Two-scale homogenisation of micro-resonant PDEs (periodic and some stochastic)

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(partly joint with Ilia Kamotski, UCL)

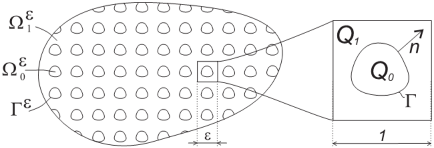
Outline:

- Background: high-contrast (= 'microresonant') homogenization (re V.V. Zhikov 2000, and followers).

- **'Partial' degeneracies** and 'generalised' micro-resonances (more of effects/ applications); A general theory for **PDE systems** under a generic 'decomposition' assumption: I. Kamotski & V.S., *Applicable Analysis* 2018, a special issue in memory of V.V. Zhikov.

- Work in progress: Stochastic micro-resonances $\Longrightarrow$ Localization/ trapping.

High-contrast homogenization and 'non-classical' two-scale limits (Zhikov 2000, 2004)



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$$a^\varepsilon(x) = \begin{cases} \delta & \text{on } \Omega_0^\varepsilon \text{ ('soft' phase)} \\ 1 & \text{on } \Omega_1^\varepsilon \text{ ('stiff' phase)} \end{cases}$$

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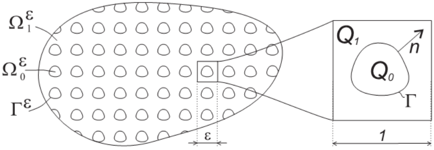

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Contrast $\delta \sim \varepsilon^2$ is a **critical scaling** giving rise to 'non-classical' effects (Khruslov 1980s; Arbogast, Douglas, Hornung 1990; Panasenko 1991; Allaire 1992; Sandrakov 1999; Brianne 2002; Bourget, Mikelic, Piatnitski 2003; Bouchitte & Felbaq 2004, ...): elliptic, spectral, parabolic, hyperbolic, nonlinear, non-periodic/ random, ...

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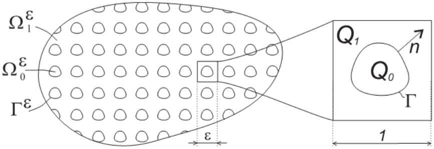

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WHY?

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$$\Rightarrow \text{Wavelength:} \quad \lambda = 2\pi/|k| = 2\pi(a/\rho)^{1/2}\omega^{-1}$$

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- $\lambda_m \sim 1$ (Macroscale) $\;\Rightarrow\; \lambda_m/\lambda_i \sim \varepsilon^{-1} \;\;\Rightarrow\;\; \delta \sim \varepsilon^2$

$$a^\varepsilon(x) = \begin{cases} \varepsilon^2 & \text{on } \Omega_0^\varepsilon \text{ (inclusions)} \\ 1 & \text{on } \Omega_1^\varepsilon \text{ (matrix)} \end{cases}$$

Two-scale formal asymptotic expansion:

$\mathrm{div}\,(a^\varepsilon(x)\nabla u^\varepsilon) + \lambda\rho u^\varepsilon = 0$ (time harmonic waves)

$\Longleftrightarrow A^\varepsilon u^\varepsilon = \lambda u^\varepsilon, \ \ \lambda = \rho\omega^2$ (spectral problem).

Seek $u^\varepsilon(x) \sim u^0(x, x/\varepsilon) + \varepsilon u^1(x, x/\varepsilon) + \ldots \ \ u^j(x,y)$ Q-periodic in y.

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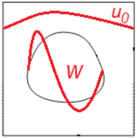
THEN:

Two-scale limit problem (Zhikov 2000, 2004)

Then

$$u^0(x,y) = \begin{cases} u_0(x) & \text{in } Q_1 \quad \text{(still low frequency)} \\ w(x,y) & \text{in } Q_0 \quad \text{(\textit{`resonance'} frequency)} \end{cases}$$



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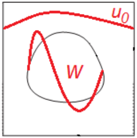

(u_0, w), $w(x,y) := u_0(x) + v(x,y)$, solves the two-scale limit spectral problem $\quad (\longleftrightarrow A^0 u^0(x,y) = \lambda u^0)$:

$$-\mathrm{div}_x(A^{hom}\nabla_x u_0(x)) = \lambda u_0(x) + \lambda\langle v\rangle_y(x) \quad \text{in } \Omega$$

$$\begin{aligned} -\Delta_y v(x,y) = \lambda(u_0(x) + v(x,y)) \qquad & \text{in } Q_0 \\ v(x,y) = 0 \qquad & \text{on } \partial Q_0, \end{aligned}$$

A^{hom} homogenized matrix for the 'perforated' domain;
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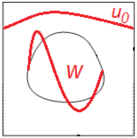

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Uncouple it $\downarrow$

Two-scale limit spectral problem

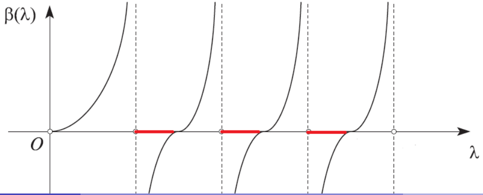
Decouple by choosing $v(x,y) = \lambda u_0(x) b(y)$

$$\begin{aligned} -\Delta_y b(y) \; - \; \lambda b = 1 \qquad & \text{in } Q_0 \\ b(y) = 0 \qquad & \text{on } \partial Q_0 \end{aligned}$$

$$-\mathrm{div}_x (A^{hom} \nabla u_0(x)) = \beta(\lambda) u_0(x), \qquad in \ \Omega,$$

$$\text{where} \quad \beta(\lambda) \; = \; \lambda + \lambda^2 \langle b \rangle = \lambda + \lambda^2 \sum_{j=1}^{\infty} \frac{\langle \phi_j \rangle_y^2}{\lambda_j - \lambda},$$

(λ_j, ϕ_j) Dirichlet eigen-values/functions of $-\Delta_y$ in inclusion Q_0 (= "micro-resonances"): $\beta < 0$: band gaps (Zhikov 2000);
$\beta(\lambda) = \mu(\omega) < 0 \longleftrightarrow$ "negative density/ magnetism" (Bouchitté & Felbacq, 2004), etc.



Analysis: Two-scale Convergence

Definition

1. Let $u_\varepsilon(x)$ be a bounded sequence in $L^2(\Omega)$. We say (u_ε) *weakly two-scale converges* to $u_0(x,y) \in L^2(\Omega \times Q)$, denoted by $u_\varepsilon \xrightharpoonup{2} u_0$, if for all $\phi \in C_0^\infty(\Omega)$, $\psi \in C_\#^\infty(Q)$

$$\int_\Omega u_\varepsilon(x)\phi(x)\psi\left(\frac{x}{\varepsilon}\right)\mathrm{d}x \longrightarrow \int_\Omega\int_Q u_0(x,y)\phi(x)\psi(y)\,\mathrm{d}x\mathrm{d}y$$

as $\varepsilon \to 0$.

2. We say (u_ε) *strongly* two-scale converges to $u_0 \in L^2(\Omega \times Q)$, denoted by $u_\varepsilon \xrightarrow{2} u_0$, if for all $v_\varepsilon \xrightharpoonup{2} v_0(x,y)$,

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as $\varepsilon \to 0$. (implies convergence of norms upon sufficient regularity)

Analysis (Zhikov 2000, 2004):

1. Two-scale limit operator:

A_0 self-adjoint in $H \subset L^2(\Omega \times Q)$, with a band-gap spectrum $\sigma(A_0)$.

$H = L^2\left(\Omega; \mathbb{C} + L^2(Q_0)\right)$. The (closed, non-negative, densely defined) form for A_0 on
$U = \left\{u(x,y) = u_0(x) + v(x,y) : u_0 \in H_0^1(\Omega), v \in L^2(\Omega; H_0^1(Q_0)\right\} \subset H$:

$$\beta(u,u) = \int_\Omega A^{hom} \nabla_x u_0 \cdot \overline{\nabla_x u_0} dx + \int_\Omega \int_{Q_0} \left|\nabla_y v(x,y)\right|^2 dy\, dx,$$

with domain $D(A_0) \subset U$.
Then, e.g. for $\Omega = \mathbb{R}^d$, the spectrum of A^0 is:

$$\sigma(A^0) = \{\lambda \geq 0 : \beta(\lambda) \geq 0\} \cup_{j=1}^{\infty} \lambda_j^D(Q_0)$$

Analysis (Zhikov 2000, 2004):

2. Two-scale ('pseudo'-)resolvent convergence:

$\forall \lambda > 0, \quad A^\varepsilon u^\varepsilon + \lambda u^\varepsilon = f^\varepsilon \in L^2(\Omega); \quad u^\varepsilon \in H^1_0(\Omega).$
If $f^\varepsilon \overset{2}{\rightharpoonup} f_0(x,y)$ then $u^\varepsilon \overset{2}{\rightharpoonup} u_0(x,y)$. (If $f^\varepsilon \overset{2}{\to} f_0(x,y)$ then $u^\varepsilon \overset{2}{\to} u_0(x,y)$.)
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3. Spectral band gaps: (Let $\Omega = \mathbb{R}^d$) cf also Hempel & Lienau 2000

$\sigma(A^\varepsilon) \to \sigma(A_0)$ in the sense of Hausdorff. (Hence a Band-gap effect:
For small enough ε, A^ε has (the smaller ε the more) gaps.
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N.B. Cherednichenko & Cooper (Arch Rat Mech Anal 2015) have improved the above *strong* two-scale resolvent convergence to an *operator* convergence, with an appropriate 'corrector' B^ε:

$$(A^\varepsilon + \alpha I)^{-1} \to (A_0 + B^\varepsilon + \alpha I)^{-1}.$$

'Frequency' gaps and time-nonlocality (memory):

$$-\mathrm{div}_x(A^{hom}\nabla u_0(x)) = \beta(\omega)u_0(x)$$

(macroscopic) Dispersion relation: $u_0 = e^{ik\cdot x - i\omega t} \quad \Rightarrow$

$$A^{hom}k\cdot k \;=\; \beta(\omega)$$

Since A^{hom} positive definite, iff $\beta(\omega) > 0$ waves propagate in **any direction** (iff $\beta(\omega) < 0$ no propagation in any direction $\iff$ Band gap).

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Nonlinearity (= dispersion)/ sign-changing of $\beta(\omega) \longrightarrow$
Fourier Transform $\omega \to$ **t** $\longrightarrow$ **time-nonlocality** (='memory')

$$\int_{-\infty}^{t} K(t-t') u_{tt}(x,t') dt' \; - \; \mathrm{div}_x(A^{hom}\nabla u(x,t)) = 0.$$

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A question:

Can one similarly get a **spatial nonlocality**? E.g. something like

$$-\operatorname{div}_x\left(\int_{\mathbb{R}^n}\mathcal{A}^{hom}(x-x')\nabla u(x',t)dx'\right).$$

If so, via (inverse) Fourier Transform $\mathbf{x}\longrightarrow\mathbf{k}$, we can, in particular, similarly expect a “spatial” dispersion/ ‘negativity’/ ‘gaps’.

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Yes, we can: e.g. $\mathbf{t} \longrightarrow \mathbf{x_{n+1}}$ then $\omega \longrightarrow \mathbf{k_{n+1}}$ etc, $\downarrow$

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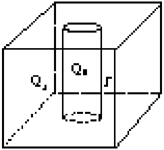
Cherednichenko, V.S., Zhikov (2006): spatial nonlocality for homogenised limit with **highly anisotropic** fibers.



$$a^\varepsilon(x) = \begin{cases} \sim & 1 & \text{in } Q_1 \text{ (matrix)} \\ \sim & \varepsilon^2 & \text{in } Q_0 \text{ “across” fibers} \\ \sim & 1 & \text{in } Q_0 \text{ “along” fibers} \end{cases}$$

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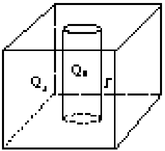

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Then (after uncoupling the two-scale limit system) there is an additional spatially nonlocal macroscopic term “along the fibers” (x_3-direction), of the form

$$-\frac{\partial}{\partial x_3}\left(\int_{\mathbb{R}} \mathcal{A}^{hom}(x_3 - x_3')\frac{\partial}{\partial x_3}u(x_1, x_2, x_3', t)dx_3'\right).$$

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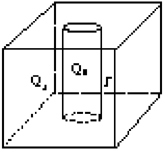

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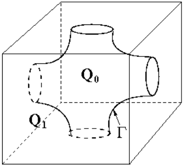
Notice, in the above fibres, $a^\varepsilon(x) = a^{(1)}(x/\varepsilon) + \varepsilon^2 a^{(0)}(x/\varepsilon)$, where

$$a^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{i.e. is partially degenerate.}$$

Macroscopic "directional" vs frequency gaps

More generally (**V.S. 2009; Linear Elasticity case:**)



$$C^{\varepsilon}(x) = \begin{cases} C^1(x/\varepsilon), & x \in Q_1^{\varepsilon} \\ \varepsilon^2 C^0(x/\varepsilon) + C^2(x/\varepsilon), & x \in Q_0^{\varepsilon} \end{cases},$$

with (as quadratic forms on symmetric matrices) $C^1, C^0 > \nu I$; but $C^2 \geq 0$ (i.e. possibly 'partially degenerate').

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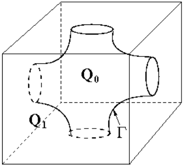

$$C^{\varepsilon}(x) = \begin{cases} C^1(x/\varepsilon), & x \in Q_1^{\varepsilon} \\ \varepsilon^2 C^0(x/\varepsilon) + C^2(x/\varepsilon), & x \in Q_0^{\varepsilon} \end{cases},$$

with (as quadratic forms on symmetric matrices) $C^1, C^0 > \nu I$; but $C^2 \geq 0$ (i.e. possibly 'partially degenerate').

Then **"directional gaps"** can occur (via formal asymptotic expansions): for certain frequency ranges macroscopic waves can propagate in some directions (e.g. along the fibers above) but cannot in others (e.g. orthogonal to the fibers). ↓

Macroscopic "directional" vs frequency gaps

Macroscopic Dispersion relation: $u = e^{ik\mathbf{n}\cdot x - i\omega t}A$, $|\mathbf{n}| = 1, k > 0 \quad \Rightarrow$

$$\det\left[\, k^2\left(C^{hom}(n) + \gamma(\mathbf{n}, k, \omega)\right) - \omega^2\beta(\mathbf{n}, k, \omega)\right] = 0, \qquad (*)$$

where $C^{hom}_{ip}(k) = C^{hom}_{ijpq}k_jk_q$ (acoustic tensor for 'half-perforated' C^{hom});

$$\gamma(\mathbf{n}, k, \omega) = \langle C^2(\mathbf{n})\zeta\rangle, \quad \beta(\mathbf{n}, k, \omega) = \langle\rho\rangle + \langle\rho_0\zeta\rangle,$$

and $\zeta(y, \mathbf{n}, k, \omega) = \zeta_{ir} = (\zeta^r)_i$ is an elastic (partially degenerating) analog of v, with ζ^r solving in the 'soft space'

$$V = \left\{v \in \left(H^1_{0,\#}(Q_0)\right)^3 \,|\, C^2\nabla v = 0\right\},$$

$$\int_{Q_0} C^0\nabla\zeta^r\cdot\nabla\eta + k^2C^2(\mathbf{n})\zeta^r\cdot\eta - \omega^2\rho_0\zeta^r\cdot\eta\, dy =$$

$$\int_{Q_0}\omega^2\rho_0\eta_r - k^2C^2(\mathbf{n})\eta\, dy, \qquad \forall\eta\in V.$$

Examples (V.S. 2009): (*) giving a 'directional localization'.

Other examples of ‘partial degeneracies’:

- ‘Easy to shear hard to compress’ elastic inclusions:

$\mu_i \sim \varepsilon^2, \quad \lambda_i \sim 1$ (Shane Cooper, 2013.)

- Photonic crystal fibers for a ‘near critical’ propagation constant (S. Cooper, I. Kamotski, V.S.: arxiv 2014).

- 3-D Maxwell with high electric permittivity (non-magnetic) inclusions (Cherednichenko, Cooper, 2015): $\epsilon_i \sim \varepsilon^{-2}$

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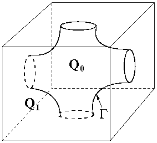
Some interesting effects in all the above, due to the ‘partial degeneracies’.

Analysis: General 'Partial' Degeneracies (I. Kamotski and V.S., a special issue in memory of V.V. Zhikov, *Applicable Analysis*, 2018)



Consider a resolvent problem:

$\Omega \subset \mathbb{R}^d, \ \ \lambda > 0,$

$-\operatorname{div}\left(a^\varepsilon(x)\nabla u^\varepsilon\right) + \lambda\rho^\varepsilon u^\varepsilon = f^\varepsilon \in L^2(\Omega),$

$u^\varepsilon \in \left(H_0^1(\Omega)\right)^n, \ n \geq 1.$

A general degeneracy:

$$a^\varepsilon(x) = a^{(1)}\left(\tfrac{x}{\varepsilon}\right) + \varepsilon^2\, a^{(0)}\left(\tfrac{x}{\varepsilon}\right), a^{(l)} \in \left(L^\infty_\#(Q)\right)^{n\times d\times n\times d}, a_{ijpq} = a_{pqij};$$

$$a^{(1)}_{ijpq}(y)\zeta_{ij}\zeta_{pq} \geq 0, \forall\zeta \in \mathbb{R}^{n\times d};\ \ a^{(1)} + a^{(0)} > \nu I : (\text{“strong ellipticity”})$$

$$\int_{\mathbb{R}^d}(a^{(1)} + a^{(0)})(y)\nabla w \cdot \nabla w \geq \nu\|\nabla w\|^2_{L^2(\mathbb{R}^d)},\ \forall w \in H^1\left(\mathbb{R}^d\right).$$

$$\rho^\varepsilon(x) = \rho\left(\tfrac{x}{\varepsilon}\right), \rho \in \left(L^\infty_\#(Q)\right)^{n\times n}, \rho_{ij} = \rho_{ji},\ \rho > \nu I.$$

Two-scale formal asymptotic expansion:

$u^\varepsilon(x) \sim u^0(x, x/\varepsilon) + \varepsilon u^1(x, x/\varepsilon) + \ldots \;\; u^j(x,y)\; Q-\textit{periodic in}\, y.$

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Then $\quad a^{(1)}(y)\nabla_y u^0(x,y) = 0.$

$$\longleftrightarrow \qquad u^0(x,\cdot) \in V := \left\{ u(u) \, : \, a^{(1)}(y)\nabla_y u^0(x,y) = 0 \right\}$$

Weak formulation:

$$\int_\Omega \left[a^{(1)}\left(\frac{x}{\varepsilon}\right) \nabla u \cdot \nabla\phi(x) + \varepsilon^2 a^{(0)}\left(\frac{x}{\varepsilon}\right) \nabla u \cdot \nabla\phi(x) + \lambda\,\rho^\varepsilon(x)\, u \cdot \phi(x) \right] dx$$

$$= \int_\Omega f^\varepsilon(x) \cdot \phi(x)\, dx, \;\; \forall \phi \in \left(H_0^1(\Omega)\right)^d.$$

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$$= \int_\Omega f^\varepsilon(x) \cdot \phi(x)\, dx, \;\; \forall \phi \in \left(H_0^1(\Omega)\right)^d.$$

A priori estimates:

$$\|u^\varepsilon\|_2 \le C\|f^\varepsilon\|_2, \quad \|\varepsilon \nabla u^\varepsilon\|_2 \le C\|f^\varepsilon\|_2, \quad \left\| \left(a^{(1)}(x/\varepsilon)\right)^{1/2} \nabla u^\varepsilon \right\|_2 \le C\|f^\varepsilon\|_2.$$

Let $\|f^\varepsilon\|_2 \le C$.

Weak two-scale limits. Key assumption on the degeneracy

Introduce

$$V := \left\{ v \in \left(H^1_\#(Q)\right)^n \,\middle|\, a^{(1)}(y)\nabla_y v = 0 \right\}$$

(subspace of "microscopic oscillations"), and

$$W := \left\{ \psi \in \left(L^2_\#(Q)\right)^{n\times d} \,\middle|\, \mathrm{div}_y \left(\left(a^{(1)}(y)\right)^{1/2} \psi(y) \right) = 0 \text{ in } \left(H^{-1}_\#(Q)\right)^n \right.$$

("microscopic fluxes")

Then, up to a subsequence, $u^\varepsilon \stackrel{2}{\rightharpoonup} u_0(x,y) \in L^2(\Omega; V)$

$$\varepsilon\nabla u^\varepsilon \stackrel{2}{\rightharpoonup} \nabla_y u_0(x,y)$$

$$\xi^\varepsilon(x) := \left(a^{(1)}(x/\varepsilon)\right)^{1/2}\nabla u^\varepsilon \stackrel{2}{\rightharpoonup} \xi_0(x,y) \in L^2(\Omega; W).$$

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$$\xi^\varepsilon(x) := \left(a^{(1)}(x/\varepsilon)\right)^{1/2} \nabla u^\varepsilon \stackrel{2}{\rightharpoonup} \xi_0(x,y) \in L^2(\Omega; W).$$

Key assumption:

There exists a constant $C > 0$ such that for all $v \in \left(H^1_\#(Q)\right)^n$ there is $v_1 \in V$ with $\|v - v_1\|_{\left(H^1_\#(Q)\right)^n} \le C \left\|a^{(1)}(y)\nabla_y v\right\|_{L^2}$ $(*)$

The key assumption examples (more in KS'2018):

The key assumption (*) holds for most of the previously considered cases:

1. Classical homogenization:
$a^{(1)}(y) \geq \nu > 0 \quad \Rightarrow \quad V = \{v = \text{const}\} \quad \Rightarrow (*) \iff$ Poincare inequality with the mean: $\|v - \langle v \rangle\|_{\left(H^1_\#(Q)\right)^n} \leq C \|\nabla_y v\|_{L^2(Q)}$

2. Double porosity models: $a^{(1)}(y) = \chi_1(y)$ (characteristic function on 'connected' phase Q_1). $\Rightarrow \quad V = \{v = \text{const} + H^1_0(Q_0)\} \quad \Rightarrow (*) \Longleftarrow$ Extension lemma: $\exists v_1 \in H^1_0$ s.t. $\|v - v_1\|_{\left(H^1_\#(Q)\right)^n} \leq C \|v\|_{H^1(Q_1)}$

3. Elasticity; 'half-soft' inclusions (Cooper 2013).
$V = \{v = \text{const}^3 + \left(H^1_0(Q_0)\right)^3 : \operatorname{div} v = 0\} \quad \Rightarrow (*) \Longleftarrow$ 'Modification' lemma (with a prescribed divergence): $\exists v_1 \in H^1_0(Q_0)$ s.t. $\operatorname{div} v_1 = 0$ and $\|\nabla(v - v_1)\|_{(L^2(Q))^n} \leq C \left(\|\nabla v\|_{L^2(Q_1)} + \|\operatorname{div} v\|_{L^2(Q_0)}\right)$

The key assumption examples (continued)

4. Elasticity with stiff fibers/ grains (cf. M. Bellieud, SIAM J Math Anal 2010): $d = n = 3$,

Single stiff cylindrical fiber: $Q_1 = \hat{Q}_1 \times [0,1)$, $\overline{\hat{Q}_1} \subset [0,1)^2$.

$$V = \left\{ v \in \left(H^1_\#(Q)\right)^3 : \, v(y) = c + \alpha y \times e_3 \text{ in } Q_1; \quad c \in \mathbb{R}^3, \alpha \in \mathbb{R} \right\}$$

($\longleftrightarrow$ translations and rotations about the cylinder's axis).

$v \in \left(H^1_\#(Q)\right)^3 \longmapsto \tilde{v}(y) = \tilde{c} + \tilde{\alpha} y \times e_3$, where $\tilde{c} \in \mathbb{R}^3$, $\tilde{\alpha} \in \mathbb{R}$ are such that

$$\int_{Q_1} \tilde{v} \, dy = \int_{Q_1} \tilde{v} \cdot (y \times e_3) \, dy = 0.$$

$\longmapsto$ can choose $v_1 = v - E\tilde{v}$, where $E : \left(H^1_\#(Q_1)\right)^3 \to \left(H^1_\#(Q)\right)^3$ is a bounded extension.

Then (*) follows from a Korn-type inequality for 'periodic' cylinders.

Similarly extended to several stiff fibers parallel to different axes and/ or isolated stiff grains, cf. M. Bellieud'10.

The key assumption examples (continued)

5. Photonic crystal fibers (Cooper, I. Kamotski, S.'14):
$V = \left\{ v \in \left(H^1_\#(Q)\right)^2 : \; v_{1,1} + v_{2,2} = v_{1,2} - v_{2,1} = 0 \;\text{ in } Q_1 \right\}$ (cf Cauchy-Riemann). $\Rightarrow (*) \iff \exists v_1 \in V$ s.t.
$\|\nabla(v - v_1)\|_{(L^2(Q))^n} \le C \left(\|v_{1,1} + v_{2,2}\|_{L^2(Q_1)} + \|v_{1,2} - v_{2,1}\|_{L^2(Q_1)} \right)$

6. 3-D Maxwell with high contrast (cf. Cherednichenko & Cooper, 2015):
$V = \left\{ v \in \left(H^1_\#(Q)\right)^3 : \operatorname{div} v = 0; \operatorname{curl} v = 0 \text{ in (simply connected)} \, Q_1 \right\}$. $\Rightarrow$
$(*) \iff \exists v_1 \in V$ s.t.
$\|\nabla(v - v_1)\|_{(L^2(Q))^n} \le C \left(\|\operatorname{curl} v\|_{L^2(Q_1)} + \|\operatorname{div} v\|_{L^2(Q)} \right)$.

7. If $a^{(1)}(y) \equiv a^{(1)}$ (a constant, not depending on y), then (*) $\iff$ $\mathcal{A}$-quasiconvexity 'constant rank' key decomposition assumption (Fonseca-Mueller).

The two-scale Limit Operator (generally 'non-local')

Let Ω be e.g. bounded Lipschitz, or $\Omega = \mathbb{R}^d$. Introduce $U \subset L^2(\Omega; V)$:

$$U := \Big\{ u(x,y) \in L^2(\Omega; V) \,\Big|\, \exists \xi(x,y) \in L^2(\Omega; W) \text{ s.t.}, \forall \Psi(x,y) \in C^\infty(\Omega; W),$$

$$\int_\Omega \int_Q \xi(x,y) \cdot \Psi(x,y) dx dy = -\int_\Omega \int_Q u(x,y) \cdot \nabla_x \cdot \left(\left(a^{(1)}(y) \right)^{1/2} \Psi(x,y) \right)$$

Define $T : U \to L^2(\Omega; W)$ by $Tu := \xi$. **Then**, $\xi_0 = Tu_0$, and

Theorem (Strong two-scale ('pseudo'-)resolvent convergence):
Let $f^\varepsilon \stackrel{2}{\to} f_0(x,y)$. Then $u^\varepsilon \stackrel{2}{\to} u_0(x,y)$, uniquely solving:

Find $u_0 \in U$ such that $\forall \phi \in U$

$$\int_\Omega \int_Q \Big\{ Tu_0(x,y) \cdot T\phi_0(x,y) + a^{(0)}(y) \nabla_y u_0(x,y) \cdot \nabla_y \phi_0(x,y) +$$

$$+ \lambda \rho(y) u_0(x,y) \cdot \phi_0(x,y) \Big\} dy\, dx = \int_\Omega \int_Q f_0(x,y) \cdot \phi_0(x,y)\, dy\, dx.$$

Two-scale limit self-adjoint operator

The above defines a self-adjoint two-scale limit operator A^0 in Hilbert space $H =$ closure of U in $L^2_\rho(\Omega \times Q)$, with domain $D(A^0) \subset U$:

$$D(A^0) \;=\; \{u(x,y) \in U \;:\; \exists w \in H \, s.t. \, \beta(u,v) = (w,v)_H \, \forall v \in U\}\,;$$

$$\beta(u,v) := \int_\Omega \int_Q Tu(x,y) \cdot \overline{Tv(x,y)} \;+\; a^{(0)}(y)\nabla_y u(x,y) \cdot \overline{\nabla_y v(x,y)}\, dy\, dx$$

Crudely, $A^0 u \;=\; T^*T \;-\; div_y\left(a^{(0)}(y)\nabla_y u\right)$,

$$T^*T \;=\; -P_V \mathrm{div}_x\left(\left(a^{(1)}(y)\right)^{1/2} P_W \left(a^{(1)}(y)\right)^{1/2} \nabla_x u(x,y)\right),$$

$P_W = L^2$-orthogonal projector on W (admissible micro-fluxes) $\longleftrightarrow$ solving the 'generalized' corrector problem:

$$\mathrm{div}_y\left(\, a^{(1)}(y)\left[\nabla_x u(x,y) + \nabla_y u_1(x,y)\right]\right) \;=\; 0,$$

$P_v = L^2$-orthogonal projector on V (admissible micro-fields).

Implications of the operator convergence

1. Strong two-scale convergence of spectral projectors. (Implies a 'part' of spectral convergence.)
2. Strong two-scale convergence of semigroups (a two-scale analogue of the Trotter-Kato theorem, cf. Zhikov 2000, Zh-Pastukhova 2007):

$$f^\varepsilon \xrightarrow{2} f_0(x,y) \in H \quad \Rightarrow \quad e^{-A^\epsilon t} f^\varepsilon \xrightarrow{2} e^{-A^0 t} f_0(x,y)$$

Hence, implications for hom-n of double porosity-type (parabolic) prblms:

$$\rho^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial t} - \mathrm{div}\left(a^\varepsilon(x)\nabla u^\varepsilon\right) = 0, \quad u^\varepsilon(x,0) = f^\varepsilon(x),$$

If $f^\varepsilon \xrightarrow{2} f_0(x,y) \in H$, then the (unique) solution $u^\varepsilon \xrightarrow{2} u_0(x,y,t)$, $\forall t \geq 0$, where u_0 is the unique solution of two-scale Cauchy problem:

$$\frac{\partial u_0}{\partial t} + A^0 u_0 = 0, \quad u_0(x,y,0) = f_0(x,y), \quad (*)$$

Implications (cf e.g. Khruslov & Co 1990s; Zhikov 2000): The limit system (*) holds under most general assumptions, and may generally give macroscopic (multi-phase) 'flows' coupled by not only temporal nonlocality (= memory) but also a 'spatial' one.

Implications of the operator convergence (continued)

2'. Strong two-scale convergence of *hyperbolic* semigroups (cf. Pastukhova 2005):
Implications for homogenisation of degenerating hyperbolic problems:

$$\rho^\varepsilon(x)\frac{\partial^2 u^\varepsilon}{\partial t^2} - \operatorname{div}\left(a^\varepsilon(x)\nabla u^\varepsilon\right) = 0, \ u^\varepsilon(x,0) = f^\varepsilon(x), \ u^\varepsilon_t(x,0) = g^\varepsilon(x),$$

$f^\varepsilon \in H^1_0$, $g^\varepsilon \in L^2$. If (for example) $f^\varepsilon \stackrel{2}{\rightharpoonup} f_0(x,y) \in U$, $g^\varepsilon \stackrel{2}{\rightharpoonup} g_0(x,y) \in H$, and

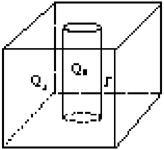
$$\limsup_{\varepsilon\to 0}\int_\Omega a^\varepsilon(x)\nabla f^\varepsilon\cdot\nabla f^\varepsilon < \infty,$$

then, for $T>0$, the (unique) solution $u^\varepsilon \stackrel{2}{\rightharpoonup} u^0(x,y,t)$ in $L^2(0,T;L^2(\Omega))$, where u^0 is the unique solution of two-scale Cauchy problem problem:

$$\frac{\partial^2 u^0}{\partial t^2} + A^0 u^0 = 0, \quad u^0(x,y,0) = f_0(x,y), \quad u^0_t(x,y,0) = Pg_0(x,y).$$

Examples with the key assumption (*) **not** held

Cherednichenko, V.S., Zhikov (2006): highly anisotropic fibers.



$$a^\varepsilon(x) = \begin{cases} \sim \ 1 & \text{in } Q_1 \text{ (matrix)} \\ \sim \ \varepsilon^2 & \text{in } Q_0 \text{ ``across'' fibers} \\ \sim \ 1 & \text{in } Q_0 \text{ ``along'' fibers} \end{cases}$$

Here $d = 3$, $n = 1$, $Q_0 = \hat{Q}_0 \times [0,1)$, $\overline{\hat{Q}_0} \subset [0,1)^2$;

$$a^{(1)}(y) = \chi_1(y)I + \chi_0(y)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \quad \alpha > 0.$$

Then

$$V = \left\{ v(y) \in H^1_\#(Q) : \ v(y) = c + \tilde{v}(\tilde{y}), \ c \in \mathbb{R}, \ \tilde{v} \in H^1_0(\hat{Q}_0), \ \tilde{y} = (y_1, y_2) \right\}$$

One can then see that (*) is not held, for e.g.
$v_n(y) = v_0(\tilde{y}) \sin(ny_1) \cos(2\pi y_3)$, $\tilde{v} \in H^1_0(\hat{Q}_0)$, when $n \to \infty$.
However the two-scale (pseudo-) resolvent convergence is still held (CSZ'06), via two-scale convergence with respect to **measures**, $d\mu_\varepsilon = \chi_1(x/\varepsilon)dx$, cf. Zhikov'00.

On the spectral convergence

The strong (two-scale) resolvent convergence implies:
$\lambda_0 \in \sigma(A^0) \quad \Rightarrow \quad \exists \lambda^\varepsilon \in \sigma(A^\varepsilon)$ such that $\lambda^\varepsilon \to \lambda_0$.

The converse property (spectral compactness) is often desired: if $\lambda^\varepsilon \in \sigma(A^\varepsilon)$ such that $\lambda^\varepsilon \to \lambda_0$ then $\lambda_0 \in \sigma(A^0)$.

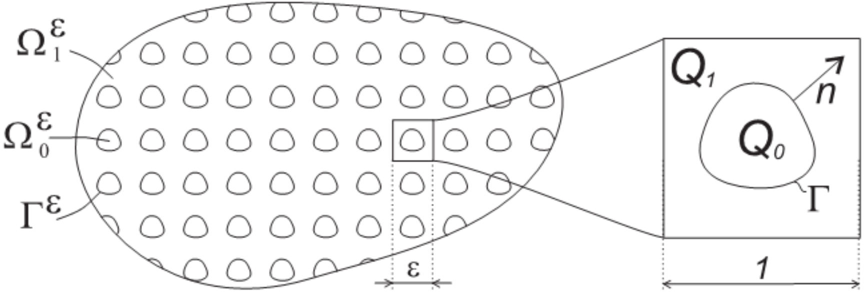
It does not hold in general. However it holds in some particular cases (which then has to be established by separate means), e.g.

- Isolated 'soft' inclusions (Zhikov 2000, 2001).

- Isolated soft elastic inclusions, icluding 'semisoft (soft in shear, stiff in compression; Cooper 2013)

Examples when it does not hold, often correspond to an 'inter-connected' soft phase, keeping supporting as $\varepsilon \to 0$ (microscopically) quasi-periodic Bloch waves, not described by the adopted two-scale (i.e. *periodic* in $y = x/\varepsilon$) framework. Nevertheless, in some cases the approach can be extended to include y-quasi-perodic limits (e.g. in photinic crystal fibers with a pre-critical propagation, Cooper, Kamotski, V.S. 2014; 1-D scalar case, Cherednichenko, Cooper, Gienneau, 2014; ...)

Dynamic problems with random micro-resonances



Let $\Omega = \mathbb{R}^3$, and $Q_0 = B_{r_0}$ (periodic balls). Consider initial value problem

$$u^\varepsilon_{tt} - \operatorname{div}\left(a^\varepsilon\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = f^\varepsilon(x,t), \quad f^\varepsilon(x,t) = f(x,t)\chi_1\left(\frac{x}{\varepsilon}\right);$$

$$\forall t \le 0,\ f(x,t) \equiv u(x,t) \equiv 0;$$

$f(x,t) \in C^\infty$, compactly supported/ rapidly decaying in x and t.

Dynamic problems with random micro-resonances

Let

$$a^\varepsilon(y,\omega) = \chi_1(y) + \varepsilon^2 \sum_m \in \mathbb{Z}^3 a(\omega,m)\chi_0(y+m),$$

where $a(\omega,m)$, $m \in \mathbb{Z}^3$, are I.I.D. with a 'nice' probability density p, e.g.

$$(\xi := \frac{a^{1/2}}{r_0}), \quad p(\xi) \in C_0^\infty(\xi_1,\xi_2), \quad 0 < \xi_1 < \xi_2 < \infty,$$

(i.e. uniformly positive and bounded),

$$\int_0^\infty p(\xi)d\xi = 1.$$

Then, formal asymptotics (believed "rigorous-able" e.g. via 'stochastic two-scale convergence, cf. e.g. Bourgeat, Mikeli, Wright (1994); Cherdantsev, Cherednichenko, Velcic (2018)), gives:

$$u^\varepsilon(x,t) \sim u^0(x,t) + v(x,t,x/\varepsilon;\omega),$$

with $\left(u^0(x,t), v(x,t,y;\omega)\right)$ coupled via: $\downarrow$

Stochastic two-scale limit problem:

↓

$$u^0_{tt} + \langle v_{tt} \rangle_{y,\omega} - \text{div}\left(A^{hom}\nabla_x u^0\right) = f(x,t),$$

$$u^0_{tt} + v_{tt} - a(\omega, m)\Delta_y v = 0,$$

where A^{hom} is a (classical) homogenized matrix for (periodic) perforated domain.

Uncoupling, then gives for (radially-symmetric in y solution of the 3-D wave equation), $v(x,y,t;\omega)$ in terms of $u^0(x,t)$: $(r := |y|)$

$$v(r,t;x,\omega) = -u^0(x,t) + \frac{r_0}{r}\Phi\left(t + a^{-1/2}(r_0 - r)\right) + \frac{r_0}{r}\Psi\left(t - a^{-1/2}(r_0 - r)\right)$$

Boundary condition for v $(r = r_0 \Rightarrow v = 0) \Rightarrow$

$$-u^0(x,t) + \Phi(t) + \Psi(t) = 0;$$

and the regularity condition at $r = 0$ gives

$$\Phi\left(t + a^{-1/2}r_0\right) + \Psi\left(t - a^{-1/2}r_0\right) = 0.$$

Uncoupling of the two-scale limit problem

As a result, $(b := a^{-1/2})$

$$v = -u^0(x,t) + \frac{r_0}{r}u^0(t - br_0 + br) - \frac{r_0}{r}u^0(t - br_0 - br) + \frac{r_0}{r}u^0(t - 3br_0 + br)$$
$$- \frac{r_0}{r}u^0\left(t - 3br_0 - br\right) + \frac{r_0}{r}u^0\left(t - 5br_0 + br\right) - \frac{r_0}{r}u^0\left(t - br_0 - 5br\right) + \ldots$$

$$v(t,r) \;=\; -u^0(x,t) + \sum_{n=0}^{\infty}\Big[\frac{r_0}{r}u^0\left(x, t - (2n+1)br_0 + br\right) - \frac{r_0}{r}u^0\left(t - (2n+1)br_0 - br\right)\Big]$$

($\forall t$ finite sum as $u^0(x,t) \equiv 0,\ t \le 0$).

To evaluate $\langle v_{tt}\rangle_{y,\omega}$,

$$v_{tt}(t,r) \;=\; -u^0_{tt}(x,t) + \sum_{n=0}^{\infty}\Big[\frac{r_0}{r}u^0_{tt}\left(x, t - (2n+1)br_0 + br\right) - \frac{r_0}{r}u^0_{tt}\left(t - (2n+1)br_0 - br\right)\Big] \quad \Downarrow$$

Uncoupling of the two-scale limit problem

$\Downarrow$ $\left(|Q_0| = \frac{4}{3}\pi r_0^3\right)$

$$\langle v_{tt}\rangle_y = -|Q_0| u^0_{tt}(x,t) + 4\pi r_0 \sum_{n=0}^{\infty}\left[\int_0^{r_0} u^0_{tt}\left(x, t-(2n+1)br_0 + br\right) r\,dr - \int_0^{r_0} u^0_{tt}\left(x, t-(2n+1)br_0 + br\right) r\,dr\right]$$

$$= -|Q_0| u^0_{tt}(x,t) + 4\pi r_0^2 b^{-1} u^0_t(x,t) - 4\pi r_0 b^{-2} u^0(x,t) + \sum_{n=1}^{\infty} b^{-1} u^0_t\left(x, t-2nbr_0\right) \qquad \Rightarrow$$

$$\langle v_{tt}\rangle_{y,\omega} = \int_{b_-}^{b_+} \langle v_{tt}\rangle_y \tilde{p}(b)\,db \qquad \left(b := a^{-1/2},\ \ 0 < b_- < b_+ < \infty\right) \ \Downarrow$$

Uncoupled two-scale limit problem

$\Downarrow$ $\left(\xi := b^{-1} r_0^{-1},\ p(\xi) \in C_0^\infty(\xi_1, \xi_2),\ \ 0 < \xi_- < \xi < \xi_+ < \infty\right)$

$$\langle v_{tt} \rangle_{y,\omega} \ = \ -|Q_0| u_{tt}^0(x,t) \ + \ 4\pi r_0^3 \left(\int_{\xi_-}^{\xi_+} \xi p(\xi) d\xi \right) u_t^0(x,t)$$

$$-\, 4\pi r_0^3 \left(\int_{\xi_-}^{\xi_+} \xi p(\xi) d\xi \right) u^0(x,t) + \int_0^t K'(\tau) u^0(x, t-\tau) d\tau,$$

$$K(\tau) \ := \ 8\pi r_0^3 \sum_{n=1}^\infty \frac{1}{\tau} \left(\frac{2n}{\tau} \right)^2 p\left(\frac{2n}{\tau} \right), \quad \tau > 0$$

(finite sum $\forall \tau > 0$).

NB: $K(\tau) \in C^\infty[0, +\infty)$, $K \geq 0$, $k'(\tau)$ 'Schwartz', $\mathrm{supp}(K) \subset \mathbb{R}^+$.

The uncoupled equation:

The equation for u^0:

$$|Q_1|u^0_{tt}(x,t) + K_1 u^0_t(x,t) - K_2 u^0(x,t) -$$
$$\int_0^\infty \mathcal{K}(\tau)u^0(x,t-\tau) - \operatorname{div}\left(A^{hom}\nabla_x u^0\right) = f(x,t),$$

with rather explicit $K_1 > 0$, $K_2 > 0$, and $\mathcal{K}(\tau)$, with "right" signs.

Taking the Fourier/ Laplace transform $t \to \omega$ etc, seems to lead (at least within certain 'frequency ranges') to a localization-type phenomenon for $u^0(x,t)$, somewhat resembling Anderson localization:

$$\left(|Q_1|\omega^2 + iK_1\omega + K_2 + \hat{\mathcal{K}}(\omega)\right)\hat{u^0} + A^h\Delta\hat{u^0} = -\hat{f}(x,\omega).$$

Possible interpretation:

The microresonances tend to 'capture' the energy at frequencies close to their eigenfrequencies; due to their randomness, a wide range of such eigen-frequencies is represented not allowing the wave to propagate.

Summary:

- A critical high contrast scaling due to **"micro-resonances"** gives rise to numerous "non-classical" effects, described by two-scale limit problems.

- **'Partial' degeneracies** often happen in physical problems, and give rise to more of such effects.

- **A general two-scale homogenization theory** can be constructed for such **partial degeneracies**, under a generically held decomposition condition. Resulting limit (homogenized) operator is generically two-scale (and macroscopically 'non-local'). Strong two-scale resolvent convergence generically holds, implying convergence of semigroups, evolution problems, etc.

- Examples when the key assumption fails, however the conclusions are still held via a two-scale convergence with respect to **measures**, Zhikov 2000.

- From convergence to (high-contrast) error bounds, etc..