Coupling between mechanics and chemistry: Multiscale modelling and analysis

Mariya Ptashnyk

LMS Durham Symposium

Homogenization in Disordered Media

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HERIOT WATT UNIVERSITY

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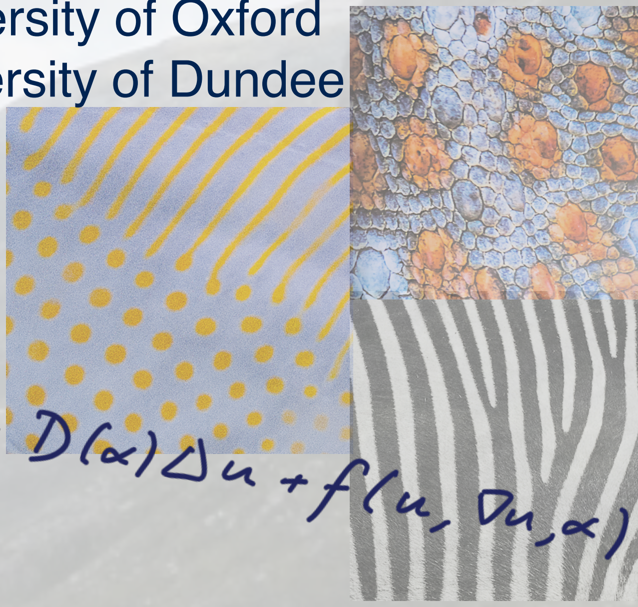
LMS-CMI Research School

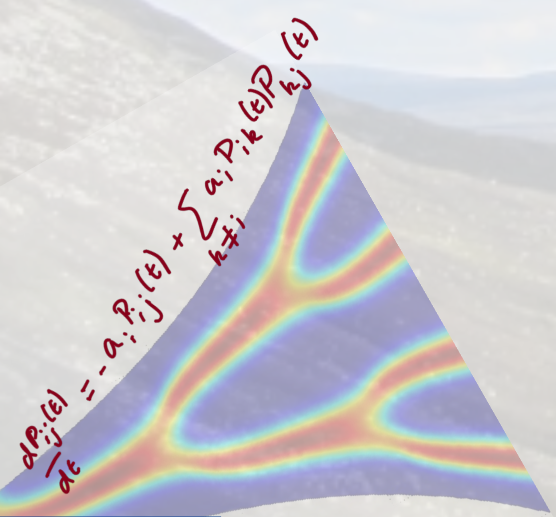
29 April – 3 May 2019
ICMS, Edinburgh

"PDEs in Mathematical Biology: Modelling and Analysis"

Lecturers
Dagmar Iber, ETH Zurich
Jonathan Potts, Sheffield University
Elaine Crooks, Swansea University
Benoit Perthame, Pierre et Marie Curie Univ.
Luigi Preziosi, Politecnico di Torino

Guest speakers
Angela Stevens, Muenster University
Alain Goriely, University of Oxford
Kees Weijier, University of Dundee





$$dv = F(D^2u, Du)\,dt + H(Du) \circ dW_t$$

$$\frac{\partial u}{\partial t} = D(\alpha)\Delta u + f(u, \nabla u, \alpha)$$

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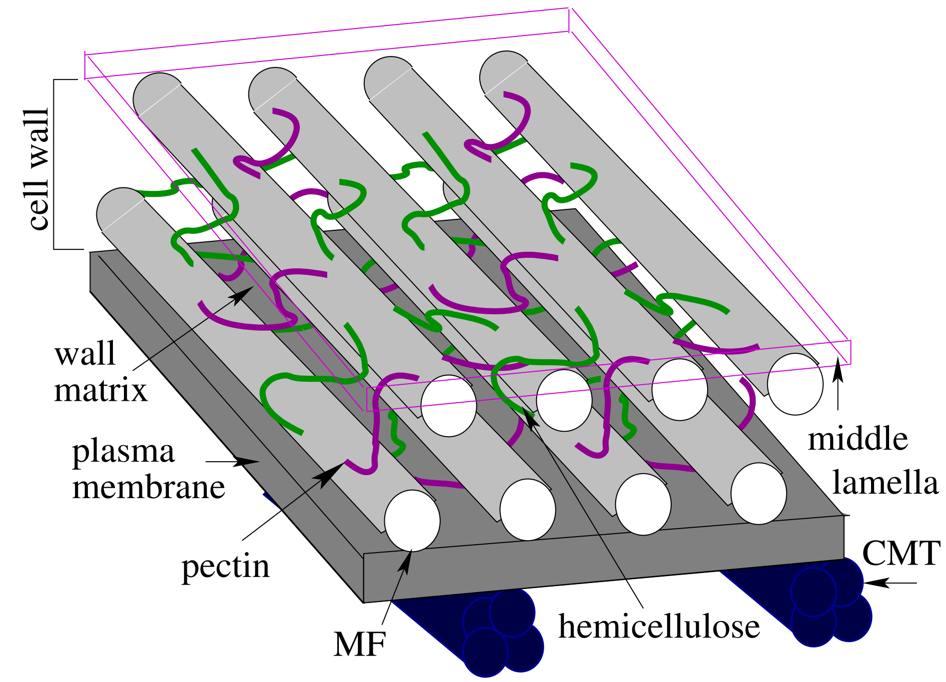
Plant cell walls: microstructure, mechanics & chemistry



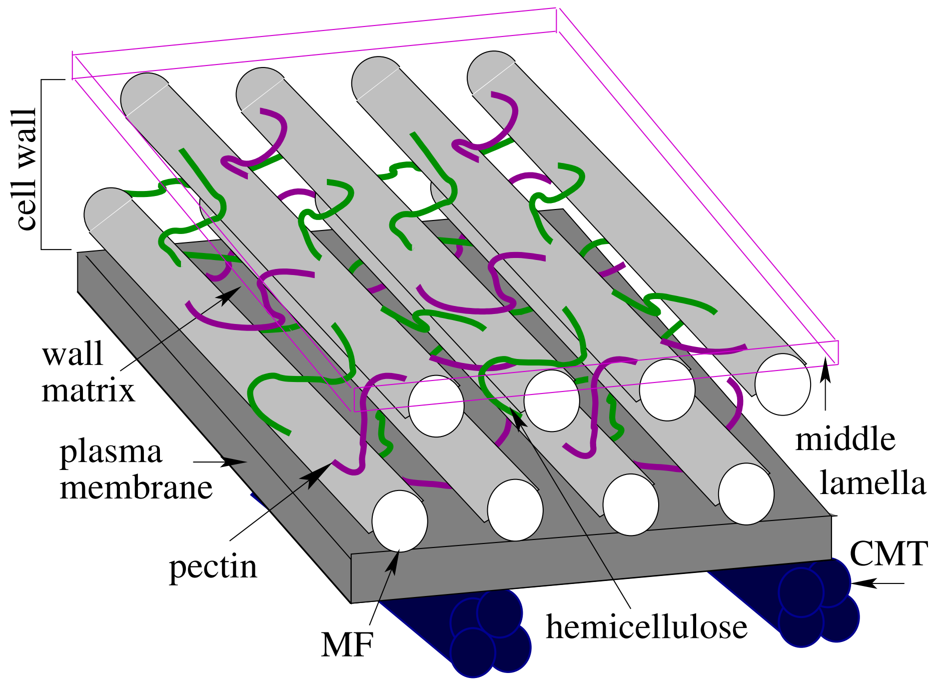
Microscopic structure of plant cell wall

- cellulose microfibrils
- cell wall matrix of pectin, hemicellulose, water, enzymes
- allows for anisotropic cell expansion

Interactions between mechanics and chemistry

- mechanical forces can break load-bearing cross-links
- dynamics of cross-links influences mechanical properties of plant cell wall matrix

Plant cell walls: microstructure, mechanics & chemistry



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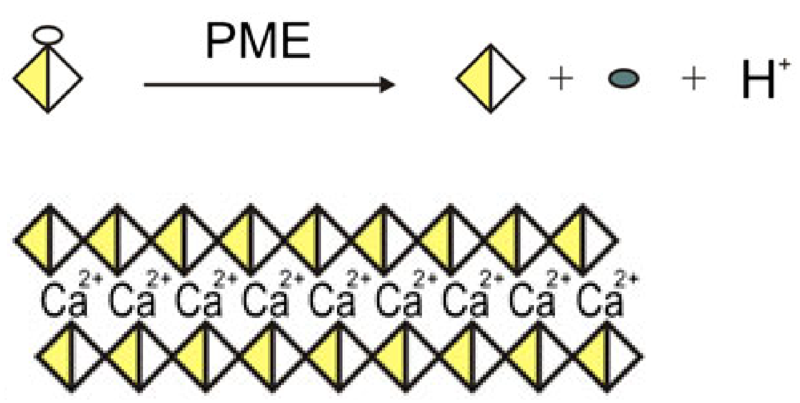


Wolf, Greiner, Protoplasma 2012

- Pectin is deposited into cell walls from Golgi apparatus in a methylesterified form
- enzyme PME interacts with methylestrified pectin to form demethylestrified pectin
- Demethylestrified pectin and calcium ions form calcium-pectin cross-links

Mechanics (hyperelastic material)

$$\operatorname{div}\mathbf{T} = 0, \qquad \mathbf{T} = J_e^{-1}\mathbf{F}_e\frac{\partial W(\mathbf{F}_e)}{\partial \mathbf{F}_e} \quad \text{or} \quad \mathbf{T} = \mathbf{F}_e\frac{\partial W(\mathbf{F}_e)}{\partial \mathbf{F}_e} - p\mathbf{I}$$

+ boundary conditions

$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}$ deformation gradient

$\mathbf{F} = \mathbf{F}_e\mathbf{F}_g$ decomposition in elastic & growth

$J_e = \det(\mathbf{F_e}),\ J_g = \det(\mathbf{F}_g)$

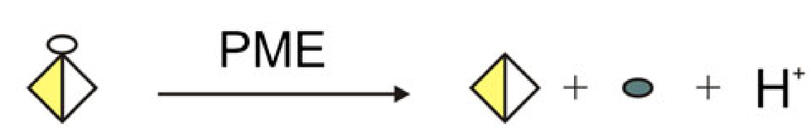
Ogden, *Nonlinear elastic deform.* 1984
Rodriguez, Hoger, McCulloch, *J Biomech.* 1994
Goriely, Moulton, Vandiver, *EPL* 2010
Goriely, Ben Amar, *J Mech.Phys.Solids* 2005
Goriely, Ben Amar, *Biomech.Model.Mechan.* 2007

Calcium-pectin chemistry

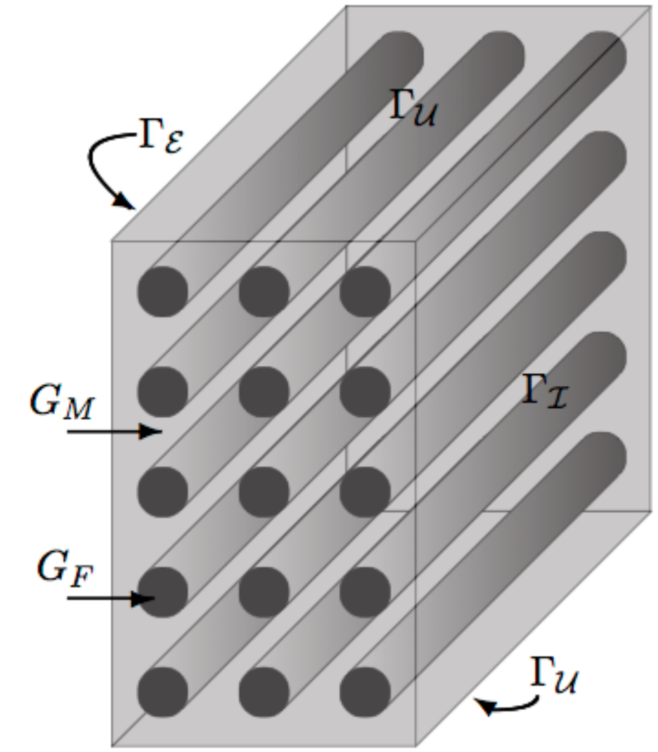
- methylestrified pectin: b_1
- demethylestrified pectin: b_2
- pectin-calcium cross links: b_3
- enzyme PME: p
- calcium ions: c

$$\partial_t n - \operatorname{div}(D_n\nabla n) = g(n, \nabla\mathbf{u})$$

$$n \in \{b_1, b_2, b_3, p, c\}$$



Plant cell walls: mechanics & chemistry



Linear elasticity

$$\operatorname{div} \mathbf{T} = 0 \quad \text{in} \quad G, \qquad \mathbf{T} \cdot \nu = -P\,\nu \quad \text{on} \quad \partial G$$

$$\mathbf{T} = \left(\mathbb{E}_M(b_3)\chi_{G_M} + \mathbb{E}_F\chi_{G_F}\right)\mathbf{e}(\mathbf{u}_e)$$

$$\mathbf{e}(\mathbf{u}_e)_{ij} = \tfrac{1}{2}(\partial_{x_i}\mathbf{u}_{e,j} + \partial_{x_j}\mathbf{u}_{e,i})$$

Reaction-diffusion equations for chemical reactions

$$\partial_t n - \operatorname{div}(D_n \nabla n) = g_n(n, \mathcal{R}(\mathbf{e}(\mathbf{u}_e))), \qquad n \in \{b_1, b_2, b_3, p, c\}$$

- demethyl-esterification of pectin by PME $\quad \dfrac{\kappa}{1+\beta b_2} b_1\, p$
- demethyl-esterified pectin can decay
- formation and destruction of calcium-pectin cross-links $\quad -2g(c)b_2 + 2\kappa\, b_3\, \mathcal{R}(\mathbf{e}(\mathbf{u}_e))$

$$\mathcal{R}(\mathbf{e}(\mathbf{u}_e)) = \left(\operatorname{tr}\left(\mathbb{E}_M(b_3)\chi_{G_M} + \mathbb{E}_F\chi_{G_F}\right)\mathbf{e}(\mathbf{u}_e)\right)^+$$

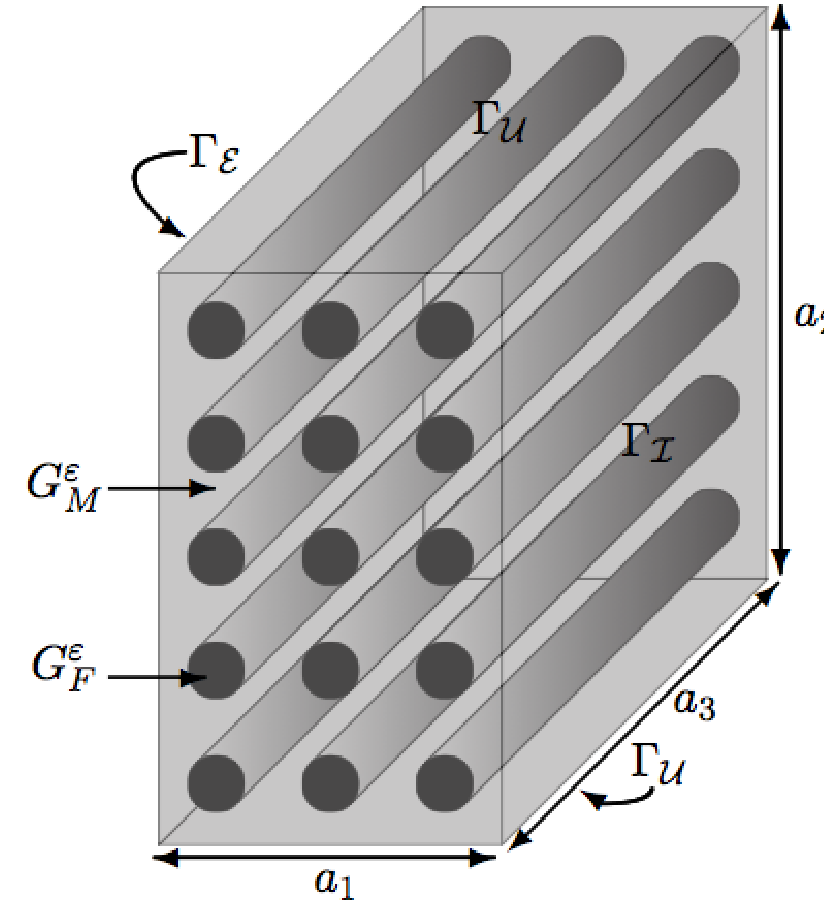
Microscopic Model

In $(0, T) \times G$

$$\mathrm{div}(\mathbb{E}^\varepsilon(b_3^\varepsilon, x)\mathbf{e}(\mathbf{u}_e^\varepsilon)) = \mathbf{0}$$

or

$$\mathrm{div}(\mathbb{E}^\varepsilon(b_3^\varepsilon, x)\mathbf{e}(\mathbf{u}_e^\varepsilon) + \mathbb{V}^\varepsilon(b_3^\varepsilon, x)\mathbf{e}(\partial_t \mathbf{u}_e^\varepsilon)) = \mathbf{0}$$



$\mathbb{E}^\varepsilon(\xi, x) = \mathbb{E}(\xi, \hat{x}/\varepsilon)$, $\quad \mathbb{V}^\varepsilon(\xi, x) = \mathbb{V}(\xi, \hat{x}/\varepsilon)$, where

$\mathbb{E}(\xi, \hat{y}) = \mathbb{E}_M(\xi)\, \chi_{\hat{Y}_M}(\hat{y}) + \mathbb{E}_F\, \chi_{\hat{Y}_F}(\hat{y})$, $\quad \mathbb{V}(\xi, \hat{y}) = \mathbb{V}_M(\xi)\, \chi_{\hat{Y}_M}(\hat{y})$ are $\hat{Y}$-periodic,

$\hat{Y} = Y \cap \{x_3 = \mathrm{const}\}$

MP, B. Seguin, *ESAIM: M2AN*, 2016

Microscopic Model

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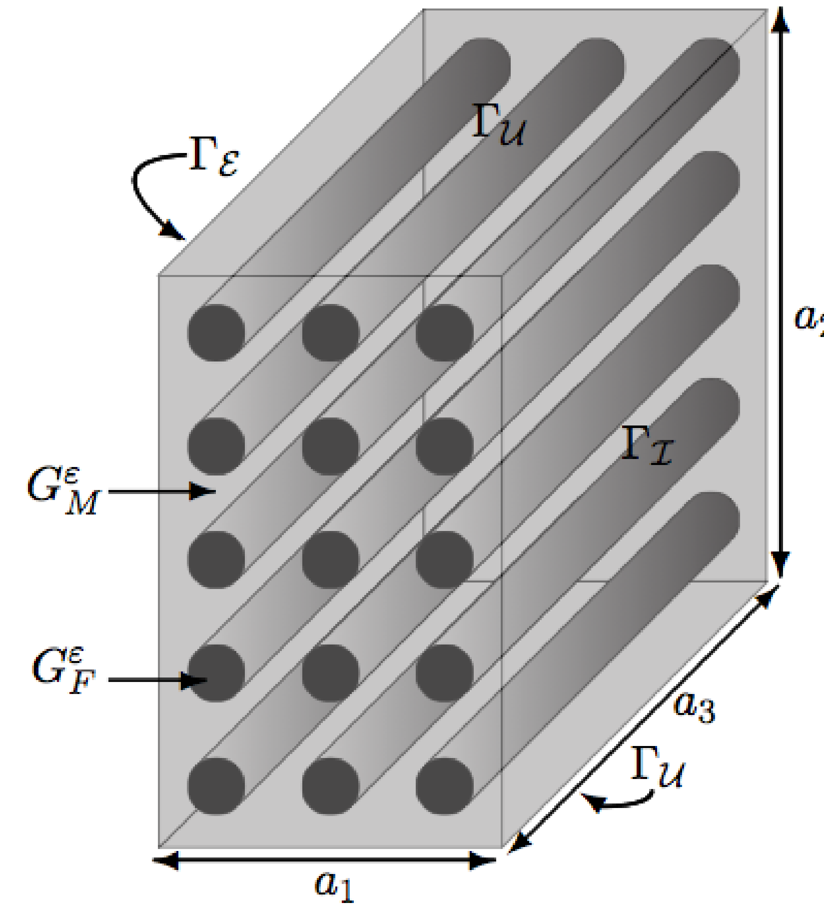

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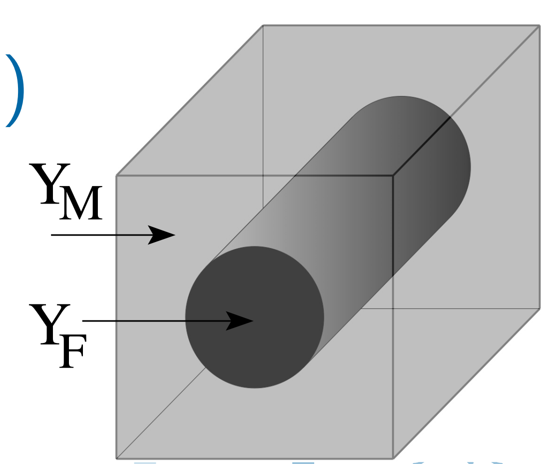
$\hat{Y} = Y \cap \{x_3 = \text{const}\}$

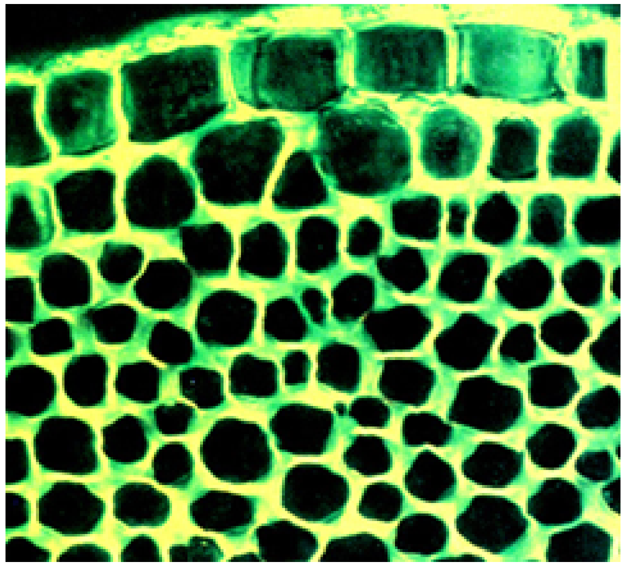
In $(0, T) \times G_M^\varepsilon$

$$\begin{aligned}
\partial_t p^\varepsilon &= \mathrm{div}(D_p \nabla p^\varepsilon) \\
\partial_t b_1^\varepsilon &= \mathrm{div}(D_{b_1} \nabla b_1^\varepsilon) - f(b_1^\varepsilon, b_2^\varepsilon, p^\varepsilon) \\
\partial_t b_2^\varepsilon &= \mathrm{div}(D_{b_2} \nabla b_2^\varepsilon) + f(b_1^\varepsilon, b_2^\varepsilon, p^\varepsilon) - 2g(c^\varepsilon)b_2^\varepsilon + 2\kappa\, b_3^\varepsilon\, \mathcal{R}(\mathbf{e}(\mathbf{u}_e^\varepsilon)) \\
\partial_t c^\varepsilon &= \mathrm{div}(D_c \nabla c^\varepsilon) - g(c^\varepsilon)b_2^\varepsilon + \kappa\, b_3^\varepsilon\, \mathcal{R}(\mathbf{e}(\mathbf{u}_e^\varepsilon)) \\
\partial_t b_3^\varepsilon &= \mathrm{div}(D_{b_3} \nabla b_3^\varepsilon) + g(c^\varepsilon)b_2^\varepsilon - \kappa\, b_3^\varepsilon\, \mathcal{R}(\mathbf{e}(\mathbf{u}_e^\varepsilon))
\end{aligned}$$



MP, B. Seguin, *ESAIM: M2AN*, 2016

Mathematical model for plant tissue



Biochemistry:

methylestrified pectin: b_1
demethylestrified pectin: b_2
calcium-pectin cross links: b_3
calcium ions: c_e and c_f

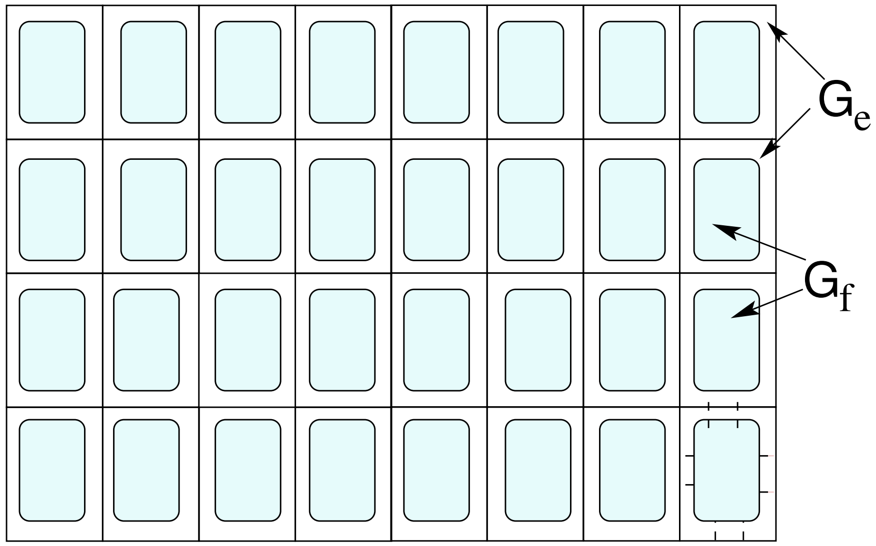
$$\partial_t b - \operatorname{div}(D_b \nabla b) = g_b(b, c_e, \mathbf{e}(\mathbf{u}_e)) \qquad \text{in } G_e$$

$$\partial_t c_e - \operatorname{div}(D_e \nabla c_e) = g_e(b, c_e, \mathbf{e}(\mathbf{u}_e)) \qquad \text{in } G_e$$

$$\partial_t c_f - \operatorname{div}(D_f \nabla c_f - \mathcal{G}(\partial_t \mathbf{u}_f) c_f) = g_f(c_f) \qquad \text{in } G_f$$

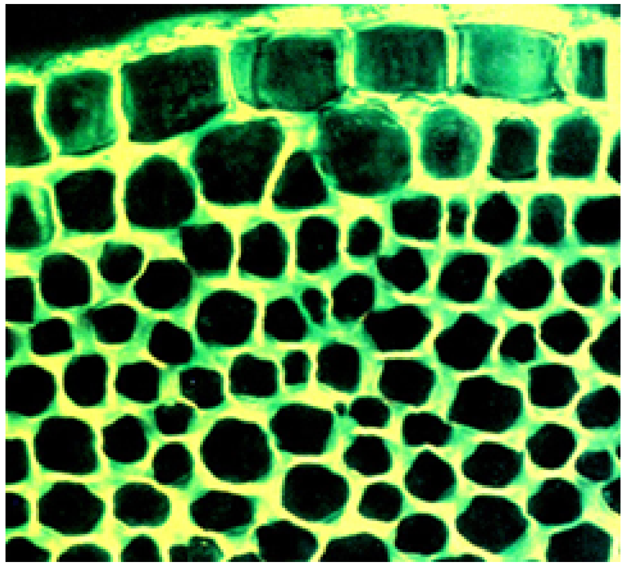
$b = (b_1, b_2, b_3)$

Mechanics: Poroelasticity



A. Piatnitski, MP, *MMS SIAM J*, 2017

Mathematical model for plant tissue



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$$\partial_t c_e - \operatorname{div}(D_e \nabla c_e) = g_e(b, c_e, \mathbf{e}(\mathbf{u}_e)) \quad \text{in } G_e$$
$$\partial_t c_f - \operatorname{div}(D_f \nabla c_f - \mathcal{G}(\partial_t \mathbf{u}_f) c_f) = g_f(c_f) \quad \text{in } G_f$$

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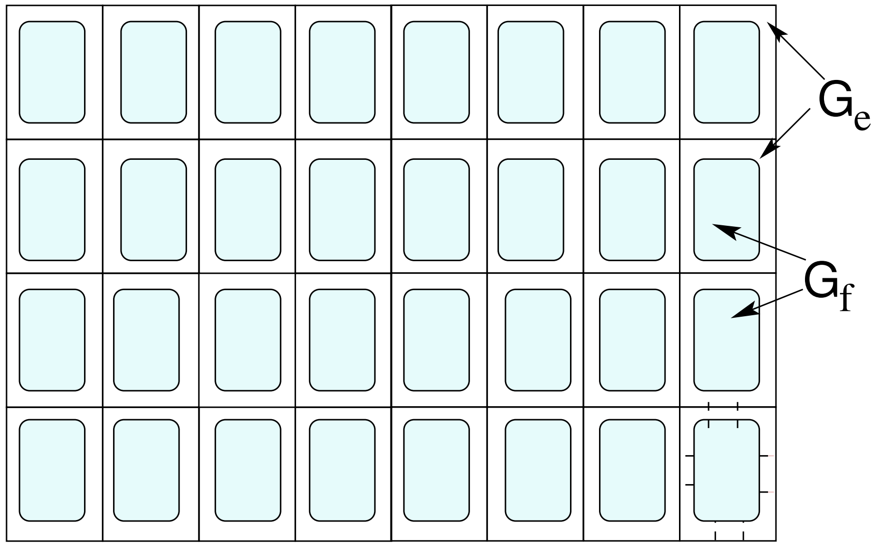
Mechanics: Poroelasticity

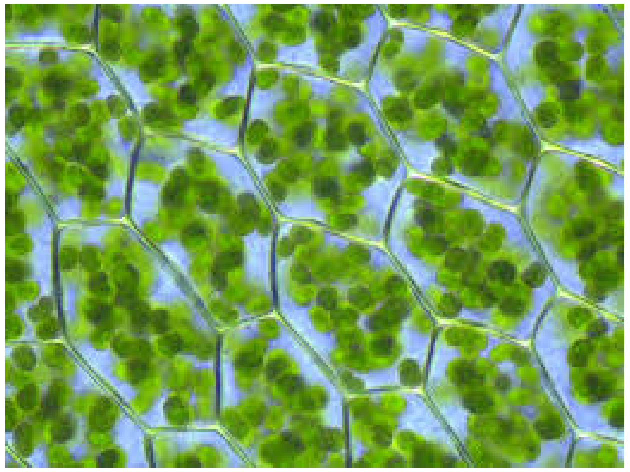


$\mathbf{u}_e$ - deformations of cell walls+middle lamella
p_e - flow pressure in cell walls+middle lamella
$\partial_t \mathbf{u}_f$ - fluid flow inside the cells

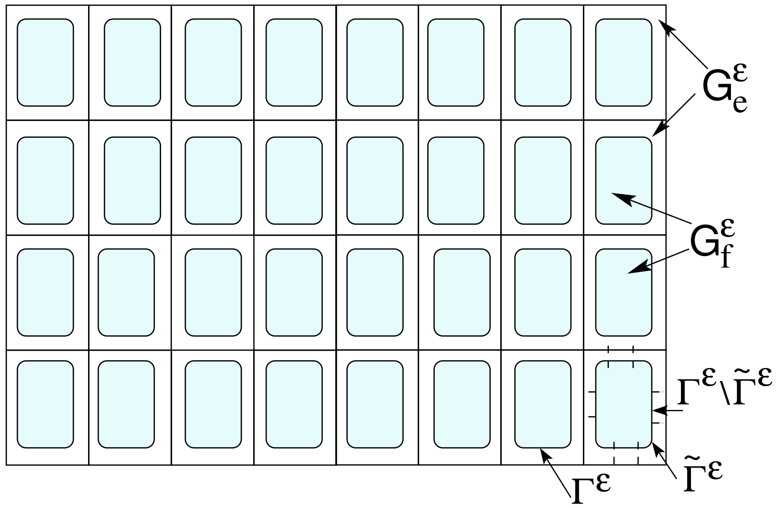
$$\operatorname{div}\left(\mathbb{E}(b_3)\, \mathbf{e}(\mathbf{u}_e) - p_e I\right) = 0 \quad \text{in } G_e$$
$$\operatorname{div}(K \nabla p_e - \partial_t \mathbf{u}_e) = 0 \quad \text{in } G_e$$
$$\partial_t(\partial_t \mathbf{u}_f) - \operatorname{div}(\mu\, \mathbf{e}(\partial_t \mathbf{u}_f) - p_f I) = 0 \quad \text{in } G_f$$

Piatnitski, MP, *MMS SIAM J*, 2017

Plant tissue biomechanics: Poroelasticity



http://commons.wikimedia.org/wiki/



$$-\operatorname{div}(\mathbb{E}^\varepsilon(b_3^\varepsilon)\mathbf{e}(\mathbf{u}_e^\varepsilon)) + \nabla p_e^\varepsilon = 0 \quad \text{in } G_e^\varepsilon$$

$$-\operatorname{div}(K_p^\varepsilon \nabla p_e^\varepsilon - \partial_t \mathbf{u}_e^\varepsilon) = 0 \quad \text{in } G_e^\varepsilon$$

$$\partial_t^2 \mathbf{u}_f^\varepsilon - \varepsilon^2 \mu \operatorname{div}(\mathbf{e}(\partial_t \mathbf{u}_f^\varepsilon)) + \nabla p_f^\varepsilon = 0 \quad \text{in } G_f^\varepsilon$$

$$\operatorname{div} \partial_t \mathbf{u}_f^\varepsilon = 0 \quad \text{in } G_f^\varepsilon$$

Transmission conditions:

$$(\mathbb{E}^\varepsilon(b_3^\varepsilon)\,\mathbf{e}(\mathbf{u}_e^\varepsilon) - p_e^\varepsilon I)\,\nu = (\varepsilon^2 \mu\,\mathbf{e}(\partial_t \mathbf{u}_f^\varepsilon) - p_f^\varepsilon I)\,\nu \quad \text{on } \Gamma^\varepsilon$$

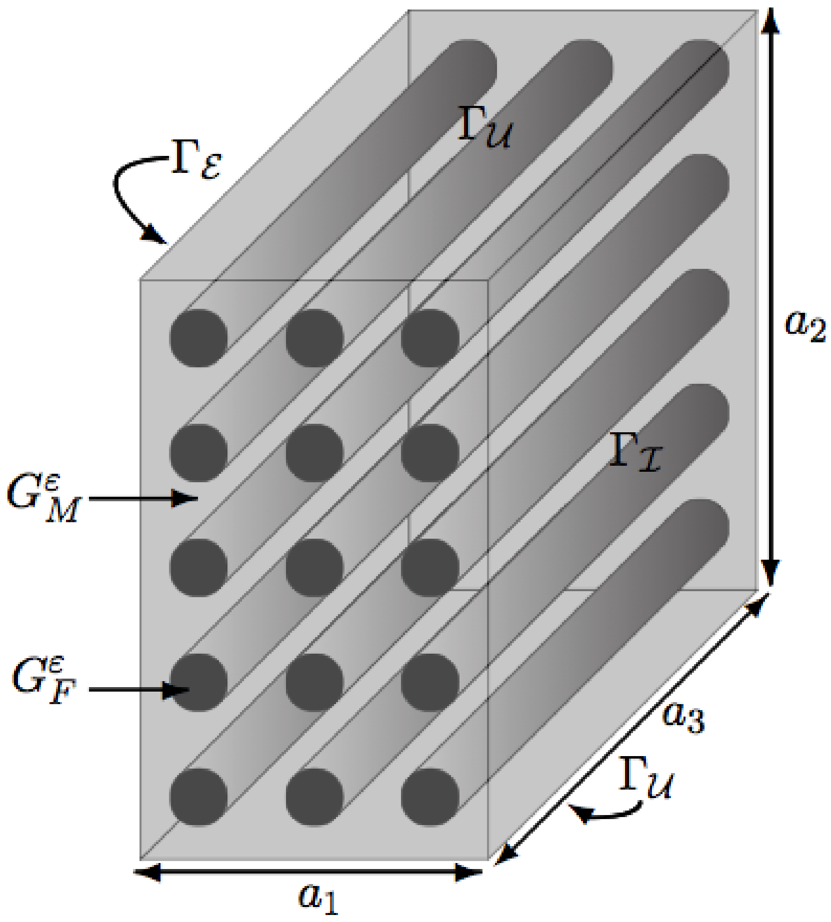
$$\Pi_\tau \partial_t \mathbf{u}_e^\varepsilon = \Pi_\tau \partial_t \mathbf{u}_f^\varepsilon \quad \text{on } \Gamma^\varepsilon$$

$$\nu \cdot (\varepsilon^2 \mu\,\mathbf{e}(\partial_t \mathbf{u}_f^\varepsilon) - p_f^\varepsilon)\,\nu = -p_e^\varepsilon \quad \text{on } \Gamma^\varepsilon$$

$$(-K_p^\varepsilon \nabla p_e^\varepsilon + \partial_t \mathbf{u}_e^\varepsilon) \cdot \nu = \partial_t \mathbf{u}_f^\varepsilon \cdot \nu \quad \text{on } \Gamma^\varepsilon$$

$\Pi_\tau w$ - tangential components

Microscopic Model



In $(0, T) \times G$

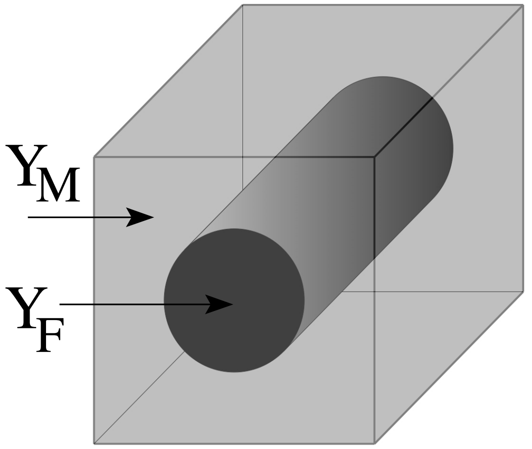
$$\mathrm{div}(\mathbb{E}^\varepsilon(b^\varepsilon, x)\mathbf{e}(\mathbf{u}_e^\varepsilon)) = \mathbf{0}$$

or

$$\mathrm{div}(\mathbb{E}^\varepsilon(b^\varepsilon, x)\mathbf{e}(\mathbf{u}_e^\varepsilon) + \mathbb{V}^\varepsilon(b^\varepsilon, x)\mathbf{e}(\partial_t \mathbf{u}_e^\varepsilon)) = \mathbf{0}$$

In $(0, T) \times G_M^\varepsilon$

$$\partial_t b^\varepsilon = \mathrm{div}(D_b \nabla b^\varepsilon) + g_b(b^\varepsilon, c^\varepsilon, \mathbf{e}(\mathbf{u}_e^\varepsilon))$$
$$\partial_t c^\varepsilon = \mathrm{div}(D_c \nabla c^\varepsilon) + g_c(b^\varepsilon, c^\varepsilon, \mathbf{e}(\mathbf{u}_e^\varepsilon))$$



Microscopic model for plant tissues

$$\mathrm{div}\left(\mathbb{E}^\varepsilon(x, b_3^\varepsilon)\, \mathbf{e}(\mathbf{u}_e^\varepsilon) - p_e^\varepsilon I\right) = 0 \qquad \text{in} \quad G_e^\varepsilon$$

$$\mathrm{div}(K\nabla p_e^\varepsilon - \partial_t \mathbf{u}_e^\varepsilon) = 0 \qquad \text{in} \quad G_e^\varepsilon$$

$$\partial_t(\partial_t \mathbf{u}_f^\varepsilon) - \mathrm{div}(\varepsilon^2 \mu\, \mathbf{e}(\partial_t \mathbf{u}_f^\varepsilon) - p_f^\varepsilon I) = 0 \qquad \text{in} \quad G_f^\varepsilon$$

and

$$\partial_t b^\varepsilon = \mathrm{div}(D_b \nabla b^\varepsilon) + g_b(b^\varepsilon, c_e^\varepsilon, \mathbf{e}(\mathbf{u}_e^\varepsilon)) \qquad \text{in} \quad G_e^\varepsilon$$

$$\partial_t c_e^\varepsilon = \mathrm{div}(D_e \nabla c_e^\varepsilon) + g_e(b^\varepsilon, c_e^\varepsilon, \mathbf{e}(\mathbf{u}_e^\varepsilon)) \qquad \text{in} \quad G_e^\varepsilon$$

$$\partial_t c_f^\varepsilon = \mathrm{div}(D_f \nabla c_f^\varepsilon - \mathcal{G}(\partial_t \mathbf{u}_f^\varepsilon) c_f^\varepsilon) + g_f(c_f^\varepsilon) \qquad \text{in} \quad G_f^\varepsilon$$

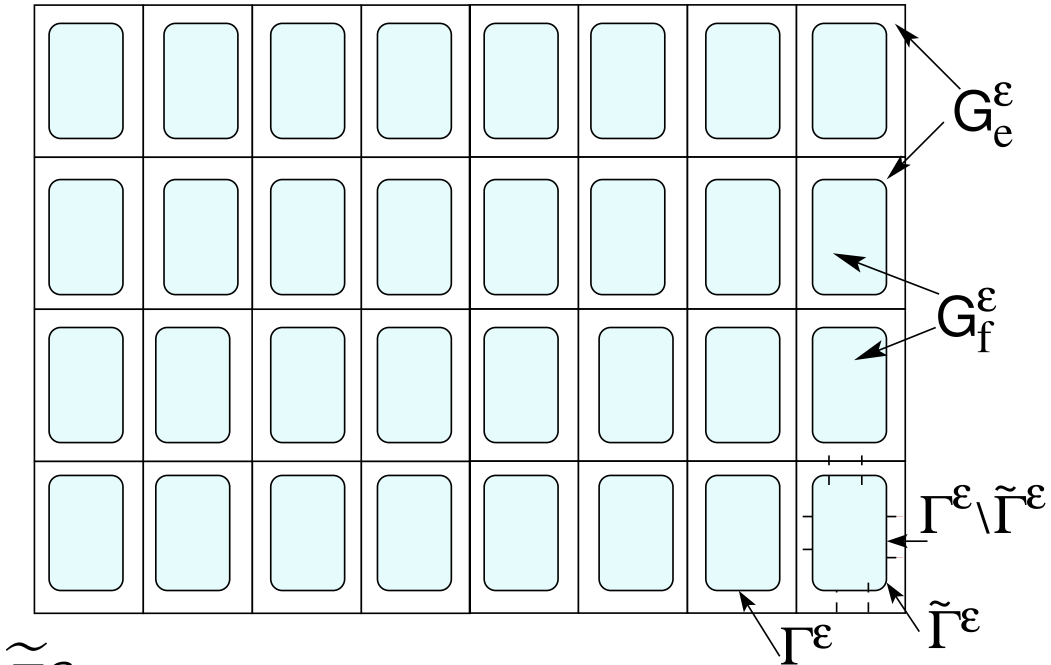
Transmission and boundary conditions:

$$\Pi_\tau \partial_t \mathbf{u}_e^\varepsilon = \Pi_\tau \partial_t \mathbf{u}_f^\varepsilon \qquad \text{on} \quad \Gamma^\varepsilon$$

$$D_b \nabla b^\varepsilon \cdot \nu = \varepsilon R(b^\varepsilon) \qquad \text{on } \Gamma^\varepsilon$$

$$c_e^\varepsilon = c_f^\varepsilon \qquad \text{on } \Gamma^\varepsilon \setminus \widetilde{\Gamma}^\varepsilon$$

$$D_e \nabla c_e^\varepsilon \cdot \nu = (D_f \nabla c_f^\varepsilon - \mathcal{G}(\partial_t \mathbf{u}_f^\varepsilon) c_f^\varepsilon) \cdot \nu \qquad \text{on } \Gamma^\varepsilon \setminus \widetilde{\Gamma}^\varepsilon$$



Existence of a weak solution for the tissue model

- Banach fixed point theorem for

$$\mathcal{K} \quad \text{over} \quad L^\infty(0,T;H^1(G_e^\varepsilon)^3) \times L^\infty(0,T;L^2(G_f^\varepsilon)^3)$$

$$\text{with} \;\; (\mathbf{u}_e^{\varepsilon,j}, \partial_t \mathbf{u}_f^{\varepsilon,j}) = \mathcal{K}(\mathbf{u}_e^{\varepsilon,j-1}, \partial_t \mathbf{u}_f^{\varepsilon,j-1})$$

- $$\|\mathbf{e}(\mathbf{u}_e^{\varepsilon,j+1} - \mathbf{u}_e^{\varepsilon,j})\|_{L^\infty(0,T;L^2(G_e^\varepsilon))} + \|\partial_t \mathbf{u}_f^{\varepsilon,j+1} - \partial_t \mathbf{u}_f^{\varepsilon,j}\|_{L^\infty(0,T;L^2(G_f^\varepsilon))} + \|\mathbf{e}(\partial_t \mathbf{u}_f^{\varepsilon,j+1} - \partial_t \mathbf{u}_f^{\varepsilon,j})\|_{L^2(0,T;L^2(G_f^\varepsilon))} \leq C \|b_3^{\varepsilon,j+1} - b_3^{\varepsilon,j}\|_{L^\infty(0,T;L^\infty(G_e^\varepsilon))}$$

- $$\|b_3^{\varepsilon,j+1} - b_3^{\varepsilon,j}\|_{L^\infty(0,T;L^\infty(G_e^\varepsilon))} \leq C\, T^\sigma \|\mathbf{e}(\mathbf{u}_e^{\varepsilon,j} - \mathbf{u}_e^{\varepsilon,j-1})\|_{L^\infty(0,T;L^2(G_e^\varepsilon))} + C_\delta\, T \|\partial_t \mathbf{u}_f^{\varepsilon,j} - \partial_t \mathbf{u}_f^{\varepsilon,j-1}\|_{L^\infty(0,T;L^2(G_f^\varepsilon))} + \delta \|\mathbf{e}(\partial_t \mathbf{u}_f^{\varepsilon,j} - \partial_t \mathbf{u}_f^{\varepsilon,j-1})\|_{L^2(0,T;L^2(G_f^\varepsilon))}$$

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- $$\|\mathbf{e}(\mathbf{u}_e^{\varepsilon,j+1} - \mathbf{u}_e^{\varepsilon,j})\|_{L^\infty(0,T;L^2(G_e^\varepsilon))} + \|\partial_t \mathbf{u}_f^{\varepsilon,j+1} - \partial_t \mathbf{u}_f^{\varepsilon,j}\|_{L^\infty(0,T;L^2(G_f^\varepsilon))}$$
$$+ \|\mathbf{e}(\partial_t \mathbf{u}_f^{\varepsilon,j+1} - \partial_t \mathbf{u}_f^{\varepsilon,j})\|_{L^2(0,T;L^2(G_f^\varepsilon))} \quad \leq C\|b_3^{\varepsilon,j+1} - b_3^{\varepsilon,j}\|_{L^\infty(0,T;L^\infty(G_e^\varepsilon))}$$

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$$+ C_\delta\, T \|\partial_t \mathbf{u}_f^{\varepsilon,j} - \partial_t \mathbf{u}_f^{\varepsilon,j-1}\|_{L^\infty(0,T;L^2(G_f^\varepsilon))}$$
$$+ \delta \|\mathbf{e}(\partial_t \mathbf{u}_f^{\varepsilon,j} - \partial_t \mathbf{u}_f^{\varepsilon,j-1})\|_{L^2(0,T;L^2(G_f^\varepsilon))}$$

- A priori estimates

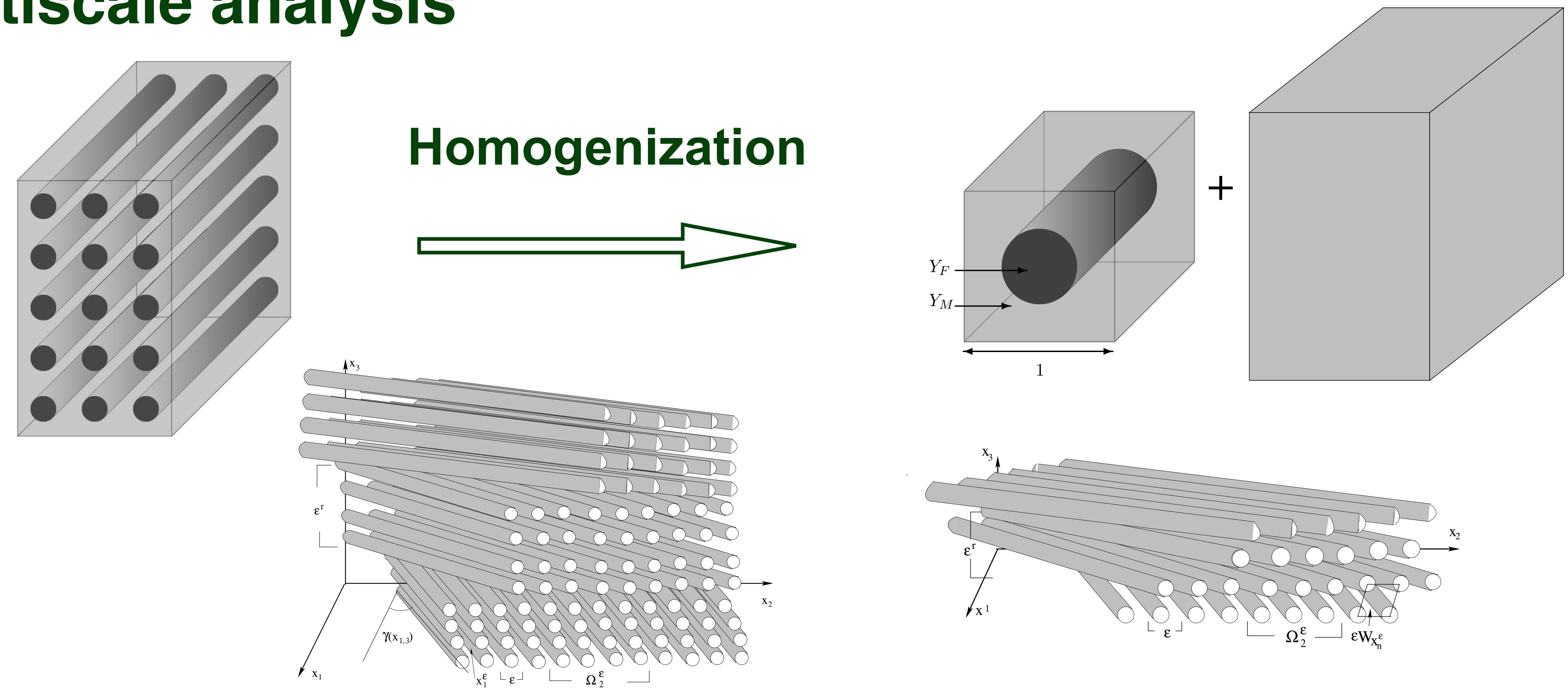
$$\|\partial_t u_e^\varepsilon\|_{L^\infty(0,T;H^1(G_e^\varepsilon))} + \|\partial_t^2 u_e^\varepsilon\|_{L^\infty(0,T;L^2(G_e^\varepsilon))} + \|\partial_t p_e^\varepsilon\|_{L^2(0,T;H^1(G_e^\varepsilon))} \leq C$$

$$\|\partial_t^2 u_f^\varepsilon\|_{L^\infty(0,T;L^2(G_f^\varepsilon))} + \varepsilon \|\nabla \partial_t u_f^\varepsilon\|_{H^1(0,T;L^2(G_f^\varepsilon))} + \|p_f^\varepsilon\|_{L^2(G_{f,T}^\varepsilon)} \leq C$$

$$\|b^\varepsilon\|_{L^\infty(0,T;L^\infty(G_e^\varepsilon))} + \|\nabla b^\varepsilon\|_{L^2(G_{e,T}^\varepsilon)} + \|c_l^\varepsilon\|_{L^\infty(0,T;L^\infty(G_l^\varepsilon))} + \|\nabla c_l^\varepsilon\|_{L^2(G_{l,T}^\varepsilon)} \leq C$$

$$l = e, f$$

Multiscale analysis



Microstructures:

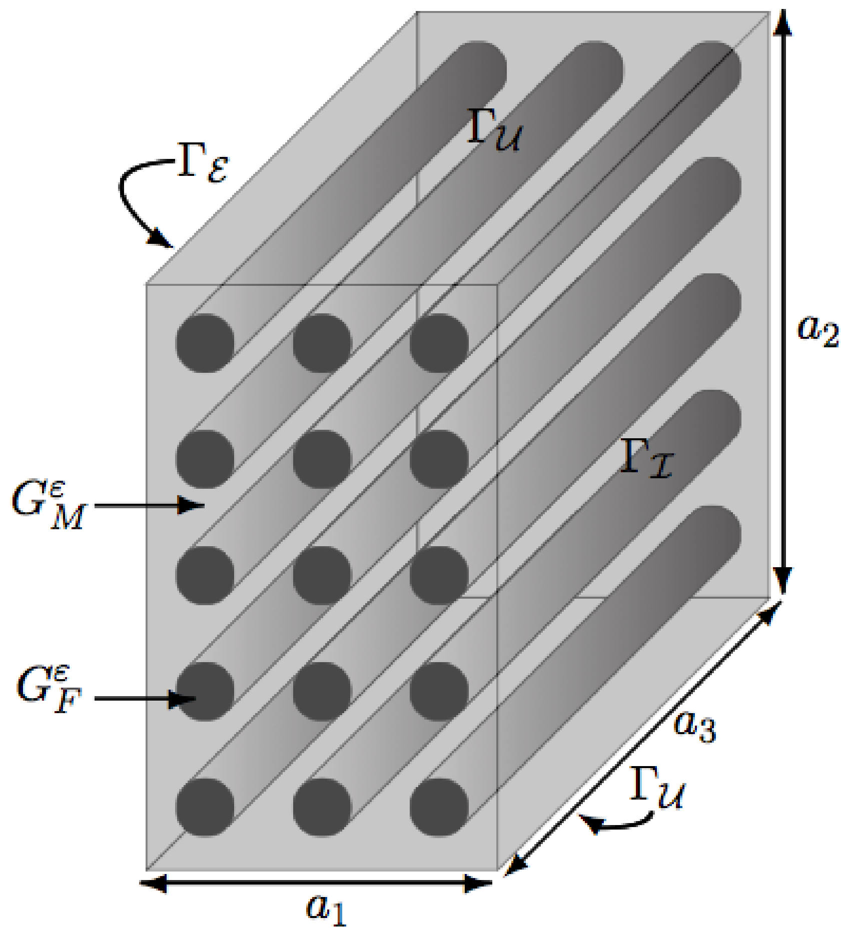
- Periodic or locally-periodic microstructures
- Stochastic microstructures

Methods:

- periodic, locally-periodic and stochastic two-scale convergence
- periodic and locally-periodic unfolding operators

periodic: Allaire, Cioranescu, Damlamian, Griso, Neues-Radu, Nguetseng, …
locally periodic: Briane, Alexandre, Mikelic, Mascarenhas, Toader, Polisevski, Arrieta, Villanueva-Pesqueira, …
stochastic: Bourgeat, Heida, Mikelic, Piatnitski, Zhikov, …

Microscopic Model



In $(0, T) \times G$

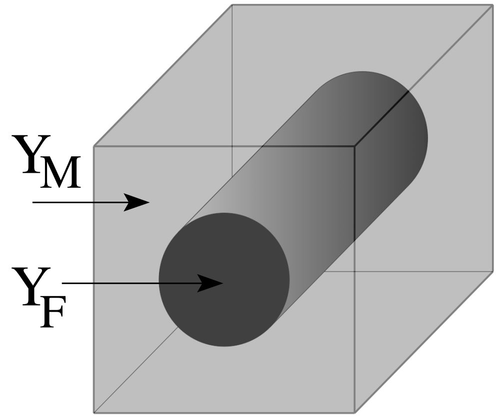
$$\mathrm{div}(\mathbb{E}^\varepsilon(b^\varepsilon, x)\mathbf{e}(\mathbf{u}_e^\varepsilon)) = \mathbf{0}$$

or

$$\mathrm{div}(\mathbb{E}^\varepsilon(b^\varepsilon, x)\mathbf{e}(\mathbf{u}_e^\varepsilon) + \mathbb{V}^\varepsilon(b^\varepsilon, x)\mathbf{e}(\partial_t \mathbf{u}_e^\varepsilon)) = \mathbf{0}$$

In $(0, T) \times G_M^\varepsilon$

$$\partial_t b^\varepsilon = \mathrm{div}(D_b \nabla b^\varepsilon) + g_b(b^\varepsilon, c^\varepsilon, \mathcal{R}(\mathbf{e}(\mathbf{u}_e^\varepsilon)))$$
$$\partial_t c^\varepsilon = \mathrm{div}(D_c \nabla c^\varepsilon) + g_c(b^\varepsilon, c^\varepsilon, \mathcal{R}(\mathbf{e}(\mathbf{u}_e^\varepsilon)))$$



$$\mathcal{R}(\mathbf{e}(\mathbf{u}_e^\varepsilon)) = \Big(\mathrm{tr}\,(\mathbb{E}_M(b^\varepsilon)\chi_{G_M} + \mathbb{E}_F \chi_{G_F})\,\mathbf{e}(\mathbf{u}_e^\varepsilon)\Big)^+$$

Strong convergence of b_3^ε, strong two-scale of $\mathbf{e}(u_e^\varepsilon)$

- In the case (no diffusion of b_3^ε)

$$\mathcal{R}(\mathbf{e}(\mathbf{u}_e^\varepsilon)) = \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}_e^\varepsilon))(t,x) = \Big(\fint_{B_\delta(x)\cap G} \operatorname{tr}\mathbb{E}^\varepsilon(b_3^\varepsilon,\tilde{x})\,\mathbf{e}(\mathbf{u}_e^\varepsilon(t,\tilde{x}))d\tilde{x}\Big)^+$$

$$\mathcal{T}^\varepsilon(b_3^\varepsilon) \to b_3 \qquad \text{strongly in } L^2(G_T \times Y_M),$$

$$b_3^\varepsilon \to b_3 \qquad \text{strongly two-scale}, \qquad \text{as } \varepsilon \to 0$$

- In the case

$$\mathcal{R}(\mathbf{e}(\mathbf{u}_e^\varepsilon)) = (\operatorname{tr}\mathbb{E}^\varepsilon(b_3^\varepsilon)\,\mathbf{e}(\mathbf{u}_e^\varepsilon))^+ \qquad (\text{ diffusion of } b_3^\varepsilon)$$

$$\mathbf{e}(\mathbf{u}_e^\varepsilon) \to \mathbf{e}(\mathbf{u}_e) + \mathbf{e}_y(\mathbf{u}_e^1) \qquad \text{strongly two-scale}$$

$$\mathbf{u}_e^1 \in L^2(G_T; H^1_{\mathrm{per}}(G)/\mathbb{R})$$

$$\partial_t \mathbf{u}_f^\varepsilon \to \partial_t \mathbf{u}_f \qquad \text{strongly two-scale}$$

$$\varepsilon\,\mathbf{e}(\partial_t \mathbf{u}_f^\varepsilon) \to \mathbf{e}_y(\partial_t \mathbf{u}_f) \qquad \text{strongly two-scale}, \qquad \text{as } \varepsilon \to 0$$

Strong convergence of b_3^ε, strong two-scale of $\mathrm{e}(u_e^\varepsilon)$

- In the case (no diffusion of b_3^ε)

$$\mathcal{R}(\mathbf{e}(\mathbf{u}_e^\varepsilon)) = \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}_e^\varepsilon))(t,x) = \Big(⨍_{B_\delta(x)\cap G} \operatorname{tr}\mathbb{E}^\varepsilon(b_3^\varepsilon,\tilde{x})\,\mathbf{e}(\mathbf{u}_e^\varepsilon(t,\tilde{x}))d\tilde{x}\Big)^+$$

$$\begin{aligned} \mathcal{T}^\varepsilon(b_3^\varepsilon) &\to b_3 && \text{strongly in } L^2(G_T\times Y_M), \\ b_3^\varepsilon &\to b_3 && \text{strongly two-scale,} \qquad \text{as } \varepsilon\to 0 \end{aligned}$$

- In the case

$$\mathcal{R}(\mathbf{e}(\mathbf{u}_e^\varepsilon)) = (\operatorname{tr}\mathbb{E}^\varepsilon(b_3^\varepsilon)\,\mathbf{e}(\mathbf{u}_e^\varepsilon))^+ \qquad \text{(diffusion of } b_3^\varepsilon)$$

$$\begin{aligned} \mathbf{e}(\mathbf{u}_e^\varepsilon) &\to \mathbf{e}(\mathbf{u}_e) + \mathbf{e}_y(\mathbf{u}_e^1) && \text{strongly two-scale} \\ & && \mathbf{u}_e^1 \in L^2(G_T; H^1_{\mathrm{per}}(G)/\mathbb{R}) \\ \partial_t\mathbf{u}_f^\varepsilon &\to \partial_t\mathbf{u}_f && \text{strongly two-scale} \\ \varepsilon\,\mathbf{e}(\partial_t\mathbf{u}_f^\varepsilon) &\to \mathbf{e}_y(\partial_t\mathbf{u}_f) && \text{strongly two-scale,} \qquad \text{as } \varepsilon\to 0 \end{aligned}$$

Macroscopic model for plant cell wall biomechanics

$$\operatorname{div}(\mathbb{E}_{\text{hom}}(b_3)\,\mathbf{e}(\mathbf{u}_e)) = \mathbf{0} \qquad \text{in } G_T$$

$$\partial_t b = \operatorname{div}(\mathcal{D}_b \nabla b) + g_b(b, c, R(\mathbf{e}(\mathbf{u}_e))) \qquad \text{in } G_T$$
$$\partial_t c = \operatorname{div}(\mathcal{D}_c \nabla c) + g_c(b, c, R(\mathbf{e}(\mathbf{u}_e))) \qquad \text{in } G_T$$

$$R(\mathbf{e}(\mathbf{u}_e)) = \Big(\fint_{B_\delta(x)\cap G} \operatorname{tr}\big(\mathbb{E}_{\text{hom}}(b_3)\,\mathbf{e}(\mathbf{u}_e)\big)\, d\tilde{x} \Big)^+ \quad \text{or} \quad \Big(\operatorname{tr}\big(\mathbb{E}_{\text{hom}}(b_3)\,\mathbf{e}(\mathbf{u}_e)\big) \Big)^+$$

$$\mathcal{D}_{\alpha,j3} = \mathcal{D}_{\alpha,3j} = D_\alpha \delta_{3j}, \quad \mathcal{D}_{\alpha,ij} = D_\alpha \fint_{\hat{Y}_M} \big[\delta_{ij} + \partial_{y_j} v_\alpha^i(y)\big]\, dy,$$

$$\alpha = b_1, b_2, b_3, c$$

$$\mathbb{E}_{\text{hom},ijkl}(b_3) = \fint_Y \big[\mathbb{E}_{ijkl}(b_3, y) + \big(\mathbb{E}(b_3, y)\mathbf{e}_y(\mathbf{w}^{ij})\big)_{kl}\big]\, dy$$

MP, B. Seguin, *ESAIM M2AN*, 2016

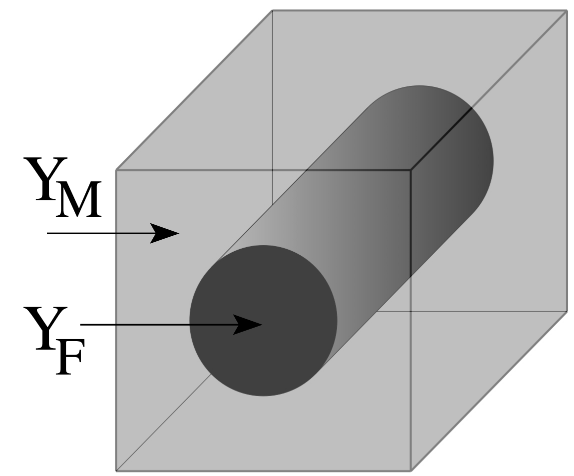
Unit cell problems

For effective diffusion coefficients

$$\mathrm{div}_{\hat{y}}(D_\alpha(\nabla_{\hat{y}} v_\alpha^j + \hat{\mathbf{b}}_j)) = 0 \quad \text{in } \hat{Y}_M, \quad j = 1, 2$$

$$D_\alpha(\nabla_{\hat{y}} v_\alpha^j + \hat{\mathbf{b}}_j) \cdot \boldsymbol{\nu} = 0 \quad \text{on } \hat{\Gamma}$$

$$\int_{\hat{Y}_M} v_\alpha^j dx = 0, \quad v_\alpha^j \quad \hat{Y} - \text{ periodic}$$



For effective elasticity tensor

$$\hat{\mathrm{div}}_y(\mathbb{E}(b_3, y)(\hat{\mathbf{e}}_y(\mathbf{w}^{ij}) + \mathbf{b}_{ij})) = \mathbf{0} \quad \text{in } \hat{Y},$$

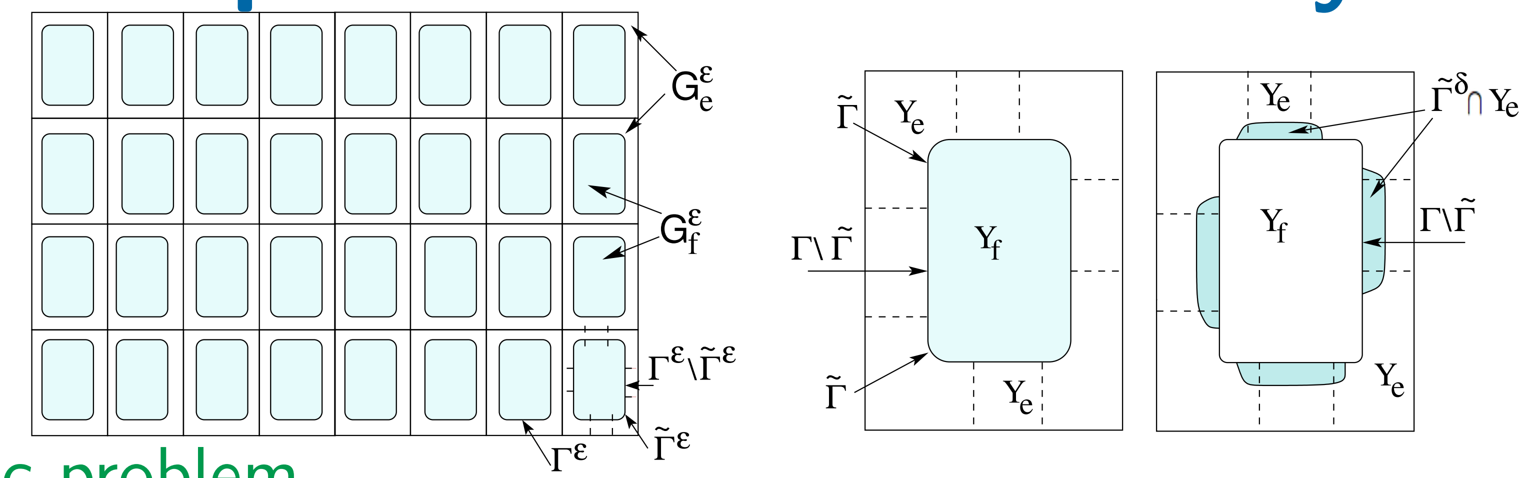
$$\int_{\hat{Y}} \mathbf{w}^{ij} dy = \mathbf{0}, \quad \mathbf{w}^{ij} \quad \hat{Y} - \text{ periodic}$$

$\hat{\mathrm{div}}_y \mathbf{v} = \partial_{y_1} \mathbf{v}_1 + \partial_{y_2} \mathbf{v}_2$ for $\mathbf{v} \in \mathbb{R}^3$

$\mathbf{b}_{jk} = \frac{1}{2}(\mathbf{b}_j \otimes \mathbf{b}_k + \mathbf{b}_k \otimes \mathbf{b}_j)$, $(\mathbf{b}_j)_{1 \le j \le 3}$ basis of $\mathbb{R}^3$, $\hat{\mathbf{b}}_1 = (1, 0)^T$, $\hat{\mathbf{b}}_2 = (0, 1)^T$

$\mathbb{E}(\xi, \hat{y}) = \mathbb{E}_M(\xi)\chi_{\hat{Y}_M}(\hat{y}) + \mathbb{E}_F \chi_{\hat{Y}_F}(\hat{y})$ is $\hat{Y}$ – periodic, $\hat{Y} = Y \cap \{x_3 = \text{const}\}$

Macroscopic equations: Poro-elasticity



Macroscopic problem

$$-\operatorname{div}(\mathbb{E}_{\rm hom}(b_3)\mathbf{e}(\mathbf{u}_e)) + \nabla p_e + \vartheta_f \fint_{Y_f} \partial_t^2 \mathbf{u}_f \, dy = 0 \quad \text{in } G_T$$

$$-\operatorname{div}\big(K_{p,\rm hom}\nabla p_e - K_u\,\partial_t \mathbf{u}_e - Q(x,\partial_t\mathbf{u}_f)\big) = 0 \quad \text{in } G_T$$

Microscopic problem

$$\partial_t^2 \mathbf{u}_f - \operatorname{div}_y(\mu\,\mathbf{e}_y(\partial_t\mathbf{u}_f) - \pi_f I) + \nabla p_e = 0 \quad \text{in } G_T \times Y_f$$

$$\operatorname{div}_y \partial_t \mathbf{u}_f = 0 \quad \text{in } G_T \times Y_f$$

$$\nu\cdot(\mu\,\mathbf{e}_y(\partial_t\mathbf{u}_f) - \pi_f I)\,\nu = -p_e^1 \quad \text{on } G_T \times \Gamma$$

$$\Pi_\tau \partial_t \mathbf{u}_f = \Pi_\tau \partial_t \mathbf{u}_e \quad \text{on } G_T \times \Gamma$$

where $\vartheta_f = |Y_f|/|Y|$, and

$$p_e^1(t,x,y) = \sum_{k=1}^3 \frac{\partial p_e}{\partial x_k}(t,x)\, w_p^k(x,y) + \sum_{k=1}^3 \partial_t \mathbf{u}_e^k(t,x)\, w_e^k(x,y) + q(x,y,\partial_t\mathbf{u}_f)$$

A. Piatnitski, MP, *MMS SIAM J*, 2017

Multiscale analysis of viscoelastic model

$$
\begin{aligned}
0 &= \operatorname{div}(\mathbb{E}^\varepsilon(b_3^\varepsilon, x)\mathbf{e}(\mathbf{u}^\varepsilon) + \mathbb{V}^\varepsilon(b_3^\varepsilon, x)\partial_t \mathbf{e}(\mathbf{u}^\varepsilon)) && \text{in } G_T,\\
(\mathbb{E}^\varepsilon(b_3^\varepsilon, x)\mathbf{e}(\mathbf{u}^\varepsilon) &+ \mathbb{V}^\varepsilon(b_3^\varepsilon, x)\partial_t \mathbf{e}(\mathbf{u}^\varepsilon))\boldsymbol{\nu} = F && \text{on } (0,T)\times\partial G,\\
\mathbf{u}^\varepsilon(0,x) &= \mathbf{u}_0(x) && \text{in } G,\\
\mathbf{u}^\varepsilon & && a_3\text{-periodic in } x_3,
\end{aligned}
$$

Multiscale analysis of viscoelastic model

Consider a perturbed problem

$$\begin{aligned}
\sigma\chi_{G_M^\varepsilon}\partial_t^2\mathbf{u}^\varepsilon &= \operatorname{div}(\mathbb{E}^\varepsilon(b_3^\varepsilon,x)\mathbf{e}(\mathbf{u}^\varepsilon)+\mathbb{V}^\varepsilon(b_3^\varepsilon,x)\partial_t\mathbf{e}(\mathbf{u}^\varepsilon)) && \text{in } G_T,\\
&(\mathbb{E}^\varepsilon(b_3^\varepsilon,x)\mathbf{e}(\mathbf{u}^\varepsilon)+\mathbb{V}^\varepsilon(b_3^\varepsilon,x)\partial_t\mathbf{e}(\mathbf{u}^\varepsilon))\boldsymbol{\nu} = F && \text{on } (0,T)\times\partial G,\\
&\mathbf{u}^\varepsilon(0,x) = \mathbf{u}_0(x) && \text{in } G,\\
&\partial_t\mathbf{u}^\varepsilon(0,x) = \mathbf{0} && \text{in } G,\\
&\mathbf{u}^\varepsilon && a_3\text{-periodic in } x_3,
\end{aligned}$$

where $\sigma > 0$ is a small perturbation parameter

Multiscale analysis of viscoelastic model

Consider a perturbed problem

$$
\begin{aligned}
\sigma \chi_{G_M^\varepsilon} \partial_t^2 \mathbf{u}^\varepsilon &= \operatorname{div}(\mathbb{E}^\varepsilon(b_3^\varepsilon, x)\mathbf{e}(\mathbf{u}^\varepsilon) + \mathbb{V}^\varepsilon(b_3^\varepsilon, x)\partial_t \mathbf{e}(\mathbf{u}^\varepsilon)) && \text{in } G_T,\\
(\mathbb{E}^\varepsilon(b_3^\varepsilon, x)\mathbf{e}(\mathbf{u}^\varepsilon) &+ \mathbb{V}^\varepsilon(b_3^\varepsilon, x)\partial_t \mathbf{e}(\mathbf{u}^\varepsilon))\boldsymbol{\nu} = F && \text{on } (0,T) \times \partial G,\\
\mathbf{u}^\varepsilon(0,x) &= \mathbf{u}_0(x) && \text{in } G,\\
\partial_t \mathbf{u}^\varepsilon(0,x) &= \mathbf{0} && \text{in } G,\\
\mathbf{u}^\varepsilon & && a_3\text{-periodic in } x_3,
\end{aligned}
$$

where $\sigma > 0$ is a small perturbation parameter

$$
\sigma^{\frac{1}{2}} \|\partial_t \mathbf{u}^\varepsilon\|_{L^\infty(0,T;L^2(G_M^\varepsilon))} + \|\mathbf{u}^\varepsilon\|_{L^\infty(0,T;\mathcal{W}(G))} + \|\partial_t \mathbf{e}(\mathbf{u}^\varepsilon)\|_{L^2(G_{M,T}^\varepsilon)} \leq C
$$

$$
\|b_3^\varepsilon\|_{W^{1,\infty}(0,T;L^\infty(G_M^\varepsilon))} \leq C
$$

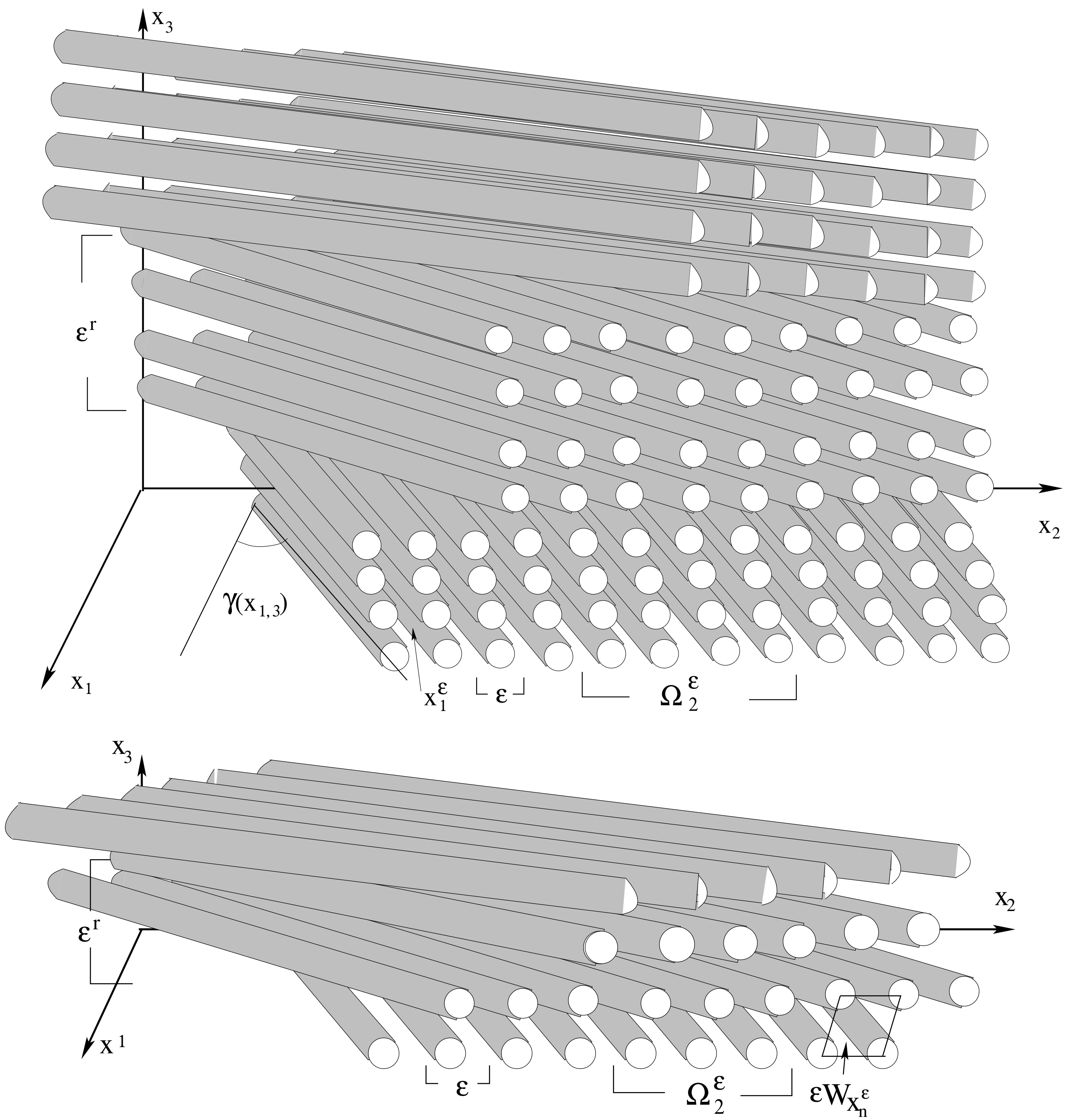
with a constant C independent of ε and σ.

$$
\mathcal{W}(G) = \left\{\mathbf{u} \in H^1(G;\mathrm{IR}^3) \;\middle|\; \int_G \mathbf{u}\, dx = \mathbf{0},\ \int_G [(\nabla \mathbf{u})_{12} - (\nabla \mathbf{u})_{21}] dx = \mathbf{0},\ \mathbf{u} \text{ periodic in } x_3 \right\}
$$

MP, B. Seguin, *ESAIM: COCV*, 2017

Locally-periodic microstructure

Locally-periodic microstructures: spatial changes of the microstructure are observed on a scale smaller than the size of the considered domain but larger than the characteristic size of the microstructure.



- fibres are of radius εa
- layers of fibres aligned in the same direction are of width ε^r
- $\varepsilon > 0, \quad 0 < a < 1/2$

  $0 < r < 1$

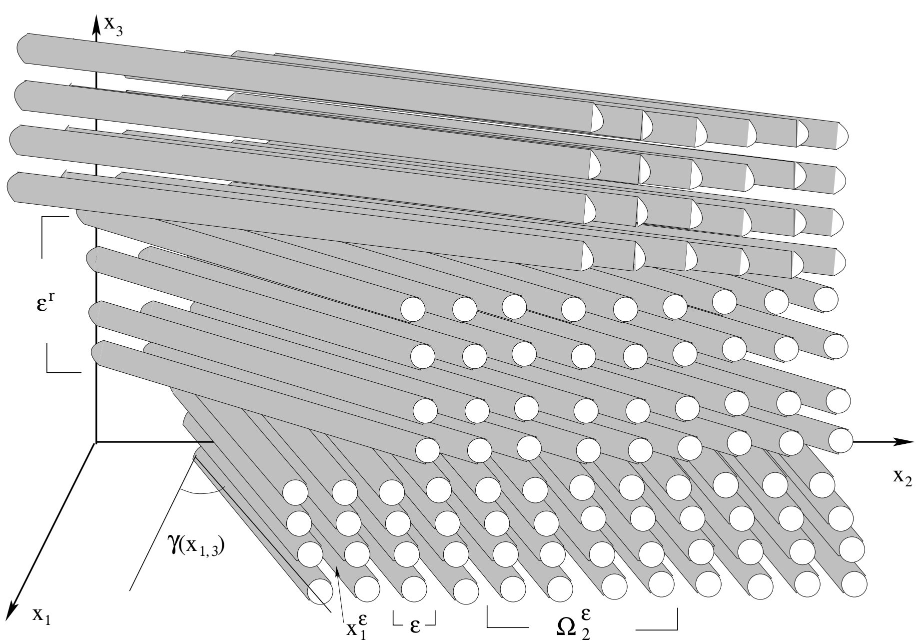
M. Briane, J. Math. Pures Appl. 1994,
& RAIRO Model. Math. Anal. Numer. 1993

Locally periodic two-scale convergence

Definition. $\{u^\varepsilon\} \subset L^p(\Omega)$ converges locally periodic two-scale (l-t-s) to $u \in L^p(\Omega; L^p(Y_x))$ if for any $\psi \in L^{p}_{q}(\Omega; C_{\text{per}}(Y_x))$

$$\lim_{\varepsilon\to 0} \int_\Omega u^\varepsilon(x) \mathcal{L}^\varepsilon \psi(x) dx = \int_\Omega \fint_{Y_x} u(x,y)\psi(x,y) dy dx,$$

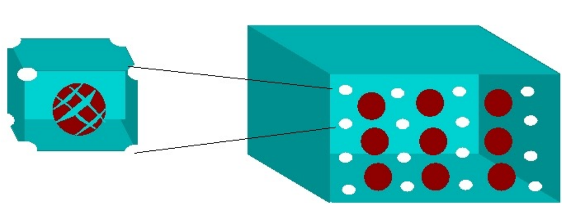
where $\mathcal{L}^\varepsilon \psi$ is the locally-periodic approximation of ψ.



Locally periodic approximation: $\mathcal{L}^\varepsilon : C(\overline{\Omega}; C_{\text{per}}(Y_x)) \to L^\infty(\Omega)$

$$(\mathcal{L}^\varepsilon \psi)(x) = \sum_{n=1}^{N_\varepsilon} \tilde{\psi}\Big(x, \frac{D^{-1}_{x_n^\varepsilon} x}{\varepsilon}\Big) \chi_{\Omega_n^\varepsilon}(x)$$

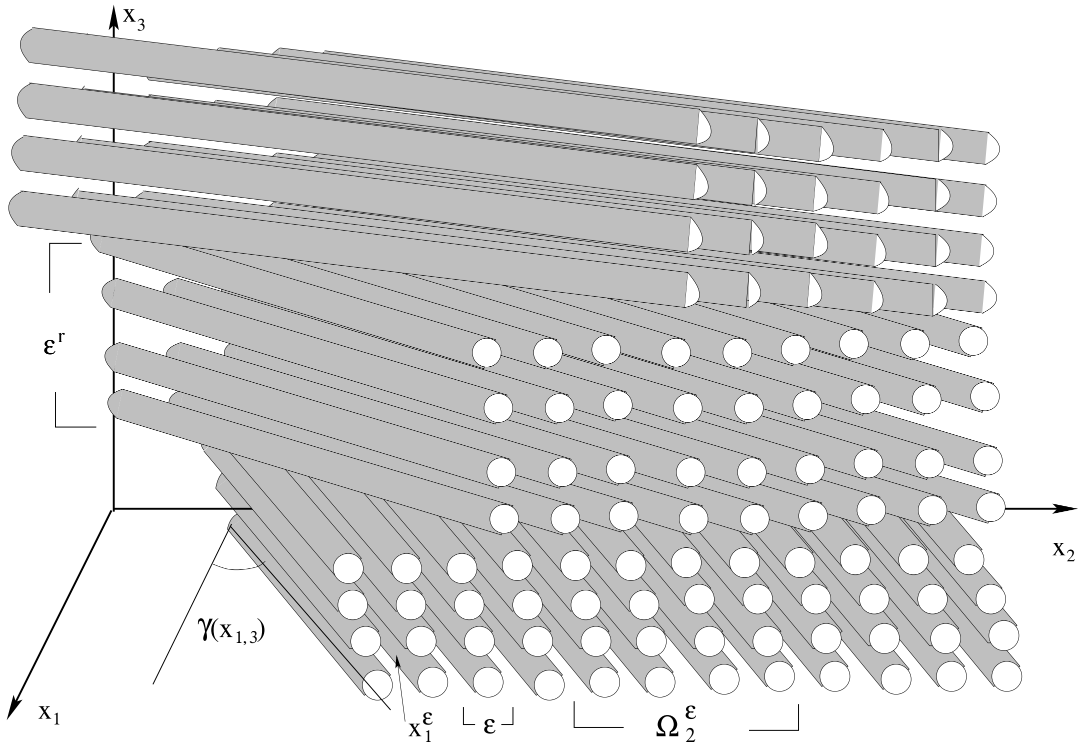
$$\tilde{\psi} \in C(\overline{\Omega}; C_{per}(Y))$$



Definition A sequence $\{u^\varepsilon\} \subset L^p(\Omega)$ two-scale converge to $u \in L^p(\Omega \times Y)$ iff for any $\phi \in L^q(\Omega, C_{per}(Y))$

$$\lim_{\varepsilon\to 0} \int_\Omega u^\varepsilon(x) \phi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \fint_Y u(x,y)\phi(x,y) dx dy.$$

Plywood like microstructures

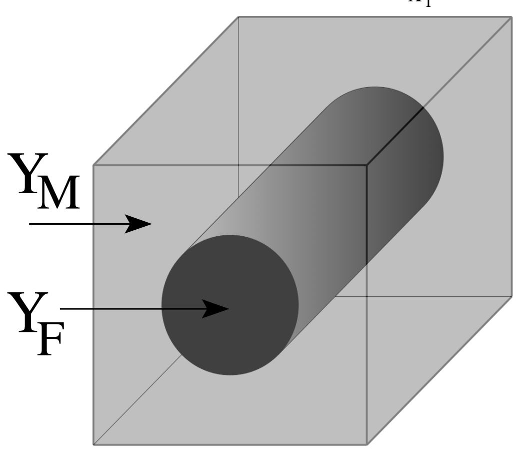


$$R_{x_3} = \begin{pmatrix} \cos(\gamma(x_3)) & \sin(\gamma(x_3)) & 0 \\ -\sin(\gamma(x_3)) & \cos(\gamma(x_3)) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tilde{\eta}(\tilde{y}) = \chi_{Y_F}(\tilde{y}) \quad \text{for} \quad \tilde{y} \in Y$$

extended Y-periodically to $\mathbb{R}^3$

$$\eta(x_3, y) = \tilde{\eta}(R_{x_3} y) \quad \text{for } y \in Y_{x_3} = R_{x_3}^{-1} Y$$



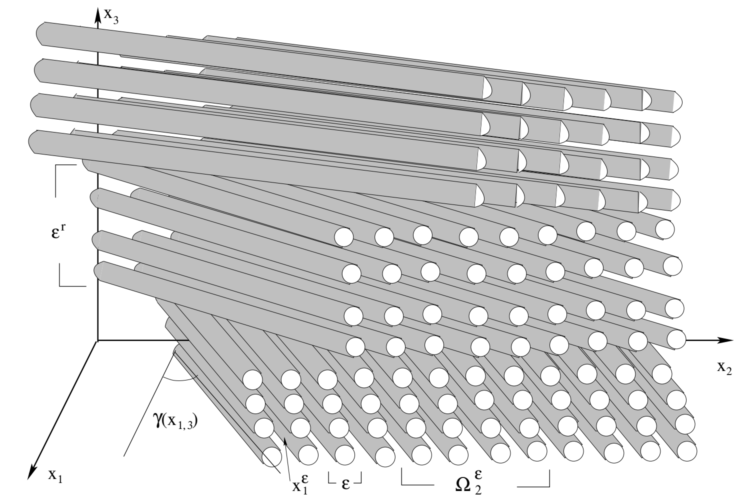
$$\tilde{\mathbb{E}}(\xi, \tilde{y}) = \mathbb{E}_M(\xi)\,(1 - \tilde{\eta}(\tilde{y})) + \mathbb{E}_F\,\tilde{\eta}(\tilde{y}) \quad \text{in} \quad Y$$

$$\mathbb{E}(\xi, x_3, y) \quad = \quad \mathbb{E}_M(\xi)\,(1 - \eta(x_3, y)) + \mathbb{E}_F\,\eta(x_3, y) \ \text{ in } \quad Y_{x_3} = R_{x_3}^{-1} Y$$

$$\overline{G} = \cup_{n=1}^{N_\varepsilon} \overline{G_n^\varepsilon}, \qquad \mathbb{E}^\varepsilon(\xi, x) = \sum_{n=1}^{N_\varepsilon} \mathbb{E}\big(\xi, x_3, \frac{x}{\varepsilon}\big)\,\chi_{G_n^\varepsilon}$$

$$-\mathrm{div}\,\left(\mathbb{E}^\varepsilon(b_3^\varepsilon, x)\,\mathbf{e}(\mathbf{u}^\varepsilon)\right) \quad = \quad 0 \quad \text{in } G_T$$

Macroscopic equations



$$-\mathrm{div}\left(\mathbb{E}_{\mathrm{hom}}(b_3, x_3)\mathbf{e}(\mathbf{u})\right) = 0 \quad \text{in } G_T$$

where

$$\mathbb{E}_{\mathrm{hom},ijkl}(b_3, x_3) = \fint_{\hat{Y}} \left(\tilde{\mathbb{E}}_{ijkl}(b_3, \hat{y}) + \left[\tilde{\mathbb{E}}(b_3, \hat{y}) \hat{e}^R_{\hat{y}}(\mathbf{w}^{ij}) \right]_{kl} \right) d\hat{y}$$

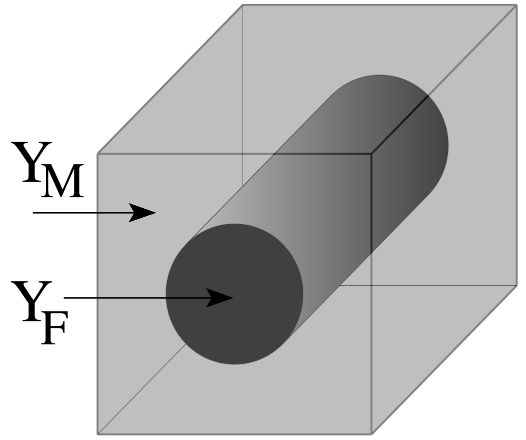
with $\mathbf{w}_{ij}$ solutions of the cell problems

$$\hat{\mathrm{div}}_y \left(\hat{R}_{x_3} \tilde{\mathbb{E}}(b_3, \hat{y}) \left(\hat{e}^R_{\hat{y}}(\mathbf{w}^{ij}) + \mathbf{b}_{ij} \right) \right) = \mathbf{0} \qquad \text{in } \hat{Y}$$

$$\mathbf{w}^{ij} \qquad \text{periodic} \qquad \text{in } \hat{Y}$$

$$\hat{e}^R_{\hat{y}}(v)_{kl} = \frac{1}{2} \left[\left(\hat{R}^T_{x_3} \nabla_{\hat{y}} v^l \right)_k + \left(\hat{R}^T_{x_3} \nabla_{\hat{y}} v^k \right)_l \right]$$

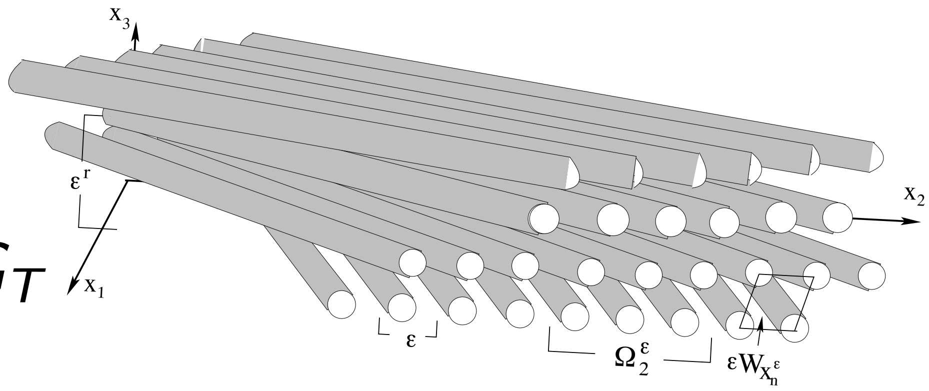
$$\hat{R}_{x_3} = \begin{pmatrix} -\sin(\gamma(x_3)) & \cos(\gamma(x_3)) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



where $\hat{Y} = Y \cap \{y_1 = \mathrm{const}\}$

MP, MMS SIAM J 2013

Fast rotating plywood

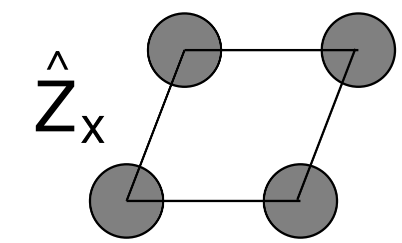


$$-\mathrm{div}\,(\mathbb{E}_{\mathrm{hom}}(b_3, x)\mathbf{e}(\mathbf{u})) \;=\; 0 \quad \text{in } G_T$$

where

$$\mathbb{E}_{\mathrm{hom},ijkl}(b_3, x) \;=\; \fint_{\hat{Z}_x} \Big(\tilde{\mathbb{E}}_{ijkl}(b_3, x, \hat{y}) + \big[\tilde{\mathbb{E}}(b_3, x, \hat{y})\hat{e}^R_{\hat{y}}(\mathbf{w}^{ij})\big]_{kl}\Big)\, d\hat{y}$$

with $\mathbf{w}_{ij}$ solutions of the cell problems



$$\hat{\mathrm{div}}_y\Big(\hat{R}_{x_3}\tilde{\mathbb{E}}(b_3, x, \hat{y})\big(\hat{e}^R_{\hat{y}}(\mathbf{w}^{ij}) + \mathbf{b}_{ij}\big)\Big) = \mathbf{0} \qquad \text{in } \hat{Z}_x$$

$$\mathbf{w}^{ij} \qquad \text{periodic} \qquad \text{in } \hat{Z}_x$$

where $\tilde{\mathbb{E}}(b_3, x, \hat{y}) = \mathbb{E}_F\hat{\vartheta}(x, \hat{y}) + \mathbb{E}_M(b_3)(1 - \hat{\vartheta}(x, \hat{y})), \quad x \in G,\ \hat{y} \in Z_x$

$$\hat{\vartheta}(x, y) = \sum_{k\in\mathbb{Z}^3} \vartheta(y - W_x k), \qquad x \in G,\ y \in Z_x,\ \ \vartheta(y) = \begin{cases} 1 & |\hat{y}| \le a \\ - & |\hat{y}| > a \end{cases}$$

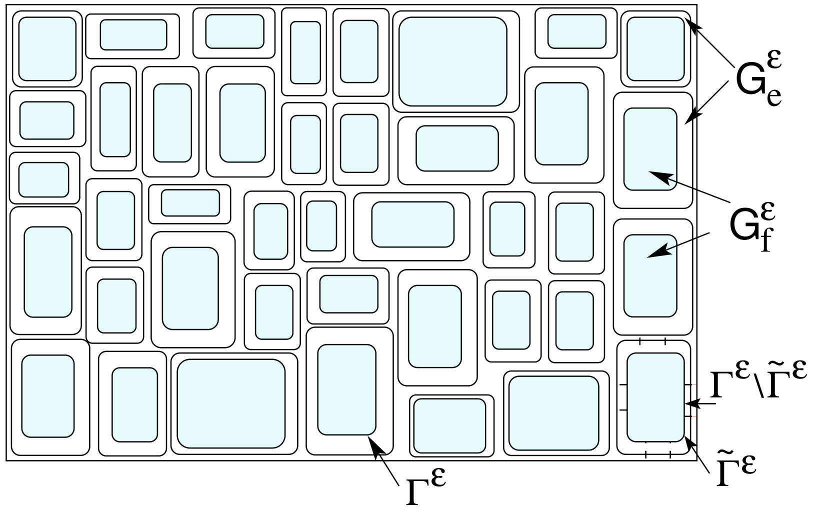
and

$$Z_x = W_x Y, \quad W_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & w(x) \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{Z}_x = Z_x \cap \{y_1 = \mathrm{const}\}$$

with $\ w(x) = \gamma'(x_3)(\cos(\gamma(x_3))x_1 + \sin(\gamma(x_3))x_2)$

MP, MMS SIAM J 2013

Plant tissues with random distribution of cells



- $(\Omega, \mathcal{F}, \mathcal{P})$ – probability space
- $\mathcal{T}(x) : \Omega \to \Omega$ dynamical system, i.e.

  for each $x \in \mathbb{R}^d$: $\mathcal{T}(x)$ is measurable and

  (i) $\mathcal{T}(0)$ is the identity map on Ω and

$$\mathcal{T}(x_1 + x_2) = \mathcal{T}(x_1)\mathcal{T}(x_2) \quad \text{for all } x_1, x_2 \in \mathbb{R}^d$$

  (ii) $\mathcal{P}$ is an invariant measure for $\mathcal{T}(x)$, i.e. for $x \in \mathbb{R}^d$ and $F \in \mathcal{F}$:

$$\mathcal{P}(\mathcal{T}^{-1}(x)F) = \mathcal{P}(F)$$

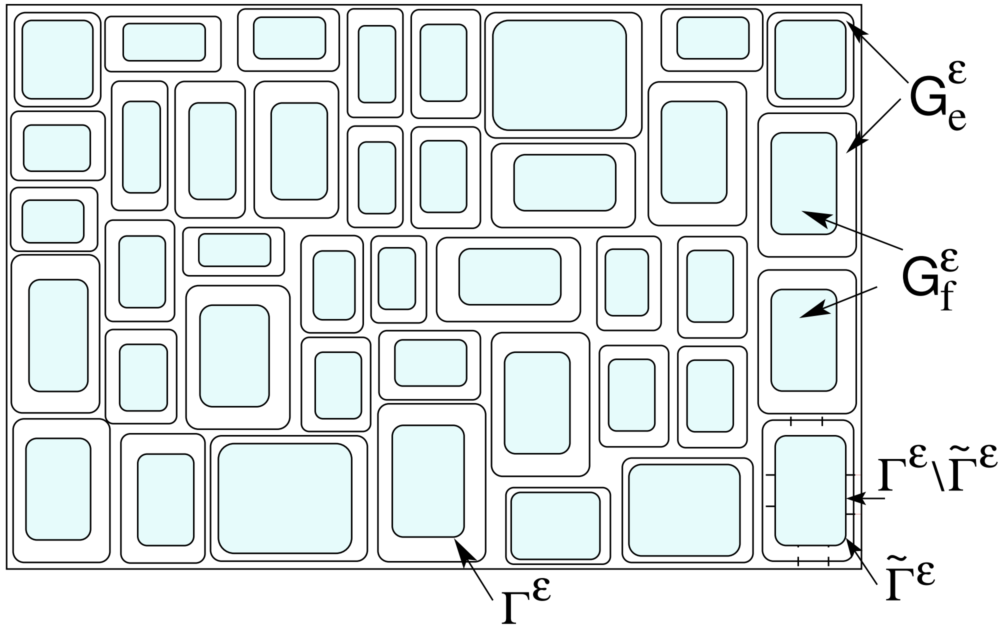
  (iii) For each $F \in \mathcal{F}$
  the set $\{(x, \omega) \in \mathbb{R}^d \times \Omega : \mathcal{T}(x)\omega \in F\}$ is a measurable

Assume : $\mathcal{T}(x)$ - ergodic
[every measurable function which is invariant for $\mathcal{T}(x)$ is $\mathcal{P}$-a.e. constant]

Plant tissues biomechanics: Random geometry

- Ω_f measurable set, $\mathcal{P}(\Omega_f) > 0$, $\mathcal{P}(\Omega \setminus \Omega_f) > 0$
- $\Omega_e = \Omega \setminus \Omega_f$
- $\Omega_\Gamma \subset \Omega$, with $\mathcal{P}(\Omega_\Gamma) > 0$ and $\mathcal{P}(\Omega_\Gamma \cap \Omega_j) > 0$, for $j = e, f$

- For $\mathcal{P}$-a.a. $\omega \in \Omega$ define a random system of subdomains in $\mathbb{R}^3$



$$G_j(\omega) = \{x \in \mathbb{R}^3 : \ \mathcal{T}(x)\omega \in \Omega_j\}, \quad j = e, f$$

$$G_\Gamma(\omega) = \{x \in \mathbb{R}^3 : \ \mathcal{T}(x)\omega \in \Omega_\Gamma\}$$

$$\Gamma(\omega) = \partial G_f(\omega), \qquad \widetilde{\Gamma}(\omega) = \Gamma(\omega) \cap G_\Gamma(\omega)$$

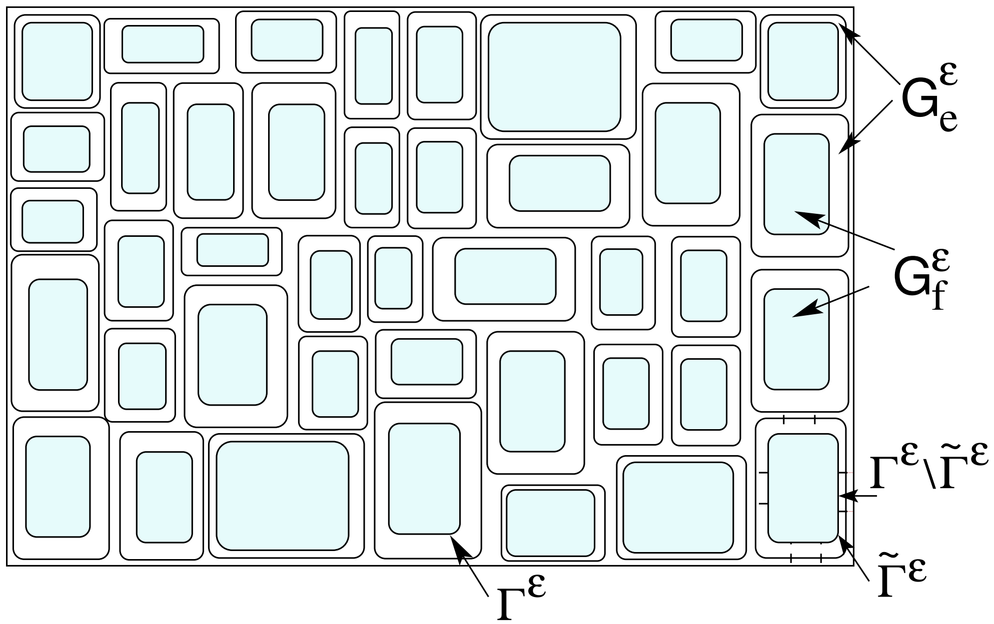
1. $G_f(\omega)$ countable number of disjoined Lipschitz domains for $\mathcal{P}$-a.s. $\omega \in \Omega$
2. The distance between two connected components of $G_f(\omega)$ and diameter of $G_f(\omega)$ are uniformly bounded from above and below
3. The surface $\widetilde{\Gamma}(\omega) \subset \Gamma(\omega)$ is open on $\Gamma(\omega)$ and Lipschitz continuous

Random measure: $d\mu_\omega(x) = \rho_e(\mathcal{T}(x)\omega)dx$, $\rho_e(\omega) = \chi_{\Omega_e}$

Plant tissues biomechanics: Random geometry

- Ω_f measurable set, $\mathcal{P}(\Omega_f) > 0$, $\mathcal{P}(\Omega \setminus \Omega_f) > 0$
- $\Omega_e = \Omega \setminus \Omega_f$
- $\Omega_\Gamma \subset \Omega$, with $\mathcal{P}(\Omega_\Gamma) > 0$ and $\mathcal{P}(\Omega_\Gamma \cap \Omega_j) > 0$, for $j = e, f$

- For $\mathcal{P}$-a.a. $\omega \in \Omega$ define a random system of subdomains in $\mathbb{R}^3$



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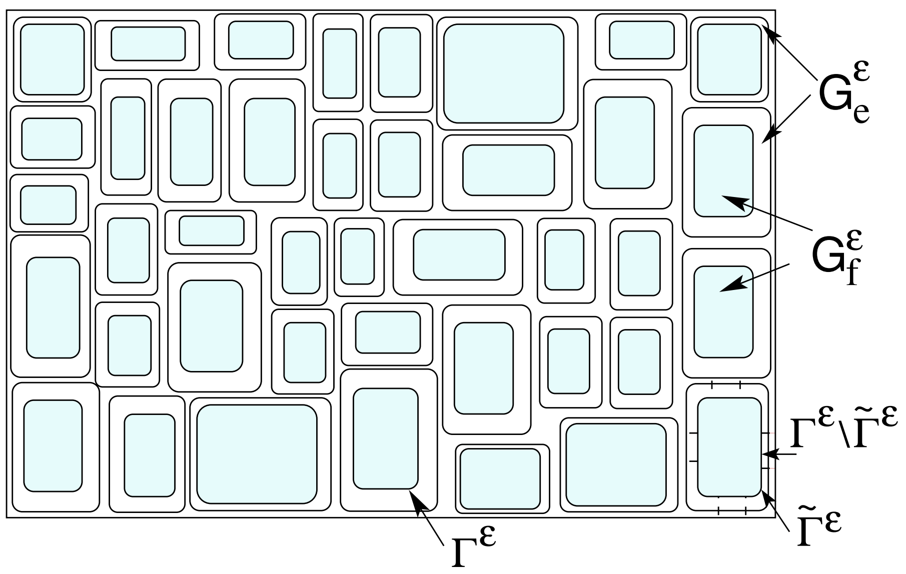
Random geometry



$G \subset \mathbb{R}^d$

$G_f^\varepsilon = \{x \in \mathbb{R}^3 : \ \mathcal{T}(x/\varepsilon)\omega \in \Omega_f\} \cap G$

cell inside

$G_e^\varepsilon = G \setminus G_f^\varepsilon$ cell wall + middle lamella

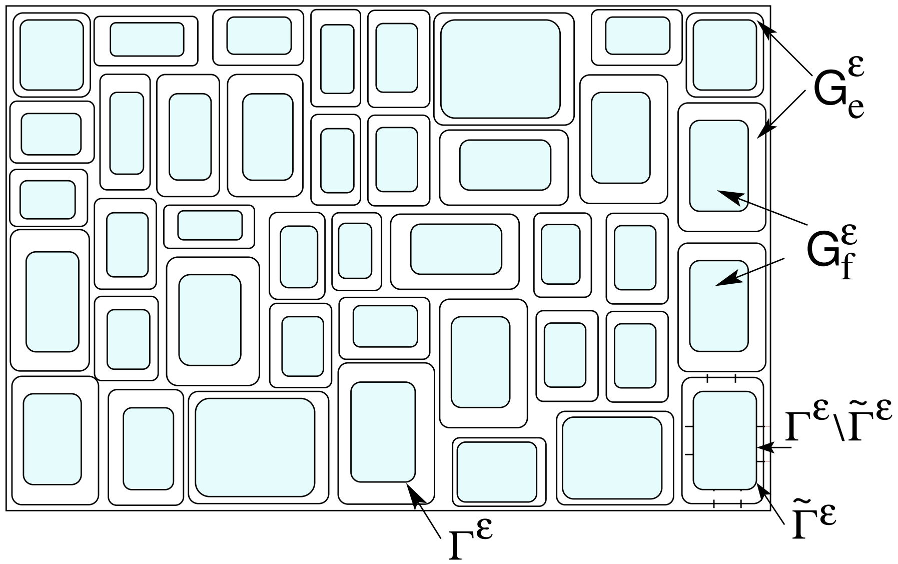
$G_\Gamma^\varepsilon = \{x \in \mathbb{R}^3 : \ \mathcal{T}(x/\varepsilon)\omega \in \Omega_\Gamma\} \cap G$

$\Gamma^\varepsilon = \partial G_f^\varepsilon$ cell membrane

$\widetilde{\Gamma}^\varepsilon = \Gamma^\varepsilon \cap G_\Gamma^\varepsilon$ part of the cell membrane

impermeable to calcium ions

Random geometry



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$\Gamma^\varepsilon = \partial G_f^\varepsilon$ cell membrane

$\widetilde{\Gamma}^\varepsilon = \Gamma^\varepsilon \cap G_\Gamma^\varepsilon$ part of the cell membrane impermeable to calcium ions

Stationary random fields

$$\mathbb{E}(x, \omega, \xi) = \widetilde{\mathbb{E}}(\mathcal{T}(x)\omega, \xi), \qquad K(x, \omega) = \widetilde{K}(\mathcal{T}(x)\omega)$$

$\widetilde{\mathbb{E}}(\cdot, \xi) : \Omega \to \mathbb{R}^{3^4}$, $\widetilde{K}(\cdot) : \Omega \to \mathbb{R}^{3\times 3}$ measurable functions, $\xi \in \mathbb{R}$

$\mathbb{E}^\varepsilon(x, \xi) = \mathbb{E}\big(x/\varepsilon, \omega, \xi\big)$, $K^\varepsilon(x) = K\big(x/\varepsilon, \omega\big)$ for $\omega \in \Omega$, $x \in \mathbb{R}^3$, $\xi \in \mathbb{R}$

Microscopic model for plant tissues

$$\operatorname{div}\left(\mathbb{E}^{\varepsilon,\omega}(x, b_3^{\varepsilon,\omega})\,\mathbf{e}(\mathbf{u}_e^{\varepsilon,\omega}) - p_e^{\varepsilon,\omega} I\right) = 0 \qquad \text{in} \quad G_e^\varepsilon$$

$$\operatorname{div}(K\nabla p_e^{\varepsilon,\omega} - \partial_t \mathbf{u}_e^{\varepsilon,\omega}) = 0 \qquad \text{in} \quad G_e^\varepsilon$$

$$\partial_t(\partial_t \mathbf{u}_f^{\varepsilon,\omega}) - \operatorname{div}(\varepsilon^2 \mu\, \mathbf{e}(\partial_t \mathbf{u}_f^{\varepsilon,\omega}) - p_f^{\varepsilon,\omega} I) = 0 \qquad \text{in} \quad G_f^\varepsilon$$

and

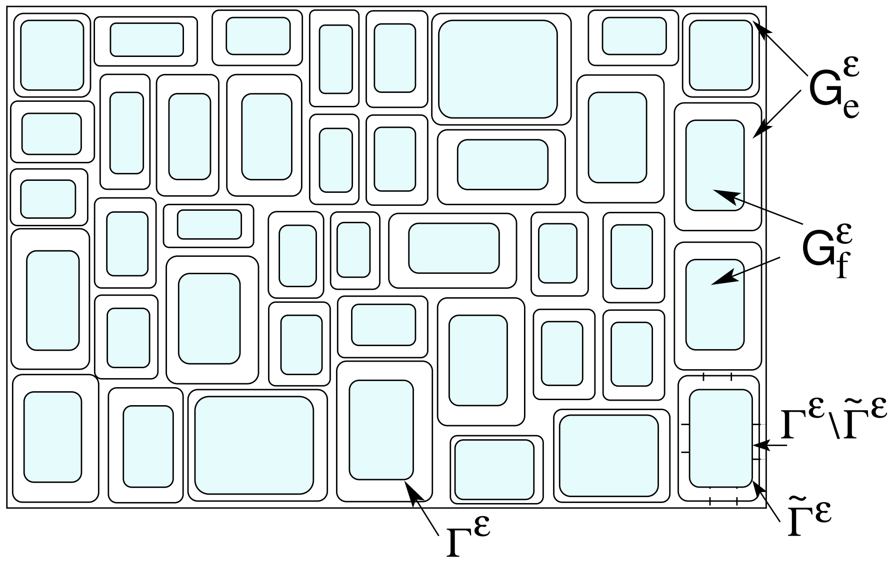
$$\partial_t b^{\varepsilon,\omega} = \operatorname{div}(D_b \nabla b^{\varepsilon,\omega}) + g_b(b^{\varepsilon,\omega}, c_e^{\varepsilon,\omega}, \mathbf{e}(\mathbf{u}_e^{\varepsilon,\omega})) \qquad \text{in} \quad G_e^\varepsilon$$

$$\partial_t c_e^{\varepsilon,\omega} = \operatorname{div}(D_e \nabla c_e^{\varepsilon,\omega}) + g_e(b^{\varepsilon,\omega}, c_e^{\varepsilon,\omega}, \mathbf{e}(\mathbf{u}_e^{\varepsilon,\omega})) \qquad \text{in} \quad G_e^\varepsilon$$

$$\partial_t c_f^{\varepsilon,\omega} = \operatorname{div}(D_f \nabla c_f^{\varepsilon,\omega} - \mathcal{G}(\partial_t \mathbf{u}_f^{\varepsilon,\omega}) c_f^{\varepsilon,\omega}) + g_f(c_f^{\varepsilon,\omega}) \qquad \text{in} \quad G_f^\varepsilon$$

Transmission and boundary conditions:

$$\Pi_\tau \partial_t \mathbf{u}_e^{\varepsilon,\omega} = \Pi_\tau \partial_t \mathbf{u}_f^{\varepsilon,\omega} \qquad \text{on} \quad \Gamma^\varepsilon$$



$$D_b \nabla b^{\varepsilon,\omega} \cdot \nu = \varepsilon R(b^{\varepsilon,\omega}) \qquad \text{on } \Gamma^\varepsilon$$

$$c_e^{\varepsilon,\omega} = c_f^{\varepsilon,\omega} \qquad \text{on } \Gamma^\varepsilon \setminus \widetilde{\Gamma}^\varepsilon$$

$$D_e \nabla c_e^{\varepsilon,\omega} \cdot \nu = (D_f \nabla c_f^{\varepsilon,\omega} - \mathcal{G}(\partial_t \mathbf{u}_f^{\varepsilon,\omega}) c_f^{\varepsilon,\omega}) \cdot \nu \qquad \text{on } \Gamma^\varepsilon \setminus \widetilde{\Gamma}^\varepsilon$$

Stochastic two-scale convergence

$\mathcal{T}(x)-$ ergodic dynamical system, $\mathcal{T}(x)\tilde{\omega}-$ 'typical trajectory', i.e.

$$\lim_{t\to\infty} \frac{1}{t^d|A|} \int_{tA} g(\mathcal{T}(x)\,\tilde{\omega})dx = \int_\Omega g(\omega)d\mathcal{P}(\omega) \quad \mathcal{P}\text{-a.s.}$$

for all bounded Borel sets A with $|A| > 0$, and all $g \in L^1(\Omega, \mathcal{P})$

Definition $\quad \{v^\varepsilon\} \subset L^2(G), \quad v \in L^2(G \times \Omega)$

$v^\varepsilon \to v$ stochastically two-scale iff

$$\limsup_{\varepsilon\to 0} \int_G |v^\varepsilon(x)|^2 \, dx < \infty \tag{1}$$

and for all $\varphi \in C_0^\infty(G)$ and $b \in L^2(\Omega)$

$$\lim_{\varepsilon\to 0} \int_G v^\varepsilon(x)\varphi(x)b(\mathcal{T}(x/\varepsilon)\tilde{\omega})\,dx = \int_G \int_\Omega v(x,\omega)\varphi(x)b(\omega)\,d\mathcal{P}(\omega)dx.$$

Theorem $\{v^\varepsilon\} \subset L^2(G)$ satisfying (1) $\implies \exists\, v \in L^2(G \times \Omega, dx \times d\mathcal{P}(\omega))$

$v^\varepsilon \to v$ stochastically two-scale.

V. Zhikov, A. Piatnitsky 2006, M. Heida 2011

Palm measures

Definition. Let $(\Omega, \mathcal{F})$ be a measurable space and $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$
$\tilde{\mu} : \Omega \times \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}_+ \cup \{\infty\}$ is a random measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ if
$\mu_\omega(A) = \tilde{\mu}(\omega, A)$ is

- $\mathcal{F}$-measurable in $\omega \in \Omega$ for each $A \in \mathcal{B}(\mathbb{R}^d)$ and
- a measure in $A \in \mathcal{B}(\mathbb{R}^d)$ for each $\omega \in \Omega$.

Definition. The random measure μ_ω is stationary if for $\phi \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \phi(y-x) d\mu_\omega(y) = \int_{\mathbb{R}^d} \phi(y) d\mu_{\mathcal{T}(x)\omega}(y)$$

Definition. The *Palm measure* of the random measure μ_ω is a measure $\boldsymbol{\mu}$ on $(\Omega, \mathcal{F})$ defined by:

$$\boldsymbol{\mu}(F) = \int_\Omega \int_{\mathbb{R}^d} \mathbb{I}_{[0,1)^d}(x)\, \mathbb{I}_F(\mathcal{T}(x)\omega)\, d\mu_\omega(x) d\mathcal{P}(\omega), \quad F \in \mathcal{F}$$

The intensity $m(\mu_\omega)$ of a random measure μ_ω :

$$m(\mu_\omega) = \int_\Omega \int_{[0,1)^d} d\mu_\omega(x) d\mathcal{P}(\omega)$$

D. J. Daley, D. Vere-Jones 1988

The ergodic theorem

Theorem Let $\mathcal{T}(x)$ be ergodic and assume that the stationary random measure μ_ω has finite intensity $m(\mu_\omega) > 0$. Then

$$\lim_{t\to\infty} \frac{1}{t^d|A|} \int_{tA} g(\mathcal{T}(x)\,\omega) d\mu_\omega(x) = \int_\Omega g(\omega) d\boldsymbol{\mu}(\omega) \quad \mathcal{P}\text{-a.s.}$$

for all bounded Borel sets A with $|A| > 0$, and all $g \in L^1(\Omega, \boldsymbol{\mu})$

- $d\mu_\omega(x) = \rho(\mathcal{T}(x)\omega)dx$ on $\mathbb{R}^d$: $\quad d\boldsymbol{\mu}(\omega) = \rho(\omega)\mathcal{P}(\omega)$ on Ω

$\mathcal{T}(x)\tilde{\omega}$ - 'typical trajectory'

Definition $\{v^\varepsilon\} \subset L^2(G, \mu^\varepsilon_{\tilde{\omega}})$ converges stochastically two-scale to

$$v \in L^2(G \times \Omega, dx \times d\boldsymbol{\mu}(\omega)) \qquad \text{if}$$

$$\limsup_{\varepsilon\to 0} \int_G |v^\varepsilon(x)|^2 \, d\mu^\varepsilon_{\tilde{\omega}}(x) < \infty$$

and

$$\lim_{\varepsilon\to 0} \int_G v^\varepsilon(x)\varphi(x)b(\mathcal{T}(x/\varepsilon)\tilde{\omega})\, d\mu^\varepsilon_{\tilde{\omega}}(x) = \int_G \int_\Omega v(x,\omega)\varphi(x)b(\omega)\, d\boldsymbol{\mu}(\omega)dx$$

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Compactness results

Theorem (Zhikov & Piatnitsky 2006) $\{v^\varepsilon\} \subset H^1(G)$

- If $\|v^\varepsilon\|_{H^1(G)} \le C(\tilde\omega) \quad \Rightarrow \quad \exists \quad v \in H^1(G), \quad v_1 \in L^2(G; L^2_{\mathrm{pot}}(\Omega))$

$$v^\varepsilon \rightharpoonup v, \qquad \nabla v^\varepsilon \rightharpoonup \nabla v + v_1 \qquad \text{stochastically two-scale}$$

- If $\|v^\varepsilon\|_{L^2(G)} + \varepsilon\|\nabla v^\varepsilon\|_{L^2(G)} \le C(\tilde\omega) \quad \Rightarrow \quad \exists \; v \in L^2(G; H^1(\Omega))$

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Lemma (Piatnitsky, M.P.) $\boldsymbol{\mu}$ - Palm measure of surface measure μ_ω of $\Gamma(\omega)$

$$u \in H^1(\Omega, \mathcal{P}) \implies u \in L^2(\Omega, \boldsymbol{\mu}), \quad \text{continuous embedding}$$

Lemma (Piatnitsky, M.P.) For $\|v^\varepsilon\|_{L^p(G_e^\varepsilon)} + \varepsilon\|\nabla v^\varepsilon\|_{L^p(G_e^\varepsilon)} \le C$:

$$\lim_{\varepsilon\to 0}\int_G v^\varepsilon(x)\phi(x)\psi(\mathcal{T}_{x/\varepsilon}\omega)d\mu^\varepsilon_\omega(x) = \int_G\int_\Omega v(x,\omega)\phi(x)\psi(\omega)d\boldsymbol{\mu}(\omega)dx$$

- $G_f(\omega) = \{x \in \mathbb{R}^d : \mathcal{T}(x)\omega \in \Omega_f\}$, $\Gamma(\omega) = \partial G_f(\omega)$, $\mathcal{P}(\Omega_f) > 0$, $\mathcal{P}(\Omega \setminus \Omega_f) > 0$

  μ_ω is the stationary random measure of $\Gamma(\omega)$, $d\mu^\varepsilon_\omega(x) = \varepsilon^d\mu_\omega(x/\varepsilon)$

Compactness results

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Macroscopic equations

$$\vartheta_e \partial_t^2 u_e - \mathrm{div}(\mathbb{E}^{\mathrm{hom}}(b)\mathbf{e}(u_e)) + \nabla p_e + \int_\Omega \partial_t^2 u_f \, \chi_{\Omega_f} d\mathcal{P}(\omega) = 0 \quad \text{in } G_T$$

$$\vartheta_e \partial_t p_e - \mathrm{div}(K^{\mathrm{hom}} \nabla p_e - K_u \partial_t u_e - Q(\partial_t u_f)) = 0 \quad \text{in } G_T$$

and

$$\int_\Omega \Big[\partial_t^2 u_f \, \varphi + \mu \, \mathbf{e}_\omega(\partial_t u_f) \mathbf{e}_\omega(\varphi) + \nabla p_e \, \varphi \Big] \chi_{\Omega_f} d\mathcal{P}(\omega) - \int_\Omega P_e^1 \, \chi_{\Omega_e} \, \varphi \, d\mathcal{P}(\omega) = 0$$

$$\mathrm{div}_\omega \partial_t u_f = 0 \quad \text{in } G_T \times \Omega, \qquad \partial_t u_f(0) = u_{f0}^1 \quad \text{in } G \times \Omega$$

$$\Pi_\tau \partial_t u_f(t, x, \mathcal{T}(\widetilde{x})\omega) = \Pi_\tau \partial_t u_e(t, x) \quad \text{for } (t, x) \in G_T \ \ \widetilde{x} \in \Gamma(\omega), \ \mathcal{P}\text{-a.s. in } \Omega$$

$$P_e^1(t, x, \omega) = \sum_{k=1}^{3} \partial_{x_k} p_e(t, x) W_p^k(\omega) + \partial_t u_e^k(t, x) W_u^k(\omega) + Q_f(\omega, \partial_t u_f)$$

for all $\varphi \in L^2(G_T; H^1(\Omega))^3$, with $\mathrm{div}_\omega \varphi = 0$ in $G_T \times \Omega$, and $\Pi_\tau \varphi(t, x, \mathcal{T}(\widetilde{x})\omega) = 0$ for $(t, x) \in G_T$, $\widetilde{x} \in \Gamma(\omega)$ and $\mathcal{P}$-a.s. in Ω.

A. Piatnitski, MP (in preparation) 2018

Macroscopic equations for b and c

$$\vartheta_e \partial_t b - \mathrm{div}(D_{b,\mathrm{eff}} \nabla b) = \int_\Omega g_b(c, b, \mathbb{U}(b,\omega)\, \mathbf{e}(u_e))\, \chi_{\Omega_e}\, d\mathcal{P}(\omega) + \int_\Omega R(b_e)\, d\boldsymbol{\mu}(\omega) \quad \text{in } G_T$$

$$\partial_t c - \mathrm{div}(D_{\mathrm{eff}} \nabla c - u_{\mathrm{eff}}\, c) = \vartheta_f g_f(c) + \int_\Omega g_e(c, b, \mathbb{U}(b,\omega)\, \mathbf{e}(u_e))\, \chi_{\Omega_e}\, d\mathcal{P}(\omega) \quad \text{in } G_T$$

where $\vartheta_j = \int_\Omega \chi_{\Omega_j}(\omega)\, d\mathcal{P}(\omega)$, for $j = e, f$, and

$$\mathbb{U}(b,\omega) = \{\mathbb{U}_{klij}(b,\omega)\}_{k,l,i,j=1,2,3} = \left\{ b_{kl}^{ij} + W_{e,\mathrm{sym},kl}^{ij} \right\}_{k,l,i,j=1,2,3}$$

W_e^{ij} solutions of the cell problems, $\mathbf{b}_{kl} = (b_{kl}^{ij})_{i,j=1,2,3}$, $\mathbf{b}_{kl} = e_k \otimes e_l$
$\boldsymbol{\mu}(\omega)$ is the Palm measure of the random measure of the surfaces $\Gamma(\omega)$

A. Piatnitski, MP (in preparation) 2018

Numerical simulations for plant cell wall model

Macroscopic model for plant cell wall biomechanics

$$\mathrm{div}(\mathbb{E}_{\mathrm{hom}}(b_3)\,\mathbf{e}(\mathbf{u}_e)) = \mathbf{0} \qquad \text{in } G_T$$

$$\partial_t b = \mathrm{div}(\mathcal{D}_b \nabla b) + g_b(b, c, R(\mathbf{e}(\mathbf{u}_e))) \qquad \text{in } G_T$$
$$\partial_t c = \mathrm{div}(\mathcal{D}_c \nabla c) + g_c(b, c, R(\mathbf{e}(\mathbf{u}_e))) \qquad \text{in } G_T$$

$$R(\mathbf{e}(\mathbf{u}_e)) = \Big(\mathrm{tr}\big(\mathbb{E}_{\mathrm{hom}}(b_3)\,\mathbf{e}(\mathbf{u}_e)\big)\Big)^+ \quad \text{or} \quad \big(\mathrm{tr}\,\mathbf{e}(\mathbf{u}_e)\big)^+$$

$$\mathcal{D}_{\alpha,j3} = \mathcal{D}_{\alpha,3j} = D_\alpha \delta_{3j}, \quad \mathcal{D}_{\alpha,ij} = D_\alpha ⨍_{\hat{Y}_M} \big[\delta_{ij} + \partial_{y_j} v^i_\alpha(y)\big]\,dy,$$

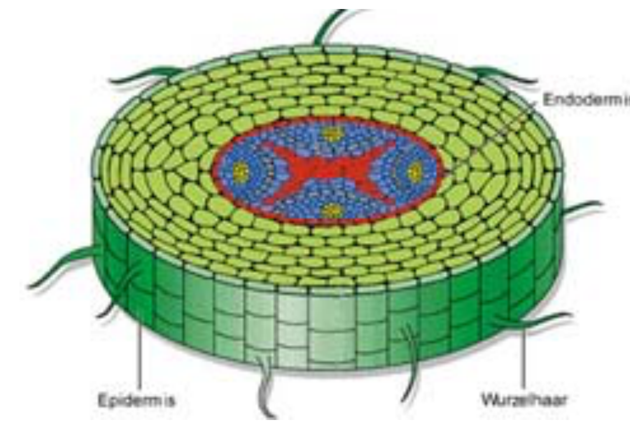
$$\alpha = b_1, b_2, b_3, c$$

$$\mathbb{E}_{\mathrm{hom},ijkl}(b_3) = ⨍_Y \Big[\mathbb{E}_{ijkl}(b_3, y) + \big(\mathbb{E}(b_3, y)\mathbf{e}_y(\mathbf{w}^{ij})\big)_{kl}\Big]\,dy$$

MP, B. Seguin, *ESAIM M2AN*, 2016

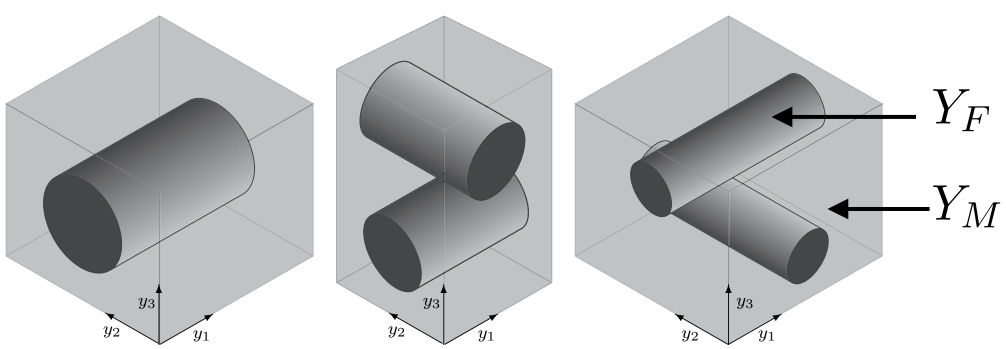
Macroscopic elasticity tensor

$$\mathbb{E}_{\hom,ijkl}(x,b_3) = \fint_Y \left[\mathbb{E}_{Y,ijkl}(b_3,y) + \mathbb{E}_{Y,ijpq}(b_3,y)\mathbf{e}_y(\mathbf{w}^{kl})_{pq}(y)\right] dy$$



$$\operatorname{div}_y\big(\mathbb{E}_Y(b_3,y)(\mathbf{e}_y(\mathbf{w}^{kl}) + \mathbf{b}^{kl})\big) = \mathbf{0} \qquad \text{in } Y$$

$$\int_Y \mathbf{w}^{kl}\, dy = \mathbf{0}, \qquad \mathbf{w}^{kl} \text{ is } Y\text{-periodic}$$



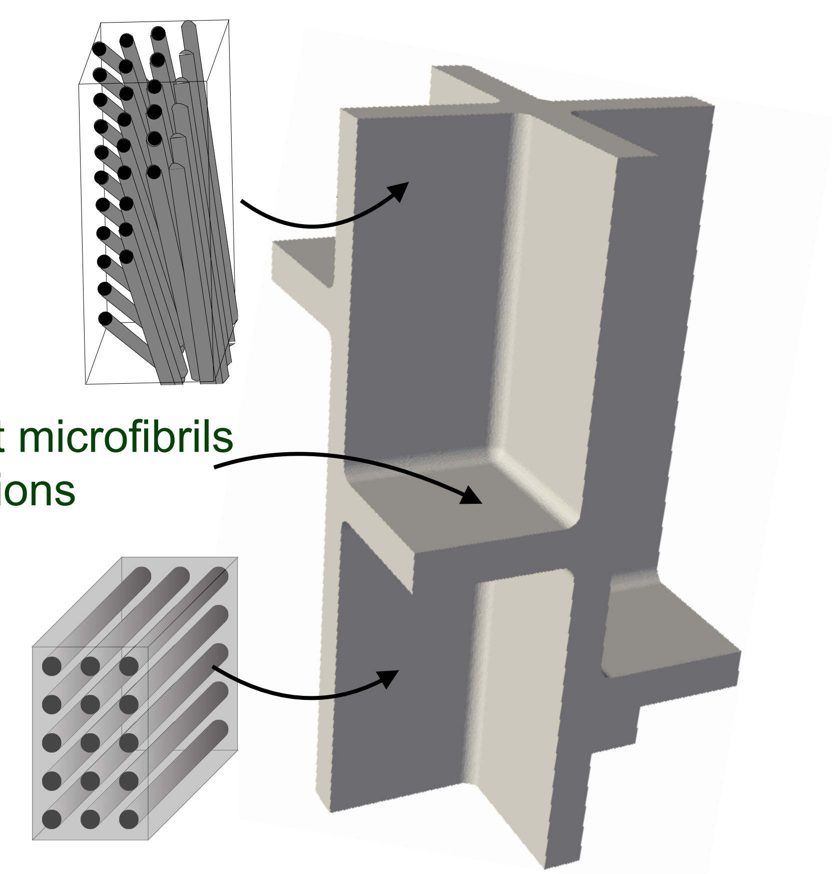
$$\mathbb{E}_Y(b_3,y) = \begin{cases} \mathbb{E}_M(b_3) & \text{if } y \in Y_M \\ \mathbb{E}_F & \text{if } y \in Y_F \end{cases}$$

Continuous rotation of microfibrils in cell walls

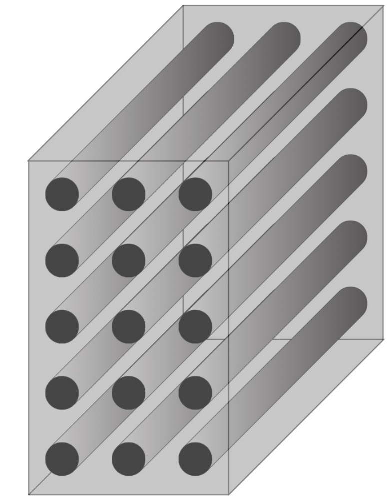
$$\mathbb{E}^\theta_{\hom,ijkl} = \mathbf{R}^\theta_{ip}\mathbf{R}^\theta_{jq}\mathbf{R}^\theta_{kr}\mathbf{R}^\theta_{ls}\mathbb{E}_{\hom,pqrs}$$

$$\mathbf{R}^\theta = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}, \qquad \mathbf{R}^\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}, \qquad \theta = \theta(x)$$

different microfibrils orientations



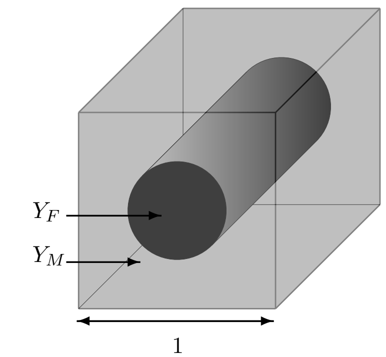
Macroscopic elasticity tensor for cell wall



$$\mathbb{E}(y, b_3) = \mathbb{E}_M(b_3)\ \chi_{Y_M}(y) + \mathbb{E}_F \chi_{Y_F}(y)$$

Cell wall matrix is assumed to be isotropic

$$\mathbb{E}_M(b_3) = E_M(b_3)\ \mathbb{E}_1 + \mathbb{E}_0 \longrightarrow \mathbb{E}_{\text{hom}}(b_3) = E_M(b_3)\ \mathbb{E}_{\text{hom},1} + \mathbb{E}_{\text{hom},0}$$



$$\mathbb{E}_M \mathbf{A} = 2\mu_M \mathbf{A} + \lambda_M (\text{tr}\, \mathbf{A})\mathbf{1}$$

Lame moduli μ_M λ_M

$$E_M = \frac{\mu_M(2\mu_M + 3\lambda_M)}{\mu_M + \lambda_M} \quad \text{and} \quad \nu_M = \frac{\lambda_M}{2(\mu_M + \lambda_M)}$$

$E_M(b_3) = 0.775\ b_3 + 8.08$ MPa

(Zsivanovits, MacDougall, Smith, Ring Carbohydrate Research, 2004)

Microfibrils are transversally isotropic

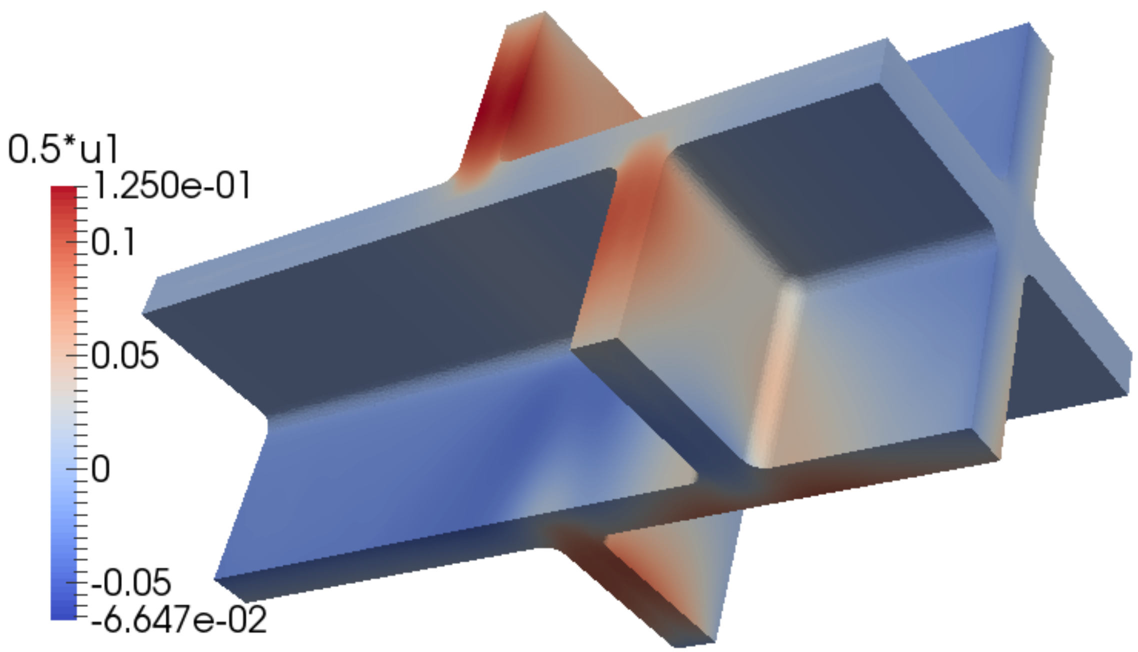
$$\begin{pmatrix} \alpha_2+\alpha_5 & \alpha_2-\alpha_5 & \alpha_3 & 0 & 0 & 0 \\ \alpha_2-\alpha_5 & \alpha_2+\alpha_5 & \alpha_3 & 0 & 0 & 0 \\ \alpha_3 & \alpha_3 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_5 \end{pmatrix}$$

$$\begin{pmatrix} 18393 & 6855 & 22277 & 0 & 0 & 0 \\ 6855 & 18393 & 22277 & 0 & 0 & 0 \\ 22277 & 22277 & \underline{259901} & 0 & 0 & 0 \\ 0 & 0 & 0 & 84842 & 0 & 0 \\ 0 & 0 & 0 & 0 & 84842 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5769 \end{pmatrix}$$

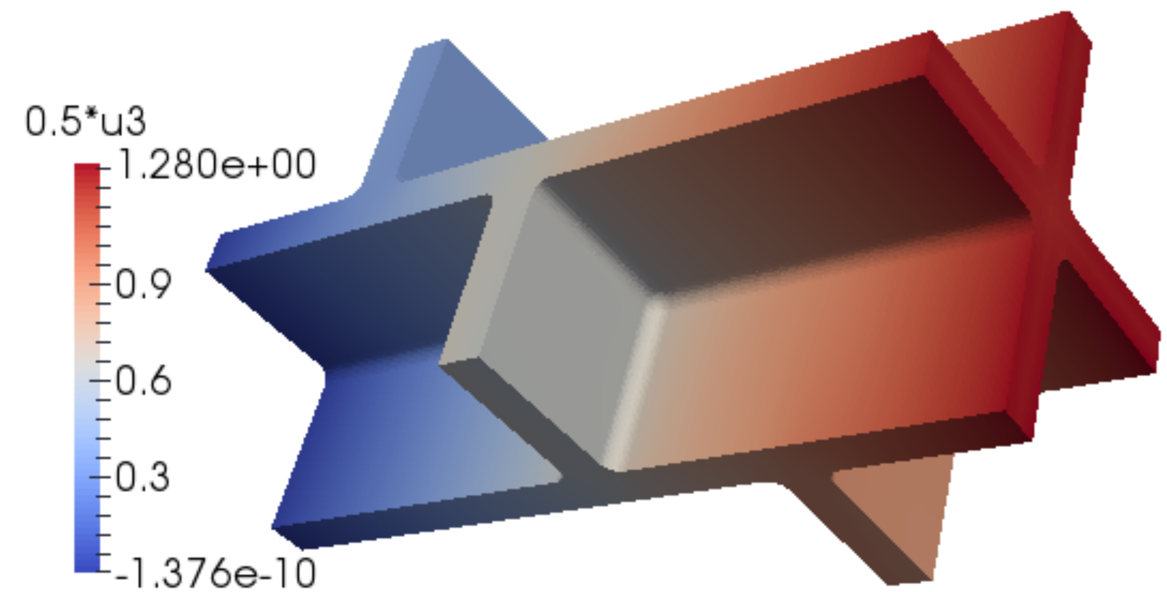
$$E_F = 15{,}000\,\text{MPa},\ \nu_{F1} = 0.3,\ n_F = 0.068,\ \nu_{F2} = 0.06,\ Z_F = 84{,}842\,\text{MPa}.$$

(Robert Moon, Review Wisconsin 2013-2014, Diddens et al., 2008)

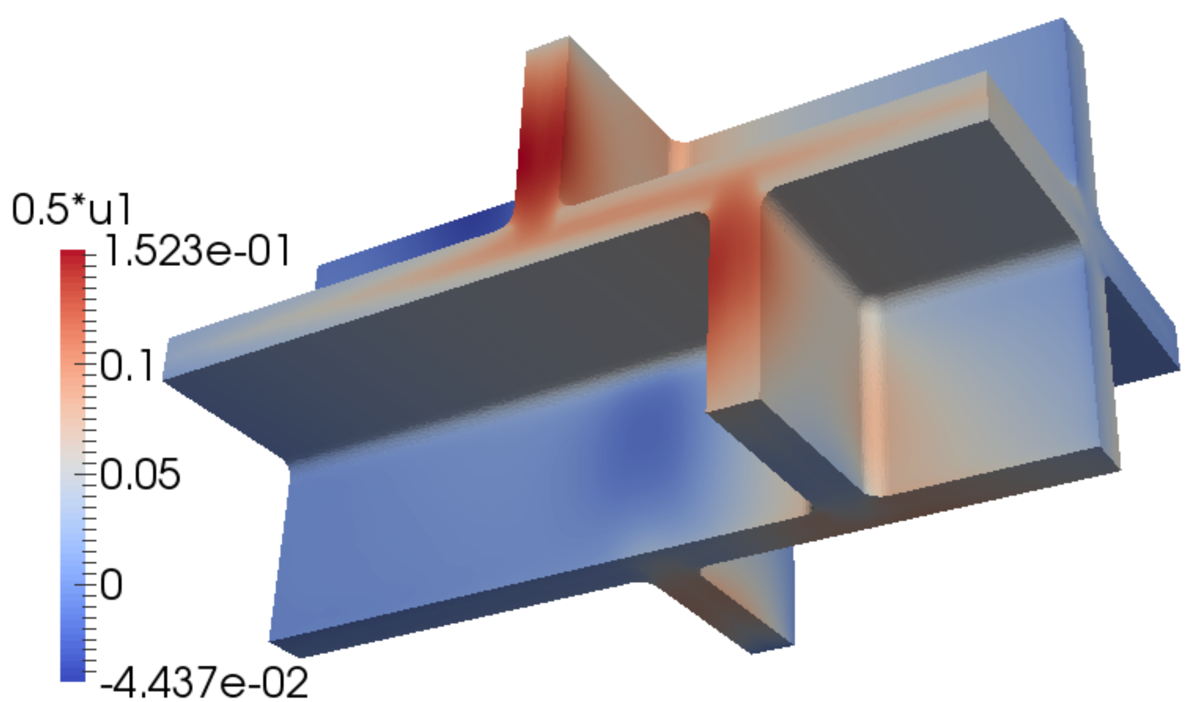
Impact of the orientation of microfibrils



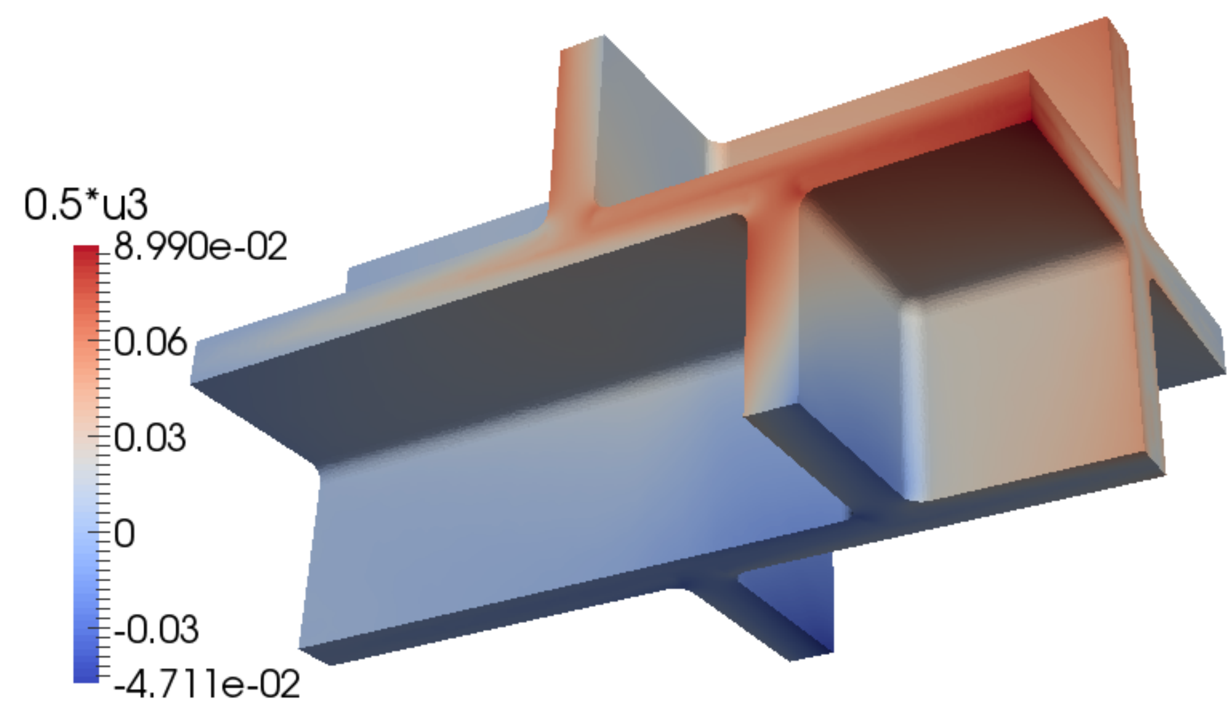
(BC1), (C1), parallel microfibrils



(BC1), (C1), parallel microfibrils



(BC1), (C1), rotating microfibrils



(BC1), (C1), rotating microfibrils

(BC1) Base case: $p = p_{\circ,1} = 0.209$ MPa and $f = 2.938 p_{\circ,1}$ MPa

MP, B. Seguin, *Bull Math Biology*, 2016

Comparison with experimental results on tissue extension and compression

RD	(BC1) parallel MF	(BC1) no shift	(BC1) 4 shifts	(BC1) rotated MF	(BC2)	(BC3)
(C1)	1.06497	1.06396	1.06984	1.00456	1.00683	1.00816
(C2)	1.06452	1.06391		1.00462	1.00671	1.00823
(C3)	1.06234	1.06294	1.06680	1.00379	1.00672	1.00831
(C4)	1.06411	1.06320		1.00497	1.00696	1.00811

Good agreement with experimental data on changes in inner or outer tissue length due to tissue tension elimination, Hejnowicz, Sievers, *J Exp.Botany* 1995:

relative displacement (RD) ranging between 0.38% and 6.98%
versus experimental data: 0.3% - 4.99%

(BC1) Base case: $p = p_{\circ,1} = 0.209$ MPa and $f = 2.938 p_{\circ,1}$ MPa

(BC2) No tensile tractions: $p = p_{\circ,1}$ and $f = 0$

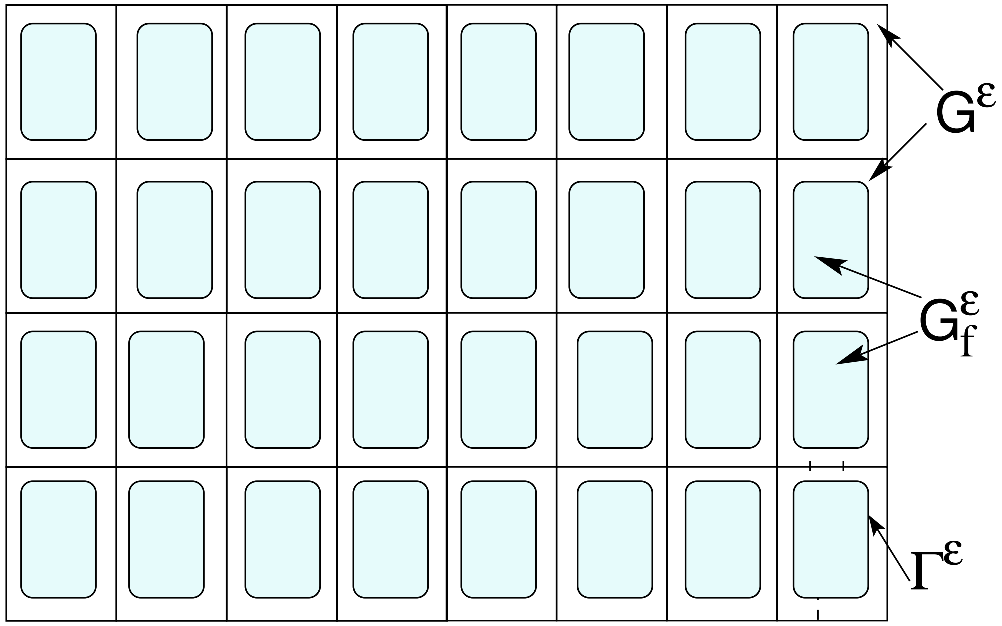
(BC3) Different turgor pressures in neighbouring cells and no tensile tractions: $p_1 = p_4 = p_5 = p_8 = p_\circ$ and $p_2 = p_3 = p_6 = p_7 = 1.3 p_{\circ,1}$, where p_i, for $i = 1, \ldots, 8$, is the pressure in cell i, and $f = 0$

MP, B. Seguin, *Bull Math Biology*, 2016

Plant tissue growth: Microscopic model

$$\mathbf{F} = \mathbf{I} + \nabla \mathbf{u}, \qquad \mathbf{F} = \mathbf{F}_e \, \mathbf{F}_g, \qquad \mathbf{T}(x, \mathbf{F}_e) = J_e^{-1} \mathbf{F}_e \frac{\partial W(x, \mathbf{F}_e)}{\partial \mathbf{F}_e}$$

$$\begin{aligned}
\operatorname{div} \mathbf{T}^\varepsilon(x, \nabla \mathbf{u}^\varepsilon, \mathbf{F}_{g,\varepsilon}) &= 0 && \text{in } G_t^\varepsilon,\ t > 0 \\
\mathbf{T}^\varepsilon(x, \nabla \mathbf{u}^\varepsilon, \mathbf{F}_{g,\varepsilon})\nu &= -\varepsilon \, P \, \nu && \text{on } \Gamma_t^\varepsilon,\ t > 0 \\
\mathbf{T}^\varepsilon(x, \nabla \mathbf{u}^\varepsilon, \mathbf{F}_{g,\varepsilon})\nu &= \mathbf{f} && \text{on } \partial G_t \setminus \Gamma_t^\varepsilon,\ t > 0
\end{aligned}$$



A.Boudaoud, MP 2018 (work in progress)

Plant tissue growth: Microscopic model

$$\mathbf{F} = \mathbf{I} + \nabla \mathbf{u}, \qquad \mathbf{F} = \mathbf{F}_{\rm e}\, \mathbf{F}_g, \qquad \mathbf{T}(x, \mathbf{F}_e) = J_e^{-1} \mathbf{F}_e \frac{\partial W(x, \mathbf{F}_e)}{\partial \mathbf{F}_e}$$

Momentum equation

$$\begin{aligned} \mathrm{div}\, \mathbf{T}^\varepsilon(x, \nabla \mathbf{u}^\varepsilon, \mathbf{F}_{g,\varepsilon}) &= 0 && \text{in } G_t^\varepsilon, \; t > 0 \\ \mathbf{T}^\varepsilon(x, \nabla \mathbf{u}^\varepsilon, \mathbf{F}_{g,\varepsilon})\nu &= -\varepsilon\, P\, \nu && \text{on } \Gamma_t^\varepsilon, \; t > 0 \\ \mathbf{T}^\varepsilon(x, \nabla \mathbf{u}^\varepsilon, \mathbf{F}_{g,\varepsilon})\nu &= \mathbf{f} && \text{on } \partial G_t \setminus \Gamma_t^\varepsilon, \; t > 0 \end{aligned}$$

Dynamics for growth

$$\mu \partial_t \mathbf{F}_{g,\varepsilon} = \begin{pmatrix} [(\mathbf{T}^\varepsilon \tau) \cdot \tau - \mathbf{T}^{\rm tresh}]_+ & 0 \\ 0 & 0 \end{pmatrix} \mathbf{F}_{g,\varepsilon} = K_2(x, \nabla \mathbf{u}^\varepsilon) \mathbf{F}_{g,\varepsilon}, \qquad t > 0$$

or

$$\mu \partial_t \mathbf{F}_{g,\varepsilon} = \begin{pmatrix} [\tilde{\mathbf{T}}_{11}^\varepsilon - \sigma^{\rm tresh}]_+ & 0 & 0 \\ 0 & [\tilde{\mathbf{T}}_{22}^\varepsilon - \sigma^{\rm tresh}]_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{F}_{g,\varepsilon} = K_3(x, \nabla \mathbf{u}) \mathbf{F}_{g,\varepsilon}$$

$$\mathbf{F}_{g,\varepsilon}(0) = \mathbf{I}$$

where

$$\tilde{\mathbf{T}}^\varepsilon = M^{-1} \mathbf{T}^\varepsilon M \quad \text{and} \quad M - \text{an appropriate transformation}$$

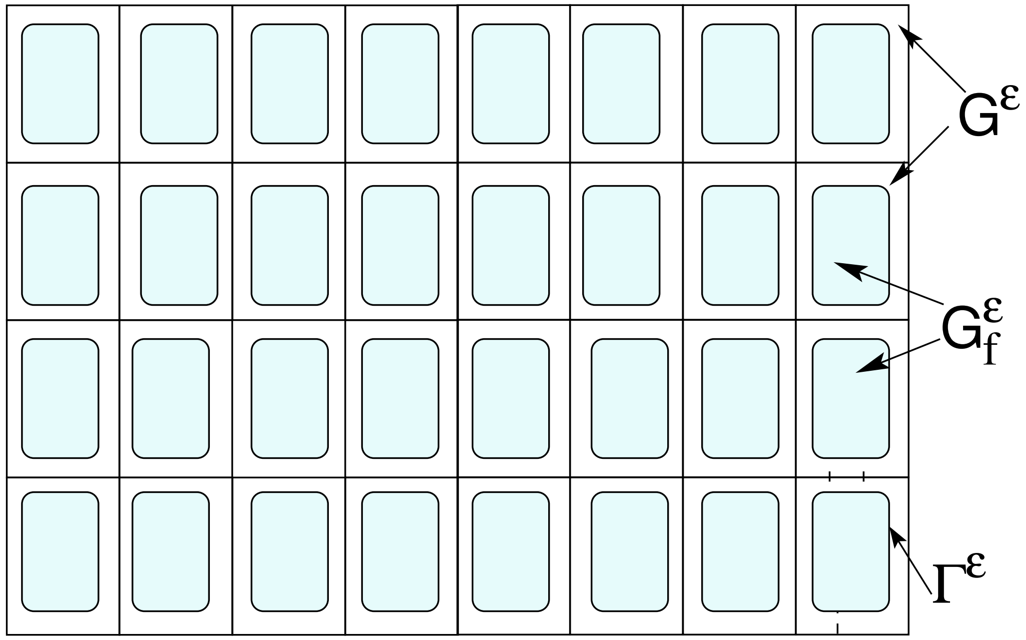
A.Boudaoud, MP 2018 (work in progress)

Plant tissue growth: Microscopic model

In reference configuration

$$
\begin{aligned}
\operatorname{div}(J_g^\varepsilon \, \mathbf{S}^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-T}) &= 0 && \text{in } \; G^\varepsilon, \; t > 0, \\
J_g^\varepsilon \, \mathbf{S}^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-T} \, N &= -\varepsilon P \, J_g^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-T} \, N && \text{on } \; \Gamma^\varepsilon, \; t > 0, \\
J_g^\varepsilon \, \mathbf{S}^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-T} \, N &= \mathbf{f} \, J_g^\varepsilon \, |\mathbf{F}_{g,\varepsilon}^{-T} N| && \text{on } \; \partial G \setminus \Gamma^\varepsilon, \; t > 0,
\end{aligned}
$$

where $\quad J_g^\varepsilon = \det(\mathbf{F}_{g,\varepsilon})$



Plant tissue growth: Microscopic model

In reference configuration

$$
\begin{aligned}
\operatorname{div}(J_g^\varepsilon \, \mathbf{S}^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-T}) &= 0 && \text{in } \; G^\varepsilon, \; t > 0, \\
J_g^\varepsilon \, \mathbf{S}^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-T} \, N &= -\varepsilon P \, J_g^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-T} \, N && \text{on } \; \Gamma^\varepsilon, \; t > 0, \\
J_g^\varepsilon \, \mathbf{S}^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-T} \, N &= \mathbf{f} \, J_g^\varepsilon \, |\mathbf{F}_{g,\varepsilon}^{-T} N| && \text{on } \; \partial G \setminus \Gamma^\varepsilon, \; t > 0,
\end{aligned}
$$

where $\quad J_g^\varepsilon = \det(\mathbf{F}_{g,\varepsilon}),$ $\quad$ linearised elasticity

$$
\mathbf{S}^\varepsilon = \mathbf{S}^\varepsilon(x, \nabla \mathbf{u}^\varepsilon, \mathbf{F}_{g,\varepsilon}) = \mathbb{E}^\varepsilon(x) \, \mathbf{e}_{\rm ell}(\nabla \mathbf{u}^\varepsilon, \mathbf{F}_{g,\varepsilon})
$$

with

$$
\mathbf{e}_{\rm ell}(\nabla \mathbf{u}^\varepsilon, \mathbf{F}_{g,\varepsilon}) = \frac{1}{2}\big[\nabla \mathbf{u}^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-1} + (\nabla \mathbf{u}^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-1})^T + \mathbf{F}_{g,\varepsilon}^{-1} + \mathbf{F}_{g,\varepsilon}^{-T} - 2\mathbf{I}\big]
$$

Plant tissue growth: Microscopic model

In reference configuration

$$\mathrm{div}(J_g^\varepsilon \, \mathbf{S}^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-T}) = 0 \qquad \text{in } \; G^\varepsilon, \; t > 0,$$

$$J_g^\varepsilon \, \mathbf{S}^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-T} N = -\varepsilon P \, J_g^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-T} N \qquad \text{on } \; \Gamma^\varepsilon, \; t > 0,$$

$$J_g^\varepsilon \, \mathbf{S}^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-T} N = \mathbf{f} \, J_g^\varepsilon \, |\mathbf{F}_{g,\varepsilon}^{-T} N| \qquad \text{on } \; \partial G \setminus \Gamma^\varepsilon, \; t > 0,$$

where $J_g^\varepsilon = \det(\mathbf{F}_{g,\varepsilon})$, linearised elasticity

$$\mathbf{S}^\varepsilon = \mathbf{S}^\varepsilon(x, \nabla \mathbf{u}^\varepsilon, \mathbf{F}_{g,\varepsilon}) = \mathbb{E}^\varepsilon(x) \, \mathbf{e}_{\rm ell}(\nabla \mathbf{u}^\varepsilon, \mathbf{F}_{g,\varepsilon})$$

with

$$\mathbf{e}_{\rm ell}(\nabla \mathbf{u}^\varepsilon, \mathbf{F}_{g,\varepsilon}) = \frac{1}{2}\left[\nabla \mathbf{u}^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-1} + (\nabla \mathbf{u}^\varepsilon \, \mathbf{F}_{g,\varepsilon}^{-1})^T + \mathbf{F}_{g,\varepsilon}^{-1} + \mathbf{F}_{g,\varepsilon}^{-T} - 2\mathbf{I}\right]$$

Growth dynamics

$$\mu \partial_t \mathbf{F}_{g,\varepsilon} = K_j(x, \nabla \mathbf{u}^\varepsilon) \mathbf{F}_{g,\varepsilon}, \qquad j = 2, 3$$

$$\mathbf{F}_{g,\varepsilon}(0) = \mathbf{I}$$

Assumptions

- $\alpha_1 |A|^2 \leq \mathbb{E}(y) A \cdot A \leq \alpha_2 |A|^2$, $\mathbb{E}_{ijkl} = \mathbb{E}_{klij} = \mathbb{E}_{jikl} = \mathbb{E}_{ijlk}$, $\mathbb{E}^\varepsilon(x) = \mathbb{E}(x/\varepsilon)$
- K_2, K_3 - bounded

Plant tissue growth: A priori estimates

Boundedness of $\mathbf{F}_{g,\varepsilon}$, extension of $\mathbf{u}^\varepsilon$, rigidity estimate for $\nabla\mathbf{u}^\varepsilon$:

$$\|(\nabla\mathbf{u}^\varepsilon\mathbf{F}_{g,\varepsilon}^{-1})_{\rm sym}\|_{L^\infty(0,T;L^2(G^\varepsilon))} + \|\partial_t(\nabla\mathbf{u}^\varepsilon\mathbf{F}_{g,\varepsilon}^{-1})_{\rm sym}\|_{L^2((0,T)\times G^\varepsilon)} \le C$$

$$\|\mathbf{F}_{g,\varepsilon}\|_{L^\infty(0,T;L^\infty(G^\varepsilon))} + \|\mathbf{F}_{g,\varepsilon}^{-1}\|_{L^\infty(0,T;L^\infty(G^\varepsilon))} \le C$$

$$\|\nabla\mathbf{u}^\varepsilon\|_{L^\infty(0,T;L^2(G))} \le C_1 + C_2\|(\nabla\mathbf{u}^\varepsilon\mathbf{F}_{g,\varepsilon}^{-1})_{\rm sym}\|_{L^\infty(0,T;L^2(G^\varepsilon))} \le C$$

Convergence:

$$\mathbf{u} \in L^\infty(0,T;H^1(G)), \quad \mathbf{u}_1 \in L^2(G_T;H^1_{\rm per}(Y)), \quad F \in L^2(G_T\times Y)$$

$$\begin{aligned} \nabla\mathbf{u}^\varepsilon &\rightharpoonup \nabla\mathbf{u} + \nabla_y\mathbf{u}_1 && \text{two-scale} \\ (\nabla\mathbf{u}^\varepsilon\mathbf{F}_{g,\varepsilon}^{-1})_{\rm sym} &\rightharpoonup F && \text{two-scale} \end{aligned}$$

Assume:

$$K_j(x,\nabla\mathbf{u}^\varepsilon) = K_j\Big(\int_{B_\delta(x)} \mathbf{S}^\varepsilon(\xi,\nabla\mathbf{u}^\varepsilon,\mathbf{F}_{g,\varepsilon})d\xi\Big), \quad j=2,3, \ \delta>0 \ \text{ fixed}$$

$\Longrightarrow$ strong convergence $\mathbf{F}_{g,\varepsilon} \to \mathbf{F}_g$, $\mathbf{F}_{g,\varepsilon}^{-1} \to \mathbf{F}_g^{-1}$

$$\mathbf{F}_g \in L^\infty(0,T;L^\infty(G))$$

Plant tissue growth: Macroscopic equations

Ansatz:

$$\mathbf{u}_1(t,x,y) = \sum_{ij=1}^{d} \partial_{x_i}\mathbf{u}^j(t,x)\mathbf{w}^{ij}(y) + \sum_{ij=1}^{d} \mathbf{F}_{g,ij}\mathbf{v}^{ij}(y) + \mathbf{h}(y)$$

Macroscopic equations in $G, \;\; t > 0$:

$$\mathrm{div}\Big(J_g\Big[\mathbb{E}_{\mathrm{hom}}(\mathbf{F}_g^{-1})\,\mathbf{e}(\mathbf{u}) + \mathbb{E}^2_{\mathrm{hom}}(\mathbf{F}_g^{-1})(\mathbf{F}_g)_{\mathrm{sym}} + \mathbb{E}^3_{\mathrm{hom}}\,(\mathbf{F}_g^{-1})_{\mathrm{sym}}\Big]\mathbf{F}_g^{-T}\Big) = -|Y_w|^{-1}\int_\Gamma J_g\,P\,\mathbf{F}_g^{-T}Nd\gamma$$

$$\mathrm{div}_y\Big[\mathbb{E}(y)(\nabla\mathbf{u} + \sum_{ij=1}^{d}\partial_{x_i}\mathbf{u}^j\,\nabla_y\mathbf{w}^{ij})\mathbf{F}_g^{-1})_{\mathrm{sym}}\Big]\mathbf{F}_g^{-T} = 0 \qquad \text{in } Y_w$$

$$\mathrm{div}_y\,\mathbb{E}(y)\Big[\big(\sum_{ij=1}^{d}\mathbf{F}_{g,ij}\nabla_y\mathbf{v}^{ij}\mathbf{F}_g^{-1}\big)_{\mathrm{sym}} - \big(\mathbf{F}_g\mathbf{F}_g^{-1}\big)_{\mathrm{sym}}\Big]\mathbf{F}_g^{-T} = 0 \qquad \text{in } Y_w$$

$$\mathrm{div}_y\,\mathbb{E}(y)\Big[\big(\nabla\mathbf{h}\,\mathbf{F}_g^{-1}\big)_{\mathrm{sym}} + \big(\mathbf{F}_g^{-1}\big)_{\mathrm{sym}}\Big]\mathbf{F}_g^{-T} = 0 \qquad \text{in } Y_w$$

A.Boudaoud, MP 2018 (work in progress)

Thank you very much !

Questions ?