Burgers equation with random forcing

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August 2018

Burgers equation

$$u_t + uu_x = \nu u_{xx} + f, \quad (x,t) \in \mathbb{R} \times \mathbb{R}$$

$\nu \geq 0$: viscosity

In this talk: space-time random f averaging to 0

- Invariant distributions
- Global stationary solutions
- One Force — One Solution Principle (1F1S)
- Infinite-volume limits for associated directed polymers ($\nu > 0$) and action minimizers ($\nu = 0$)
- $\nu \downarrow 0$
- Compact/periodic case (1990's–2000's)
- Noncompact case (2010's)
- We will start with reminders about Burgers equation

Burgers equation: fluid dynamics interpretation

Evolution of velocity field u in $\mathbb{R}^1$

$$u_t + uu_x = \nu u_{xx} + f, \quad (t, x) \in \mathbb{R} \times \mathbb{R}$$

l.h.s.= acceleration of particle at (t, x):

$$\dot{x}(t) = u(t, x(t))$$
$$\ddot{x}(t) = (\textit{chain rule}) = u_t + u_x \dot{x} = u_t + uu_x$$

- $\nu = 0$: particles do not interact until they bump into each other creating shock waves.
- $\nu > 0$: smoothing of the velocity profile

Energy: pumped in by f, dissipated through friction

Burgers equation: via HJ equation

$$u_t + uu_x = \nu u_{xx} + f$$

$$u = U_x, \; f = F_x$$

HJ with quadratic Hamiltonian; KPZ

$$U_t + \frac{(U_x)^2}{2} = \nu U_{xx} + F, \quad (t,x) \in \mathbb{R} \times \mathbb{R}$$

F : external potential. If space-time white noise, then KPZ

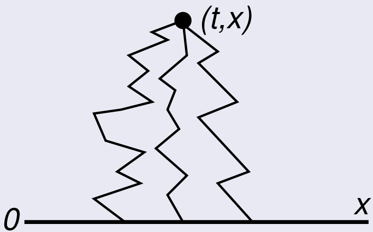
Cauchy problem for $\nu > 0$

Hopf–Cole substitution (1950-1951), also Florin (1948)

$$u = U_x = -2\nu(\log v)_x \quad \Longrightarrow \quad v_t = \nu v_{xx} - \frac{F}{2\nu} v$$

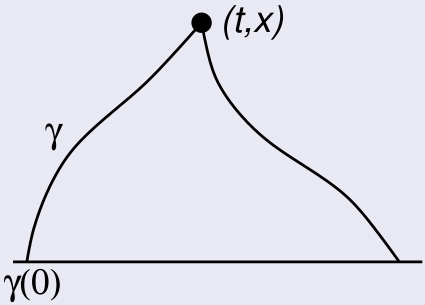
Feynman–Kac formula

$$v(t,x) = \mathbb{E}\left[e^{-\frac{1}{2\nu}\int_0^t F(t-s, x+\sqrt{2\nu}W_s)ds} v(0, x+\sqrt{2\nu}W_t)\right]$$



HJBHLO variational principle for $\nu = 0$

$$U(t,x) = \inf_{\substack{\gamma:[0,t]\to\mathbb{R}\\ \gamma(t)=x}} \left\{ U_0(\gamma(0)) + \frac{1}{2}\int_0^t \dot\gamma^2(s)ds + \int_0^t F(s,\gamma(s))ds \right\}.$$



$$u(t,x) = \begin{cases} U_x(t,x) \\ \dot\gamma(t). \end{cases}$$

Euler–Lagrange equation

$$\ddot\gamma(t) = f(t,\gamma(t))$$

$F = 0 \implies$ straight lines

Ergodic theory for random forcing, inviscid case

E,Khanin,Mazel,Sinai (Ann.Math. 2000)

Potential forcing on the circle $\mathbb{T}^1$:

$$F(t,x) = \sum F_j(x)\dot{W}_j(t)$$

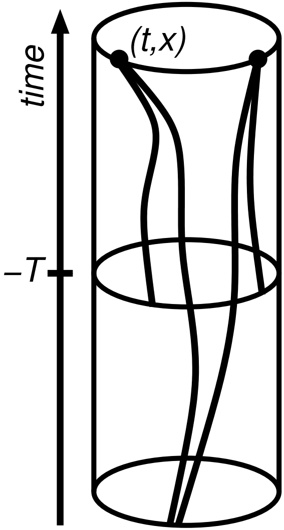
Theorem

Ergodic components:

$$\left\{ u : \int_{\mathbb{T}^1} u = c \right\}$$

One Force – One Solution Principle (1F1S) on each component.

One Force – One Solution (1F1S)



Initial conditions at time $-T$:
identical 0 or other. Take $-T$ to $-\infty$.

Slope stabilizes to some $u(t,x)$,
global attracting solution

$$u(t,x) = \Phi(\textit{forcing in the past})$$
$$\textit{Law}(u(t,x)) = \textit{stationary disribution}$$

Hyperbolicity: exponential closeness of minimizers in reversed time.

Solutions of HJB: Busemann functions

Bounds on velocity of minimizers

Invariant measures for Markov processes generated by random dynamical systems

Ledrappier–Young (1980's)

In general, any invariant measures of a Markov process generated by a random dynamical system can be represented via *sample measures*

$$\mu(\cdot) = \int_\Omega \mathsf{P}(d\omega)\mu_\omega(\cdot),$$

where μ_ω depends only on $\omega\big|_{(-\infty,0]}$

1F1S

μ_ω are Dirac δ-measures for almost every ω.

Other results in compact setting

Compact setting

- Gomes, Iturriaga, Khanin, Padilla (2000's): On $\mathbb{T}^d$
- Bakhtin (2007): On $[0, 1]$ with random boundary conditions
- Boritchev, Khanin (2013): Simplified proof of hyperbolicity
- Khanin, Zhang (2017): Hyperbolicity in $\mathbb{T}^d$
- Mixing rates based on hyperbolicity: Boritchev (2018) on $\mathbb{T}^1$; Iturriaga, Khanin, Zhang (recent) on $\mathbb{T}^d$:
- Hairer, Mattingly(2018) Strong Feller property for space-time white-noise KPZ
- Dirr, Souganidis (2005), Debussche and Vovelle (2015): extensions by "PDE methods"
- Chueshov, Scheutzow, Flandoli, Gess(2004,...) Synchronization by noise in monotone systems

Noncompact Setting

Quasi-compact setting

Hoang, Khanin (2003), Suidan (2005), Bakhtin (2013)

Truly noncompact setting

Two models where ergodic program goes through.

- Bakhtin, Cator, Khanin (JAMS 2014): space-time homogeneous Poissonian forcing
- Bakhtin (EJP, 2016): space-homogeneous i.i.d. kick forcing

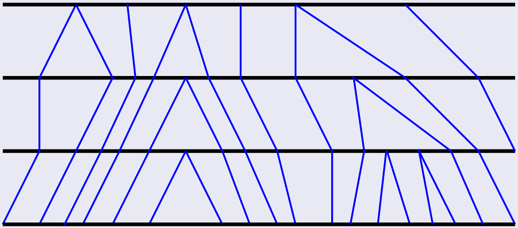
Space-continuous kick forcing

Forcing applies only at times $n \in \mathbb{Z}$

$$F(t,x) = \sum_{n\in\mathbb{Z}} F_{n,\omega}(x)\delta(t-n),$$

$(F_n)_{n\in\mathbb{Z}}$: i.i.d., stationary, decorrelation, tails

$$\text{Action}(\gamma) = U(\gamma(0)) + \frac{1}{2}\sum_{k=m}^{n-1}(\gamma_{k+1}-\gamma_k)^2 + \sum_{k=m}^{n-1} F_{k,\omega}(\gamma_k)$$



Minimizers (geodesics) for FPP,LPP-type models; Busemann functions

- C.D.Howard, C.M.Newman (late 1990's)
- M.Wüthrich (2002)
- E.Cator, L.Pimentel (2010–2012)

Theorem

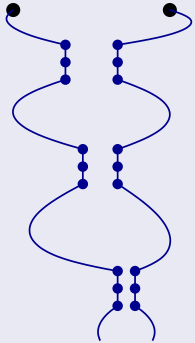
- *For each* $v \in \mathbb{R}$ *there is a unique global solution* $u_v(t,x)$ *with average velocity* v*.*
- $u_v(t,\cdot)$ *is determined by the history of the forcing up to* t *(1F1S)*
- u_v *is a one-point pullback attractor for initial conditions with average velocity* v*.*
- *For any* t*,* $\quad u_v(t,x)$ *is a stationary mixing process in* x*.*

Results in terms of one-sided minimizers

Theorem

Let $v \in \mathbb{R}$. Then, with probability one:

- *For most (t, x) there is a unique one-sided minimizer with slope v. Finite minimizers converge to infinite ones.*
- $\liminf_{m \to -\infty} \frac{|\gamma_m^1 - \gamma_m^2|}{|m|^{-1}} = 0.$
- *Busemann functions and global solutions are uniquely defined by partial limits*



Shape function for point-to-point minimizers

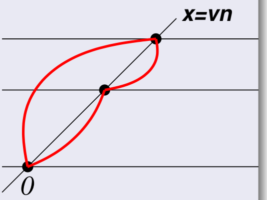
Best p2p action: $A^{m,n}(x,y) = \inf\{A^{m,n}(\gamma) : \gamma_m = x, \gamma_n = y\}$

Subadditivity:

$$A^{0,n}(0,vn) \le A^{0,m}(0,vm) + A^{m,n}(vm,vn)$$

so

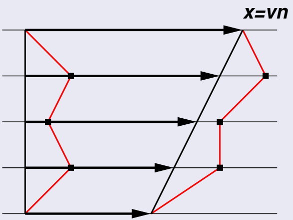
$$\lim_{t\to\infty} \frac{A^{0,n}(0,vn)}{n} = \alpha(v)$$



Shape function α
(effective Lagrangian)

$$\alpha(v) = \alpha(0) + \frac{v^2}{2}$$

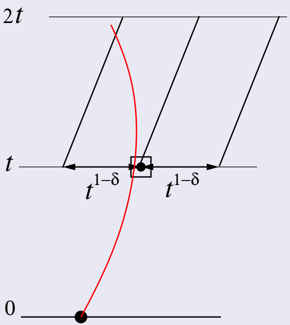
(due to shear invariance)



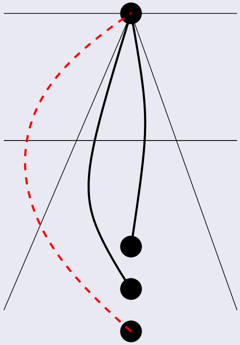
Theorem

For $u \in (c_3 n^{1/2} \ln^2 n, c_4 n^{3/2} \ln n]$*,*

$$\mathsf{P}\left\{|A^{0,n}(0, vn) - \alpha(v)n| > u\right\} \le c_1 \exp\left\{-c_2 \frac{u}{n^{1/2} \ln n}\right\},$$



Prob $< c_1 \exp\left(-c_2 t^{1/2-2\delta}\right)$



The rest of the program for zero viscosity/temperature

- Uniquenesss, with countably many exceptions (shocks) — uses shear invariance
- Weak hyperbolicity (minimizers approach each other in reverse time) — a soft lack-of-space argument
- Global solutions as Busemann functions for partial limits
- 1F1S: uniqueness of solution, attraction.
  - Potentials converge in LU;
  - $x \mapsto x - u(x)$ is a monotone function with discontinuities; convergence at every continuity point.

Positive viscosity

$$u_t + uu_x = \nu u_{xx} + f$$

Compact case

Sinai (1991), Gomes, Iturriaga, Khanin, Padilla (2000's)

Non-compact case: Kifer (1997)

$\mathbb{R}^d$, $d \geq 3$, small forcing,
perturbation theory series
(weak disorder)

Randomly kicked Burgers equation, $\nu > 0$

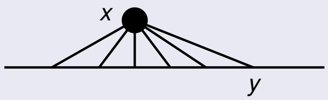
$$u_t + uu_x = \nu u_{xx} + \sum_{n\in\mathbb{Z}} f_n(x)\delta_n(t)$$

Hopf–Cole: $u = -2\nu(\ln v)_x = -2\nu\, v_x/v$

$$v(1-,x) = \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{4\nu}}}{\sqrt{4\pi\nu}} e^{-\frac{F_{0,\omega}(y)}{2\nu}} v(0-,y)dy$$
$$= \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{4\nu}}}{\sqrt{4\pi\nu}} e^{-\frac{F_{0,\omega}(y)}{2\nu} - \frac{U(0-,y)}{2\nu}} dy$$

$$u(1-,x) = \int_{\mathbb{R}} (x-y)\mu_x(dy)$$

$$\mu_x(dy) = \frac{e^{-\frac{(x-y)^2}{4\nu} - \frac{U(0,y)}{2\nu}} dy}{Z_x}$$



Multiple kicks, $\nu > 0$

Feynman–Kac evolution operator

$$\Psi_\omega^{0,n} v(y) = \int_{\mathbb{R}} Z_\omega^{0,n}(x,y) v(x) dx, \quad y \in \mathbb{R}$$

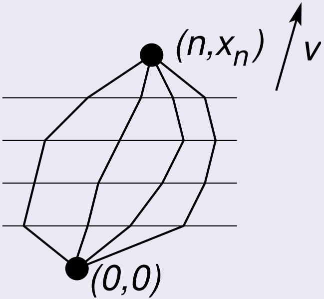
Point-to-point partition function:

$$Z_\omega^{0,n}(x_0, x_n) = \int_{\mathbb{R}\times\ldots\times\mathbb{R}} \cdots \int \prod_{k=0}^{n-1} \left[\frac{e^{-\frac{(x_{k+1}-x_k)^2}{4\nu}}}{\sqrt{4\pi\nu}} e^{-\frac{F_{k,\omega}(x_k)}{2\nu}} \right] dx_1 \ldots dx_{n-1}$$

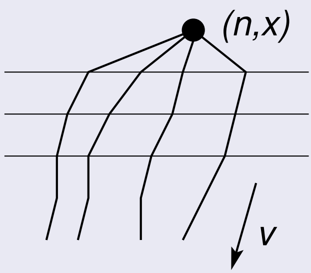
[Similar to product of positive random matrices]

Burgers Polymers Thermodynamic limit?

$$\mu^{0,n}_{x_0,x_n,\omega}(dx_1\ldots dx_{n-1}) = \frac{\prod_{k=0}^{n-1}\left[\frac{e^{-\frac{(x_{k+1}-x_k)^2}{4\nu}}}{\sqrt{4\pi\nu}} e^{-\frac{F_{k,\omega}(x_k)}{2\nu}}\right]}{Z^{0,n}_{\omega}(x_0,x_n)} dx_1\ldots dx_{n-1}$$



Burgers Polymers. Thermodynamic limit



Theorem (Bakhtin, Li: CPAM, 2018)

Fix any $v \in \mathbb{R}$. With probability 1,

- If $\lim_{m\to-\infty} \frac{x_m}{m} = v$, then

$$\lim_{m\to-\infty} \mu^{m,n}_{x_n,x,\omega} = \mu_\omega$$

  [Also point-to-line limits]

- μ_ω is a unique infinite volume polymer measure (DLR condition) with endpoint (n, x) and slope v:

$$\mu_\omega \left\{ \gamma : \lim_{m\to-\infty} \frac{\gamma_m}{m} = v \right\} = 1$$

- Asymptotic overlap of infinite volume polymer measures

Free energy per unit time

$$\lim_{n\to\infty}\left(-2\nu\frac{\ln Z^{0,n}(0,vn)}{n}\right) \overset{a.s.}{=} \alpha_\nu(v) = \alpha_\nu + \frac{v^2}{2}$$

Shear invariance

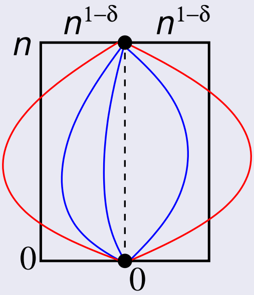
Concentration inequality

A straightness estimate for polymers

Lemma

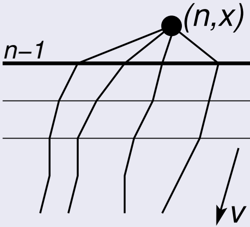
Let $\delta \in (0, 1/4)$. There are $\alpha, \beta > 0$: for large n,

$$\mathsf{P}\left\{\omega :\ \mu^{0,n}_{0,0,\omega}\left\{\gamma : \max_{0\le k\le n} |\gamma_k| > n^{1-\delta}\right\} \ge e^{-n^{\alpha}}\right\} \le e^{-n^{\beta}}.$$



First result on transversal exponent $\xi \le 3/4$: Mejane (2004)

Stationary solutions for viscous Burgers



$S(dy) :=$ distribution of the polymer location at time $n-1$

$$u_v(n,x) := \int_{\mathbb{R}} (x-y) S(dy)$$

Theorem

- *u_v is a unique global solution with slope v.*
- *1F1S: LU-convergence for u. In terms of the heat equation: the role of Busemann function is played by convergent ratios of partition functions.*

Zero viscosity (zero-temperature) limit

Theorem (Bakhtin,Li: JSP, 2018)

As $\nu \to 0$,

- *one-sided polymers converge to one-sided minimizers*
- *Global solutions of Burgers equation converge to inviscid global solutions [convergence at every continuity point of the monotone function* $x \mapsto x - u(x)$*]*

Based on tightness.
Events of interest are of the form: $\{$for all $\nu, \ldots\ldots\}$

Relevant recent results for discrete FPP/LPP/polymers

In discrete settings, Busemann functions and stationary solutions for positive and zero temperature polymers were studied by Rassoul-Agha, Georgiou, Seppäläinen, Yilmaz

Open problems

- More general HJB equations and Lax–Oleinik semigroups.
- General HJB equations with positive viscosity: generalized directed polymers via stochastic control
- Continuous non-white forcing, no shear invariance
- Higher dimensions: which form of hyperbolicity?
- Quantitative results
- KPZ equation, KPZ universality. CLT for solutions of Burgers HJB
- Statistics of shocks and concentration of minimizers
- Stochastic Navier–Stokes in noncompact setting

[Bakhtin, Khanin: Nonlinearity, 2018]