Stochastic Homogenization for Reaction-Diffusion Equations

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Joint Work with Andrej Zlatoš

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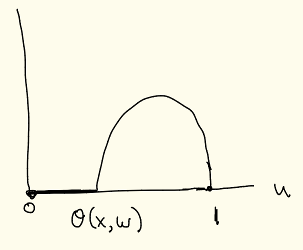
The Model

Consider

$$\begin{cases} u_t^\varepsilon - \varepsilon \Delta u^\varepsilon = \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}, u^\varepsilon, \omega\right) & \text{in} \quad (0,\infty)\times\mathbb{R}^d, \\ u^\varepsilon(0,x) \approx \chi_A & \text{on} \quad \mathbb{R}^d, \end{cases}$$

where

- $u^\varepsilon : (0,\infty)\times\mathbb{R}^d \to \mathbb{R}$,
- $A \subseteq \mathbb{R}^d$ open and bounded,
- $f(x,\cdot,\omega)$ is a random ignition reaction, typically looking like:



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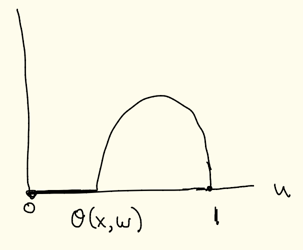

The Random Environment

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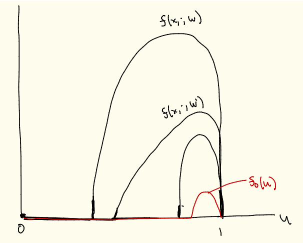
For each $\omega \in \Omega$, $f(x, u, \omega)$ satisfies

- $f(x, \cdot, \omega)$ is an ignition reaction
- There exists a fixed $f_0 : [0, 1] \to \mathbb{R}$ homogeneous ignition reaction such that $f(x, u, \omega) \geq f_0(u)$.
- $f(\cdot, \cdot, \omega)$ is Lipschitz continuous with Lipschitz constant M



Stationarity and Ergodicity (SE):

- $f(\cdot, u, \cdot)$ is stationary, i.e. there exists a measure-preserving group of transformations $\{\mathcal{T}_y\}_{y \in \mathbb{R}^d} : \Omega \to \Omega$ so that for all $u \in \mathbb{R}$,

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- $(\Omega, \mathcal{F}, \mathbb{P})$ is ergodic with respect to $\mathcal{T}_y$. In other words, if there exists an event $E \in \mathcal{F}$ so that

$$E = \mathcal{T}_y E \quad \text{for all} \quad y \in \mathbb{R}^d,$$

then $\mathbb{P}[E]$ is either 0 or 1.

Interpretation of This Model

Observe

$$\begin{cases} u_t^\varepsilon - \varepsilon \Delta u^\varepsilon = \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}, u^\varepsilon, \omega\right) & \text{in} \quad (0,\infty) \times \mathbb{R}^d, \\ u^\varepsilon(0,x) \approx \chi_A & \text{on} \quad \mathbb{R}^d, \end{cases}$$

Then

$$u^\varepsilon(t,x,\omega) = u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \omega\right)$$

where u solves

$$\begin{cases} u_t - \Delta u = f(x,u,\omega) & \text{in} \quad (0,\infty) \times \mathbb{R}^d, \\ u(0,x,\omega) \approx \chi_{\frac{1}{\varepsilon}A}(x) & \text{on} \quad \mathbb{R}^d. \end{cases}$$

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Homogenization Goal: Identify a deterministic $\overline{u}$ such that for $\mathbb{P}$-a.e. ω, $u^\varepsilon \to \overline{u}$, which represents a large-scale, long-time limit of the unscaled RD equation with random right hand side.

What does $\overline{u}$ look like?

Since

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Thus, $\overline{u} = 0 \quad \text{or} \quad 1$, so we expect

$$\overline{u}(t,x) = \chi_{\{t\}\times A_t}(t,x) = \chi_{A_t}(x)$$

for sets $\{A_t\}_{t>0} \subseteq \mathbb{R}^d$.

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More Specific Goal: Identify deterministic open sets $\{A_t\}_{t>0}$ such that almost surely and locally uniformly away from the boundary,

$$\lim_{\varepsilon\to 0} u^\varepsilon(t,x,\omega) = \begin{cases} 1 & \text{if} \quad (t,x) \in \{t\} \times A_t \\ 0 & \text{if} \quad (t,x) \in \left((0,\infty)\times\mathbb{R}^d\right) \setminus \left(\{t\} \times \overline{A_t}\right). \end{cases}$$

$\{A_t\}_{t>0}$ represents the effective front propagation taking place **on average** in the random, heterogeneous environment.

Identifying $\overline{u}(t,x)$ with a PDE

What PDE can $\overline{u}(x,t) = \chi_{A_t}(x)$ solve?

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$$\lim_{\varepsilon \to 0} u^{\varepsilon}(t,x,\omega) = \overline{u}(t,x),$$

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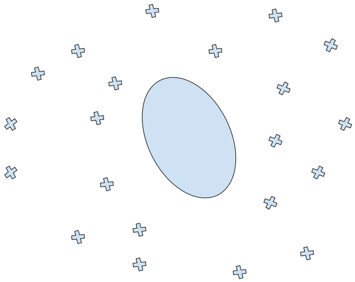
Barles, Soner, and Souganidis: For c^* "nice enough", $\overline{u}(t,x) = \chi_{A_t}(x)$ is the unique discontinuous viscosity solution.

Motivation: Forest Fires



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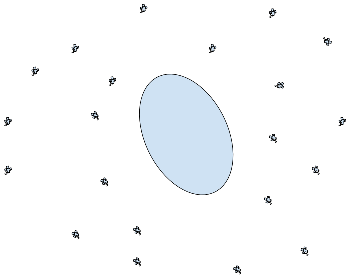

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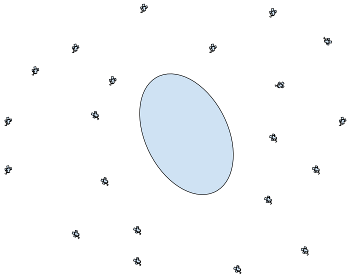

Homogenization says that for $\mathbb{P}$-almost every configuration of trees in the forest, the fire will spread like the function $\overline{u}(t,x)$ (burnt and unburnt state) (once the heterogeneities are sufficiently small and asymptotically in time)

Literature Context

Many works on periodic/random homogenization/identification of front speeds with different scalings: Barles, Soner, and Souganidis; Barles and Souganidis; Berestycki and Hamel; Majda and Souganidis; Nolen and Ryzhik; Nolen and Xin; Weinberger; Xin; Zlatoš...

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- Stochastic homogenization for viscous HJ equations with convex Hamiltonians is well-understood.

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So how will we identify $c^*(e)$ for $e \in \mathbb{S}^{d-1}$?

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The front speed $c^*(e) > 0$ is the deterministic constant such that for $\mathbb{P}$-a.e. ω, for any $K \subseteq \mathbb{R}^d$ compact, for any $\delta > 0$,

$$\lim_{t\to\infty} \inf_{K\subseteq\{x\cdot e\le c^*(e)-\delta\}} u(t,xt,\omega) = 1$$

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Observe: Initial data is invariant with respect to hyperbolic scaling, so can re-write this definition in u^ε scaling.

Seeing $c^*(e)$ from a Solution to a PDE

- If the right hand side is $f(u)$, a traveling front with speed c is an entire solution of the form

$$u(t,x) = U(x \cdot e - ct)$$

where

$$\lim_{s\to-\infty} U(s) = 1 \quad \lim_{s\to\infty} U(s) = 0.$$

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- There is an analogous type of solution (pulsating front) for right hand side $f(x,u)$ when $f(\cdot, u)$ is periodic.
- No such solutions exist for general heterogeneous right hand side $f(x,u,\omega)$.

Challenges Specific to Ignition

(H1) u has to spread. Ignition has the possibility that

- If temperature is too low (i.e. $u(0,x) \leq \theta(x,\omega)$), then $f(x,u,\omega) = 0$ so RD becomes the heat equation.
- If initial data supported on a small set, the solution may not spread.

We need there is θ_0, R such that

$$\begin{cases} u_t - \Delta u = f(x,u,\omega) & \text{in} \quad (0,\infty) \times \mathbb{R}^d, \\ u(0,x,\omega) = \theta_0 B_R & \text{on} \quad \mathbb{R}^d, \end{cases}$$

then locally uniformly in x,

$$\lim_{t\to\infty} u(t,x,\omega) = 1.$$

(this is ok for ignition by $f_0(u)$ lower bound).

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1D:

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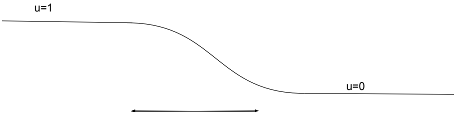

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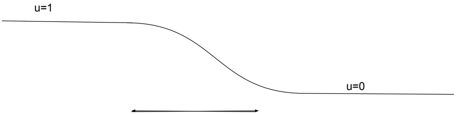

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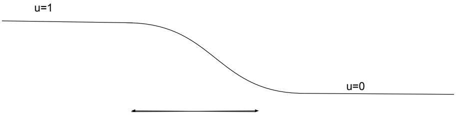

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For $d > 3$, this is not in general true! There exist reactions $f(\cdot,\cdot,\omega)$ with $\omega \in \Omega$ such that

$$L_{u,\eta,\omega}(t) \sim Ct$$

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Strategy: Track where $u \approx 1$ and $u \approx 0$.

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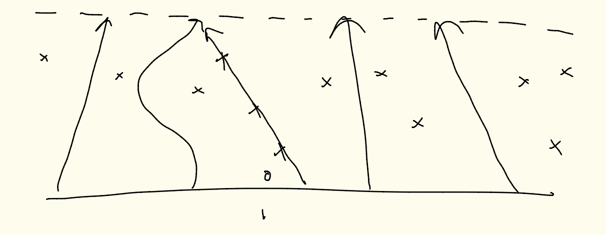
Front-Like Initial Data and Higher Dimensions:



Front speeds in random media in one dimension: Nolen and Ryzhik, Zlatoš

Definition: Spreading Speeds

Fix $e \in \mathbb{S}^{d-1}$, and let $u(\cdot, \cdot, \omega)$ solve

$$\begin{cases} u_t - \Delta u = f(x, u, \omega) & \text{in} \quad (0, \infty) \times \mathbb{R}^d, \\ u(0, x) = \theta_0 \chi_{B_R} & \text{on} \quad \mathbb{R}^d. \end{cases}$$

Then we say $w(e)$ is the *spreading speed* in direction e if for $\mathbb{P}$-a.e. ω, for any $\delta > 0$,

$$\lim_{t \to \infty} u(t, (w(e) - \delta)te, \omega) = 1,$$
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The homogenized PDE should be Hamilton-Jacobi, so try to use some ideas from stochastic homogenization for Hamilton-Jacobi equations...but no corrector equation.

First Passage Times for Reaction-Diffusion Equations

Define

$$\tau(0, y, \omega) := \inf \left\{ t : u(t, x, \omega) \geq \theta_0 \chi_{B_R(y)} \right\}.$$

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Then

$$w(e) := \frac{1}{\overline{\tau}(e)}$$

satisfies the definition of spreading speed.

All Directions at Once: The Wulff Shape

Proposition

Let $u(\cdot,\cdot,\omega)$ *solve*

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Define

$$\mathcal{S} := \left\{ se : 0 \le s \le w(e); e \in \mathbb{S}^{d-1} \right\} \subseteq \mathbb{R}^d,$$

a convex set. For $\mathbb{P}$*-a.e.* ω*, for every* $\delta > 0$*, for* t *sufficiently large,*

$$(1-\delta)t\mathcal{S} \subseteq \left\{ x : u(t,x,\omega) = \frac{1}{2} \right\} \subseteq (1+\delta)t\mathcal{S}.$$

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Remark: Can also obtain the Wulff shape for any solution with initial condition $u(0, x, \omega) = \theta_0 \chi_{B_R(y)}$ for $|y| \leq \Lambda t$.

Recovery of Front Speeds

In the periodic setting, Freidlin-Gärtner formula says:

$$w(e) = \inf_{\substack{e' \in \mathbb{S}^{d-1}, \\ e' \cdot e > 0}} \frac{c^*(e')}{e' \cdot e}$$

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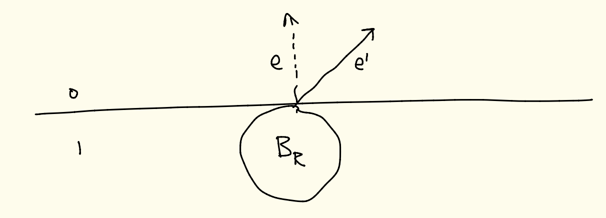

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Lemma (Local Comparison)

There are $m', c' > 0$ such that if u, u' solve the RDE with

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If $\mathcal{S}$ has a tangent in direction $e \in \mathbb{S}^{d-1}$, then the reverse Freidlin-Gartner formula holds at e!

Convergence of $u^\varepsilon \to u$

After defining $c^*(e)$ by this formula, we show $c^*(e)$ is nice enough to adapt the method of Barles and Souganidis (generalized front propagation) to show that

$$\lim_{\varepsilon \to 0} u^\varepsilon(t,x) = \overline{u}(t,x).$$

Theorem (L., Zlatoš)

Suppose u^ε *solves*

$$\begin{cases} u^\varepsilon_t - \varepsilon \Delta u^\varepsilon = \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}, u^\varepsilon, \omega\right) & \textit{in} \quad (0,\infty)\times\mathbb{R}^d, \\ u^\varepsilon(0,x) \approx \chi_A & \textit{on} \quad \mathbb{R}^d, \end{cases}$$

with $d \leq 3$, *and* f *a stationary-ergodic ignition reaction satisfying the above hypotheses. If the Wulff shape* $\mathcal{S}$ *has no corners, then homogenization holds.*

Example where Homogenization Holds: Isotropic Environment

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Canonical Example: Poisson Point Process

Let $\mathcal{P}(\omega) := \{x_n(\omega)\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^d$ denote a collection of points distributed by a Poisson point process with intensity 1. Then we have

$$f(x, u, \omega) \approx f_1(u)\chi_{B_1(\mathcal{P}(\omega))} + f_0(u)(1 - \chi_{B_1(\mathcal{P}(\omega))})$$

Common Theme: Convexity

Let

$$\overline{H}(p) := c^* \left(\frac{p}{|p|} \right) |p|.$$

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- For general ignition, will likely need to strengthen some assumptions to obtain general homogenization.

Final Comments and Future Directions:

- This approach works for very general homogenization problems of the form $u_t^\varepsilon = \varepsilon^{-1} F\left(\varepsilon^2 D^2 u^\varepsilon, \varepsilon D u^\varepsilon, u^\varepsilon, \frac{x}{\varepsilon}, \omega\right)$ satisfying some general conditions (like (H1) and (H2)).

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- Even if the front speeds $c^*(e)$ are not necessarily "nice enough," can still always homogenize initial data supported on a convex set.
- The Canadian Forest Fire Behavior Prediction System uses the Huygen's principle $A_t = A \oplus t\mathcal{S}$ for $\mathcal{S}$ an ellipse (called Richards' equation) to predict the spread of forest fires.

Thank you very much for your attention!