and Roberts '16].

- Remains to worry about particles in a ‘small’ ball around $Z_t$.

Strategy:
- Need to control when exactly the good point $Z_t$ is hit for the first time $\sim$ Stopping lines.
- Then show that it is too expensive to leave the good point! (here we very much rely on the extreme growth of potential!)
Proof of the one-point localisation

• From scaling limit theorem, we now that at a typical large time $t$, we have

$$\frac{1}{a_t} \log N(t, r_t^{-1}z) \approx$$

This implies that there is localisation in the rescaled picture, i.e. there exists $\varepsilon > 0$ and a process $Z_t$ such that

$$\frac{\sum_{z \in B(Z_t, \varepsilon r_t)} N(t, z)}{\sum_{w \in \mathbb{Z}^d} N(t, w)} \to 1 \text{ in prob.}$$

Here $Z_t$ is defined as the maximizer of the corresponding lilypad process, see [O. and Roberts ’16].

• Remains to worry about particles in a ‘small’ ball around $Z_t$.

Strategy:

• Need to control when exactly the good point $Z_t$ is hit for the first time $\leadsto$ Stopping lines.

• Then show that it is too expensive to leave the good point! (here we very much rely on the extreme growth of potential!)
Comparison to parabolic Anderson model

- Recall: The solution $u(t, x)$ of the parabolic Anderson model describes the **expected number** of particles in the branching random walk (when averaging over branching/migration).

Our methods also give a scaling limit for

$$\Lambda_T(t, z) = \frac{1}{aT} \log u(tT, rTz), \quad z \in \mathbb{R}^d$$

using a description via a ‘modified lilypad process’.

- New hitting times $\tau_T(z)$ ($= \text{time such that } \Lambda(t, z) > 1$) depend for peaks only on position and potential (and otherwise only on nearest peak).

- **Support can be disconnected!**
Comparison to parabolic Anderson model

- Recall: The solution $u(t, x)$ of the parabolic Anderson model describes the **expected number** of particles in the branching random walk (when averaging over branching/migration).

Our methods also give a scaling limit for

$$\Lambda_T(t, z) = \frac{1}{a_T T} \log u(tT, r_T z), \quad z \in \mathbb{R}^d$$

using a description via a ‘modified lilypad process’.

- New hitting times $\tau_T(z)$ (= time such that $\Lambda(t, z) > 1$) depend for peaks only on position and potential (and otherwise only on nearest peak).

- **Support can be disconnected!**
Comparison of the support in dimension 2

- Support of the BRW: green.
- “Support” of the parabolic Anderson model: blue.
Comparison of the support in dimension 2

- Support of the BRW: green.
- “Support” of the parabolic Anderson model: blue.
Comparison of the support in dimension 2

- Support of the BRW: green.
- “Support” of the parabolic Anderson model: blue.
Comparison of the support in dimension 2

- Support of the BRW: green.
- “Support” of the parabolic Anderson model: blue.
Comparison of the support in dimension 2

- Support of the BRW: green.
- “Support” of the parabolic Anderson model: blue.
Comparison of the support in dimension 2

- Support of the BRW: green.
- “Support” of the parabolic Anderson model: blue.
Comparison: Number of particles

- Branching random walk: green.
- Expected number of particles (PAM): blue.
Comparison: Number of particles

- Branching random walk: green.
- Expected number of particles (PAM): blue.
Comparison: Number of particles

- Branching random walk: green.
- Expected number of particles (PAM): blue.
Comparison: Number of particles

- Branching random walk: green.
- Expected number of particles (PAM): blue.
Comparison: Number of particles

- Branching random walk: green.
- Expected number of particles (PAM): blue.
Comparison: Number of particles

- Branching random walk: green.
- Expected number of particles (PAM): blue.
Comparison: Number of particles

- Branching random walk: green.
- Expected number of particles (PAM): blue.
BRW in Weibull environment

So far all results have been for Pareto potential.

Next step: **Weibull potentials**:

\[
\text{Prob}\{\xi(0) > z\} \sim e^{-z^\gamma}.
\]

Localisation and asymptotics of total mass of the parabolic Anderson model well understood:

- [Gärtner, Molchanov '98, van der Hofstad, Sidorova, Mörters '08, Lacoin, H, Mörters '12, Sidorova, Twarowski '14, Fidorov, Muirhead '14].
- This class includes heavy-tailed and non-heavy tailed distributions.
- For any \( \gamma > 0 \): one-point localisation (in probability).
So far all results have been for Pareto potential.

Next step: **Weibull potentials**:

\[
\text{Prob}\{ \xi(0) > z \} \sim e^{-z^\gamma}.
\]

Localisation and asymptotics of total mass of the parabolic Anderson model well understood:

- [Gärtner, Molchanov ’98, van der Hofstad, Sidorova, Mörters ’08, Lacoin, H, Mörters ’12, Sidorova, Twarowski ’14, Fidorov, Muirhead ’14 ].
- This class includes heavy-tailed and non-heavy tailed distributions.
- For any \( \gamma > 0 \): one-point localisation (in probability).
Rescaling the environment

Extreme value theory tells us to rescale differently this time:

Spatial rescaling:

\[ r_T = \frac{T (\log T)^{\frac{1}{\gamma}}}{\log \log T}. \]

For the potential we need:

\[ a_T = (d \log r_T)^{\frac{1}{\gamma}}, \quad b_T = (d \log r_T)^{\frac{1}{\gamma}} - 1. \]

Then, the rescaled point process

\[ \Pi_T = \sum_{z \in \mathbb{Z}^d} \delta_{\left( \frac{z}{r_T}, \frac{\xi(z) - a_T}{b_T} \right)}, \]

converges to a Poisson point process on \( \mathbb{R}^d \times \mathbb{R} \).

Note the leading order of maximal value of \( \Pi_T \) on a compact set is deterministic!

Also it is known that there exists \( Z^1_T \):

\[ \frac{1}{T} \log \sum_z u(T, z) \sim \frac{1}{T} \log u(T, Z^1_T) \sim a_T + b_T \text{ random term.} \]
Rescaling the environment

Extreme value theory tells us to rescale differently this time:

Spatial rescaling:

\[ r_T = \frac{T (\log T)^{\frac{1}{\gamma}}}{\log \log T}. \]

For the potential we need:

\[ a_T = (d \log r_T)^{\frac{1}{\gamma}}, \quad b_T = (d \log r_T)^{\frac{1}{\gamma}} - 1. \]

Then, the rescaled point process

\[ \Pi_T = \sum_{z \in \mathbb{Z}^d} \delta(\frac{z}{r_T}, \frac{\xi(z) - a_T}{b_T}), \]

converges to a Poisson point process on \( \mathbb{R}^d \times \mathbb{R} \).

Note the leading order of maximal value of \( \Pi_T \) on a compact set is deterministic!

Also it is known that there exists \( Z^1_T \):

\[ \frac{1}{T} \log \sum_z u(T, z) \sim \frac{1}{T} \log u(T, Z^1_T) \sim a_T + b_T \text{ random term.} \]
Rescaling the environment

Extreme value theory tells us to rescale differently this time:

Spatial rescaling:

\[ r_T = \frac{T (\log T)^{\frac{1}{\gamma} - 1}}{\log \log T}. \]

For the potential we need:

\[ a_T = \left( d \log r_T \right)^{\frac{1}{\gamma}}, \quad b_T = \left( d \log r_T \right)^{\frac{1}{\gamma} - 1}. \]

Then, the rescaled point process

\[ \Pi_T = \sum_{z \in \mathbb{Z}^d} \delta_{\left( \frac{z}{r_T}, \frac{\xi(z) - a_T}{b_T} \right)}, \]

converges to a Poisson point process on \( \mathbb{R}^d \times \mathbb{R} \).

Note the leading order of maximal value of \( \Pi_T \) on a compact set is deterministic!

Also it is known that there exists \( Z^{1}_T \):

\[ \frac{1}{T} \log \sum_{z} u(T, z) \sim \frac{1}{T} \log u(T, Z^{1}_T) \sim a_T + b_T \text{ random term}. \]
Rescaling the environment

Extreme value theory tells us to rescale differently this time:

Spatial rescaling:

\[ r_T = \frac{T \left( \log T \right)^{\frac{1}{\gamma}-1}}{\log \log T}. \]

For the potential we need:

\[ a_T = \left( d \log r_T \right)^{\frac{1}{\gamma}}, \quad b_T = \left( d \log r_T \right)^{\frac{1}{\gamma}-1}. \]

Then, the rescaled point process

\[ \Pi_T = \sum_{z \in \mathbb{Z}^d} \delta \left( \frac{z}{r_T}, \frac{\xi(z)-a_T}{b_T} \right), \]

converges to a Poisson point process on \( \mathbb{R}^d \times \mathbb{R} \).

Note the leading order of maximal value of \( \Pi_T \) on a compact set is deterministic!

Also it is known that there exists \( Z_{1T}^1 \):

\[ \frac{1}{T} \log \sum_z u(T, z) \sim \frac{1}{T} \log u(T, Z_{1T}^1) \sim a_T + b_T \text{ random term.} \]
Q: Are BRW and PAM still different?

**Proposition 3**

For Weibull potential with $\gamma$ small, we have that

$$\frac{1}{Tb_T} \left( \log \sum_z u(T, z) - \log \sum_z N(T, z) \right) \to 0,$$

in probability. I.e. PAM and BRW agree to first orders (including the random term).

Moreover, there exists $\varepsilon > 0$ and a site $X_T$ with

$$|X_T| \geq r_T \log \log(T)^\varepsilon.$$

such that $N(T, X_T) \geq 1$.

- Recall for the maximizer in the PAM $|Z_T^1|/r_T$ converges.
- So the support of the BRW grows on different scale from maximizer.
- Claim: On the scale of the maximizer, there are particles everywhere.
Q: Are BRW and PAM still different?

**Proposition 3**

*For Weibull potential with $\gamma$ small, we have that*

$$\frac{1}{Tb_T} \left( \log \sum_z u(T, z) - \log \sum_z N(T, z) \right) \to 0,$$

*in probability. I.e. PAM and BRW agree to first orders (including the random term).*

Moreover, there exists $\varepsilon > 0$ and a site $X_T$ with

$$|X_T| \geq r_T \log \log(T)^\varepsilon.$$

such that $N(T, X_T) \geq 1$.

- Recall for the maximizer in the PAM $|Z_T^1|/r_T$ converges.
- So the support of the BRW grows on different scale from maximizer.
- Claim: On the scale of the maximizer, there are particles everywhere.
Q: Are BRW and PAM still different?

**Proposition 3**

*For Weibull potential with $\gamma$ small, we have that*

$$\frac{1}{T b_T} \left( \log \sum_z u(T, z) - \log \sum_z N(T, z) \right) \to 0,$$

*in probability. i.e. PAM and BRW agree to first orders (including the random term).*

Moreover, there exists $\varepsilon > 0$ and a site $X_T$ with

$$|X_T| \geq r_T \log \log(T)^\varepsilon.$$

such that $N(T, X_T) \geq 1$.

- Recall for the maximizer in the PAM $|Z_T^1|/r_T$ converges.
- So the support of the BRW grows on different scale from maximizer.
- Claim: On the scale of the maximizer, there are particles everywhere.
Q: Are BRW and PAM still different?

**Proposition 3**

*For Weibull potential with $\gamma$ small, we have that*

\[
\frac{1}{Tb_T} \left( \log \sum_z u(T, z) - \log \sum_z N(T, z) \right) \to 0,
\]

*in probability. I.e. PAM and BRW agree to first orders (including the random term).*

Moreover, there exists $\varepsilon > 0$ and a site $X_T$ with

\[
|X_T| \geq r_T \log \log(T)^\varepsilon.
\]

such that $N(T, X_T) \geq 1$.

- Recall for the maximizer in the PAM $|Z_T^1|/r_T$ converges.
- So the support of the BRW grows on different scale from maximizer.
- Claim: On the scale of the maximizer, there are particles everywhere.
Q: Are BRW and PAM still different?

**Proposition 3**

*For Weibull potential with $\gamma$ small, we have that*

$$\frac{1}{Tb_T} \left( \log \sum_z u(T, z) - \log \sum_z N(T, z) \right) \to 0,$$

*in probability. I.e. PAM and BRW agree to first orders (including the random term).*

Moreover, there exists $\varepsilon > 0$ and a site $X_T$ with

$$|X_T| \geq r_T \log \log(T)^\varepsilon.$$

such that $N(T, X_T) \geq 1$.

- Recall for the maximizer in the PAM $|Z^1_T|/r_T$ converges.
- So the support of the BRW grows on different scale from maximizer.
- Claim: On the scale of the maximizer, there are particles everywhere.
Proof idea for Weibull case

Identify the optimal strategy for BRW:

- Try to get to a good site $z$ with $z_T := z/r_T$ and $\xi_T(z) = \frac{\xi(z) - a_T}{b_T}$ of order one.
- Taking the route via a decent site $w$ near the origin, we can show that the first particle arrives at $z$ no later than

$$\frac{|z_T|}{\gamma d^{1/\gamma}} \frac{T}{\log T}.$$ 

- Then, by time $T$, we have at least the following number of particles:

$$\exp \left\{ \xi(z) \left( T - \frac{|z_T|}{\gamma d^{1/\gamma}} \frac{T}{\log T} \right) \right\}$$

$$= \exp \left\{ a_T T + b_T T \left( \xi_T(z) - \frac{|z_T|}{\gamma d^{1/\gamma-1}} \right) + o(b_T T) \right\}$$

- This gives the same optimization problem as for the PAM.
Proof idea for Weibull case

Identify the optimal strategy for BRW:

- Try to get to a good site \( z \) with \( z_T := z/r_T \) and \( \xi_T(z) = \frac{\xi(z) - a_T}{b_T} \) of order one.
- Taking the route via a decent site \( w \) near the origin, we can show that the first particle arrives at \( z \) no later than

\[
\frac{|z_T|}{\gamma d^{1/\gamma}} \frac{T}{\log T}.
\]

- Then, by time \( T \), we have at least the following number of particles:

\[
\exp \left\{ \xi(z) \left( T - \frac{|z_T|}{\gamma d^{1/\gamma}} \frac{T}{\log T} \right) \right\}
\]

\[
= \exp \left\{ a_T T + b_T T \left( \xi_T(z) - \frac{|z_T|}{\gamma d^{1/\gamma - 1}} \right) + o(b_T T) \right\}
\]

- This gives the same optimization problem as for the PAM.
Proof idea for Weibull case

Identify the optimal strategy for BRW:

- Try to get to a good site $z$ with $z_T := z/r_T$ and $\xi_T(z) = \frac{\xi(z) - a_T}{b_T}$ of order one.

- Taking the route via a decent site $w$ near the origin, we can show that the first particle arrives at $z$ no later than

$$|z_T| \frac{T}{\gamma d^{1/\gamma} \log T}.$$ 

- Then, by time $T$, we have at least the following number of particles:

$$\exp \left\{ \xi(z) \left( T - \frac{|z_T|}{\gamma d^{1/\gamma} \log T} \right) \right\}$$

$$= \exp \left\{ a_T T + b_T T \left( \xi_T(z) - \frac{|z_T|}{\gamma d^{1/\gamma - 1}} \right) + o(b_T T) \right\}$$

- This gives the same optimization problem as for the PAM.
Proof idea for Weibull case

Identify the optimal strategy for BRW:

- Try to get to a good site $z$ with $z_T := z/r_T$ and $\xi_T(z) = \frac{\xi(z) - a_T}{b_T}$ of order one.
- Taking the route via a decent site $w$ near the origin, we can show that the first particle arrives at $z$ no later than
  \[
  \frac{|z_T|}{\gamma d^{1/\gamma}} \frac{T}{\log T}.
  \]
- Then, by time $T$, we have at least the following number of particles:
  \[
  \exp \left\{ \xi(z) \left( T - \frac{|z_T|}{\gamma d^{1/\gamma}} \frac{T}{\log T} \right) \right\} = \exp \left\{ a_T T + b_T T \left( \xi_T(z) - \frac{|z_T|}{\gamma d^{1/\gamma-1}} \right) + o(b_T T) \right\}
  \]
- This gives the same optimization problem as for the PAM.
Conjecture:

For the parabolic Anderson model / branching random walks:

$$\log u(tT, r_T x) \sim t Ta_T + T b_T \Lambda_T(t, x),$$

where $\Lambda_T$ converges to the following functional of a Poisson point process (taking a supremum at each spatial position):

$$\Lambda(t, x) = \sup_{z \in \Pi} \left\{ t \xi(z) - \frac{|z - x|}{\gamma d^{1/\gamma - 1}} \right\}.$$
Conjecture:

For the parabolic Anderson model / branching random walks:

$$\log u(tT, r_T x) \sim t T a_T + T b_T \Lambda_T(t, x),$$

where $\Lambda_T$ converges to the following functional of a Poisson point process (taking a supremum at each spatial position):
Conjecture:

For the parabolic Anderson model / branching random walks:

\[
\log u(tT, rTx) \sim tTa_T + Tb_T\Lambda_T(t, x),
\]

where \( \Lambda_T \) converges to the following functional of a Poisson point process (taking a supremum at each spatial position):
Conjecture:

For the parabolic Anderson model / branching random walks:

\[ \log u(tT, r_T x) \sim t Ta_T + T b_T \Lambda_T(t, x), \]

where \( \Lambda_T \) converges to the following functional of a Poisson point process (taking a supremum at each spatial position):
Conjecture:

For the parabolic Anderson model / branching random walks:

$$\log u(tT, r_T x) \sim tT a_T + T b_T \Lambda_T(t, x),$$

where $\Lambda_T$ converges to the following functional of a Poisson point process (taking a supremum at each spatial position):
For the parabolic Anderson model / branching random walks:

\[ \log u(tT, r_T x) \sim tT a_T + T b_T \Lambda_T(t, x), \]

where \( \Lambda_T \) converges to the following functional of a Poisson point process (taking a supremum at each spatial position):
Conjecture:

For the parabolic Anderson model / branching random walks:

\[
\log u(tT, r_Tx) \sim tT a_T + T b_T \Lambda_T(t, x),
\]

where \( \Lambda_T \) converges to the following functional of a Poisson point process (taking a supremum at each spatial position):

\[
\lim_{T \to \infty} \Lambda_T(t, x) = \Lambda(t, x).
\]
Conjecture:

For the parabolic Anderson model / branching random walks:

$$\log u(tT, r_T x) \sim tT a_T + T b_T \Lambda_T(t, x),$$

where $\Lambda_T$ converges to the following functional of a Poisson point process (taking a supremum at each spatial position):
Open problems:

For branching random walks in random environment

- Double exponential potential?
- Branching rate 1 and (soft or hard) killing according to random potential?
  \( \sim \) corresponds to parabolic Anderson model with bounded potential. [Engländer 2011, 2015]
- Correlated potentials? \( \sim \) any new effects?

Related (more realistic) models of population growth in random environment:

- In Pareto case: the population growth is super-exponential and front of particles is driven by extreme large-deviations events.
- Is there an interesting model with more realistic particle behaviour that shows similar effect as our lilypad model?
- Incorporate local competition to restrain population growth?
Open problems:

For branching random walks in random environment

- Double exponential potential?
- Branching rate 1 and (soft or hard) killing according to random potential?
  \(\leadsto\) corresponds to parabolic Anderson model with bounded potential. [Engländer 2011, 2015]
- Correlated potentials? \(\leadsto\) any new effects?

Related (more realistic) models of population growth in random environment:

- In Pareto case: the population growth is super-exponential and front of particles is driven by extreme large-deviations events.
- Is there an interesting model with more realistic particle behaviour that shows similar effect as our lilypad model?
- Incorporate local competition to restrain population growth?
Open problems:

For branching random walks in random environment

- Double exponential potential?
- Branching rate 1 and (soft or hard) killing according to random potential?
  \[\sim\] corresponds to parabolic Anderson model with bounded potential. \[\textit{Engländer 2011, 2015}\]
- Correlated potentials? \[\sim\] any new effects?

Related (more realistic) models of population growth in random environment:

- In Pareto case: the population growth is super-exponential and front of particles is driven by extreme large-deviations events.
- Is there an interesting model with more realistic particle behaviour that shows similar effect as our lilypad model?
- Incorporate local competition to restrain population growth?
Open problems:

For branching random walks in random environment

- Double exponential potential?
- Branching rate 1 and (soft or hard) killing according to random potential?
  \[ \rightsquigarrow \] corresponds to parabolic Anderson model with bounded potential. [Engländer 2011, 2015]
- Correlated potentials? \[ \rightsquigarrow \] any new effects?

Related (more realistic) models of population growth in random environment:

- In Pareto case: the population growth is super-exponential and front of particles is driven by extreme large-deviations events.
- Is there an interesting model with more realistic particle behaviour that shows similar effect as our lilypad model?
- Incorporate local competition to restrain population growth?
Open problems:

For branching random walks in random environment

- Double exponential potential?
- Branching rate 1 and (soft or hard) killing according to random potential?
  \( \sim \) corresponds to parabolic Anderson model with bounded potential. \textbf{[Engländer 2011, 2015]}
- Correlated potentials? \( \sim \) any new effects?

Related (more realistic) models of population growth in random environment:

- In Pareto case: the population growth is super-exponential and front of particles is driven by extreme large-deviations events.
- Is there an interesting model with more realistic particle behaviour that shows similar effect as our lilypad model?
- Incorporate local competition to restrain population growth?
Open problems:

For branching random walks in random environment

- Double exponential potential?
- Branching rate 1 and (soft or hard) killing according to random potential?
  \[ \sim \] corresponds to parabolic Anderson model with bounded potential. [Engländer 2011, 2015]
- Correlated potentials? \[ \sim \] any new effects?

Related (more realistic) models of population growth in random environment:

- In Pareto case: the population growth is super-exponential and front of particles is driven by extreme large-deviations events.
- Is there an interesting model with more realistic particle behaviour that shows similar effect as our lilypad model?
- Incorporate local competition to restrain population growth?