Large-Scale Regularity of Random Elliptic Operators on the Half-Space

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MAX-PLANCK-GESELLSCHAFT

- We are interested in the large-scale regularity of $u \in H^1_{loc}(\overline{\mathbb{H}}^d_+)$ solving

$$
\begin{aligned}
-\nabla \cdot (a|_{\mathbb{H}^d_+} \nabla u) &= 0 \quad \text{in } \mathbb{H}^d_+, \\
u &= 0 \quad \text{on } \partial\mathbb{H}^d_+ \\
&\text{or} \\
e_d \cdot a\nabla u &= 0 \quad \text{on } \partial\mathbb{H}^d_+.
\end{aligned}
$$

$\uparrow$

"a-harmonic" on $\mathbb{H}^d_+$

- Here $a|_{\mathbb{H}^d_+}$ is the restriction of a heterogeneous coefficient-field $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, which is uniformly elliptic and bounded.

What do we mean by large-scale regularity?

- We summarize our large-scale regularity result in terms of a first-order Liouville principle.

- For a constant coefficient-field a_{hom}... the space of subquadratic functions that are a_{hom}-harmonic on $\mathbb{H}^d_+$ and vanish on $\partial\mathbb{H}^d_+$ consists of linear functions of the form $b \cdot x_d$ with $b \in \mathbb{R}$.

  Proof:

  Caccioppoli estimate + Regularity of a_{hom}-harmonic functions

  $$\frac{1}{r^2} \fint_{B_r^+} |\nabla u|^2 \lesssim \frac{1}{r^4} \fint_{B_{2r}^+} |u|^2 \, dx \qquad \sup_{x \in B_{r/2}^+} |\nabla^2 u|^2 \lesssim \frac{1}{r^2} \fint_{B_r^+} |\nabla u|^2 \, dx$$

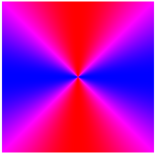
- Can we show a similar statement for heterogeneous coefficient-fields?

When can we expect such a Liouville principle to hold?

- There is a well-known counterexample:
  - Meyers ('73): For any $\lambda \in (0,1)$ there exists a bounded, λ- uniformly elliptic coefficient-field $a : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ such that $u(x) = \frac{x_2}{|x|} \cdot |x|^{\sqrt{\lambda}}$ is a-harmonic.
  - u contradicts a zeroth-order Liouville principle:

    $\rightarrow$ It is a-harmonic on $\mathbb{R}^2$ and sublinear ... but not constant.
- Take a closer look: For $\lambda \in (0,1)$ the coefficient-field in his counterexample is

$$a(x) = \begin{bmatrix} \left(\frac{x_2^2}{\lambda} + x_1^2\right)|x|^{-2} & \frac{\lambda-1}{\lambda} x_2 x_1 |x|^{-2} \\ \frac{\lambda-1}{\lambda} x_2 x_1 |x|^{-2} & \left(\frac{x_1^2}{\lambda} + x_2^2\right)|x|^{-2} \end{bmatrix}$$



Picture from J. Fischer

- Maybe we can show such a Liouville principle for a generic coefficient-fields...

  $\rightarrow$ In an almost-sure sense for a stationary and *ergodic* ensemble of coefficient fields.

Take inspiration from the whole-space case.
(Gloria, Neukamm, and Otto, 2014)

Inspiration from the whole-space case (GNO): Tools

- The corrector ϕ_i in the direction e_i is a distributional solution of

$$-\nabla \cdot (a\nabla(\phi_i + x_i)) = 0 \quad \text{in} \quad \mathbb{R}^d.$$

- The flux corrector σ_{ijk} is a distributional solution of

$$\nabla_\bullet \cdot \sigma_{ij\bullet} = e_j \cdot (a\nabla(\phi_i + x_i) - a_{hom} e_i) \quad \text{in} \quad \mathbb{R}^d$$

  and is skew-symmetric in the last two indices.

- We are interested in sublinear pairs (ϕ, σ)... which exist $\langle\cdot\rangle$ -almost surely for stationary and ergodic ensembles (Gloria, Neukamm and Otto, 2014).
- We mean "sublinearity" in an averaged L^2-sense... A pair (ϕ, σ) is called sublinear if

$$\delta_r = \frac{1}{r}\left(\fint_{B_r}\left|(\phi - \fint_{B_r}\phi\,\mathrm{d}x, \sigma - \fint_{B_r}\sigma\,\mathrm{d}x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \xrightarrow{r\uparrow\infty} 0$$

- The heterogeneous solution u is approximated by the two-scale expansion as

$$u_{2-scale} := u_{hom} + \sum_{i=1}^{d} \phi_i \partial_i u_{hom}.$$

- The "homogenization error" $w := u - u_{2-scale}$ solves

$$-\nabla \cdot (a\nabla w) = \nabla \cdot \left(\sum_{i=1}^{d} (\phi_i a - \sigma_i)\partial_i \nabla u_{hom}\right)$$

- For a function u that is a-harmonic on $\mathbb{R}^d$ the "excess of u on B_r" is given by

$$\mathrm{Exc}(r) = \inf_{b\in\mathbb{R}^d} \fint_{B_r} |\nabla(u - b\cdot(x+\phi))|^2\, dx.$$

"homogenization-adapted"

$\rightarrow$ Compare u to the space that you expect to see in the first-order Liouville principle in the squared energy norm.

Want: A large-scale $C^{1,\alpha}$-excess decay... *i.e.* on large scales $\mathrm{Exc}(r) \lesssim \left(\frac{r}{R}\right)^{2\alpha} \mathrm{Exc}(R)$.

Main step: For all $R \geq r > 0$ there exists $b \in \mathbb{R}^d$ such that

$$\fint_{B_r} |\nabla(u - b\cdot(x+\phi))|^2\,\mathrm{d}x \lesssim \left(\left(\frac{r}{R}\right)^2 (1+\varepsilon) + \left(\frac{R}{r}\right)^d \varepsilon\right) \fint_{B_R} |\nabla u|^2\,\mathrm{d}x, \qquad (*)$$

where $\varepsilon = \varepsilon(\delta_r, \delta_R) \xrightarrow{\delta_r,\delta_R\downarrow 0} 0$.

Post-processing the main step:

1) Since $\tilde{u}_c = u + c\cdot(x+\phi)$ is a-harmonic we have that (*) holds for every $c \in \mathbb{R}^d$. Letting $\theta = \frac{r}{R}$ this yields

$$\mathrm{Exc}(\theta R) \leq \left(\theta^2(1+\varepsilon) + \theta^{-d}\varepsilon\right) \mathrm{Exc}(R).$$

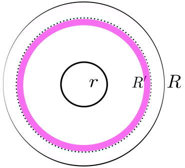
2) Choose θ and ε such that $\theta^2(1+\varepsilon) + \theta^{-d}\varepsilon \leq \theta^{2\alpha}$ for $\alpha \in (0,1)$, which gives the desired $C^{1,\alpha}$-excess decay for large enough $r, R > 0$ such that $\theta = \frac{r}{R}$.
3) Iterate.

Recall that we would like to show that... for $R \geq r > 0$ there exists $b \in \mathbb{R}^d$ such that

$$\fint_{B_r} |\nabla(u - b \cdot (x + \phi))|^2 \, \mathrm{d}x \lesssim \left(\left(\frac{r}{R}\right)^2 (1 + \varepsilon) + \left(\frac{R}{r}\right)^d \varepsilon \right) \fint_{B_R} |\nabla u|^2 \, \mathrm{d}x.$$

Main idea: Wolog $r \leq \frac{R}{4}$. Choose $R' \in (\frac{R}{2}, R)$ such that

$$\fint_{\partial B_{R'}} |\nabla^{tan} u|^2 \, \mathrm{d}x \lesssim \frac{1}{R} \fint_{B_R} |\nabla u|^2 \, \mathrm{d}x$$



and let u_{hom} solve

$$\begin{aligned} -\nabla a_{hom} \nabla u_{hom} &= 0 \quad \text{in } B_{R'} \\ u_{hom} &= u \quad \text{on } \partial B_{R'}. \end{aligned}$$

Take the ansatz $b = \nabla u_{hom}(0)$ and apply the triangle inequality:

$$\begin{aligned} &\fint_{B_r} |\nabla u - \nabla u_{hom}(0) \cdot \nabla(x + \phi))|^2 \, \mathrm{d}x \\ \leq &\fint_{B_r} |\nabla u - \nabla u_{hom} \cdot (\mathrm{id} + \nabla\phi))|^2 \, \mathrm{d}x + \fint_{B_r} |(\nabla u_{hom} - \nabla u_{hom}(0)) \cdot (\mathrm{id} + \nabla\phi))|^2 \, \mathrm{d}x \end{aligned}$$

"homogenization error"; $\sim \nabla w$ regularity of u_{hom}

$\rightarrow$ To treat the homogenization error notice that $w = u - u_{hom} - \eta \partial_i u_{hom} \phi_i$ solves

$$\begin{aligned} -\nabla \cdot a \nabla w &= \nabla \cdot ((1 - \eta)(a - a_{hom}) \nabla u_{hom} + (\phi_i a - \sigma_i) \nabla(\eta \partial_i u_{hom})) && \text{in } B_{R'} \\ w &= 0 && \text{on } \partial B_{R'}, \end{aligned}$$

where η is a smooth cut-off for $B_{R'-\delta}$ in $B_{R'}$, and then optimize the radius δ.

Homogenization | Regularity theory

Idea: Sublinear (ϕ, σ) $\xrightarrow{\text{Campanato iteration}}$ large-scale $C^{1,\alpha}$ - excess decay

This is inspired by:
$\rightarrow$ Avellaneda and Lin ('87): $C^{0,1}$- theory up to the boundary for Dirichlet data in the periodic setting.
$\rightarrow$ Armstrong and Smart (2016): a large-scale $C^{0,1}$ theory for stationary ensembles satisfying a finite range of dependence assumption.

- **Whole-space large-scale excess decay,** (Gloria, Neukamm and Otto, 2014): Assuming that there exists a sublinear pair (ϕ, σ), for every Hölder exponent $\alpha \in (0,1)$ there exists a minimal radius $r^* > 0$ such that for $R \geq r \geq r^*$:

$$\mathrm{Exc}(r) \leq C(d,\lambda,\alpha)\left(\frac{r}{R}\right)^{2\alpha}\mathrm{Exc}(R).$$

Corollaries:

- For $\alpha = 1/2$ this yields for $R \geq r \geq r^*(1/2)$ the mean value property:

$$\fint_{B_r} |\nabla u|^2\, dx \leq C_{Mean}(d,\lambda) \fint_{B_R} |\nabla u|^2\, dx.$$

- This yields a $\langle\cdot\rangle$ - almost sure $C^{1,\alpha}$- Liouville principle: Let u be a -harmonic and satisfy $|u(x)| \leq C(1+|x|^{1+\alpha})$ then for any $R > r > r^*(\alpha)$:

$$\mathrm{Exc}(r) \lesssim \left(\frac{r}{R}\right)^{2\alpha}\fint_{B_R}|\nabla u|^2\,\mathrm{d}x \lesssim \frac{r^{2\alpha}}{R^{2+2\alpha}}\fint_{B_R}|u|^2\,\mathrm{d}x \xrightarrow{R\uparrow\infty} 0$$

The half-space case.

Half-space case with homogeneous Dirichlet boundary data

- We expect that $\langle\cdot\rangle$- almost surely: If u is a-harmonic on $\mathbb{H}^d_+$ with homogeneous Dirichlet boundary conditions and satisfies the growth condition $|u(x)| \leq C(1+|x|^{1+\alpha})$, then $u = b(x_d + \phi_d^{\mathbb{H}_D})$ for some $b \in \mathbb{R}$.

- Here, $\phi_d^{\mathbb{H}_D}$ is the Dirichlet half-space corrector in the direction e_d and solves

$$\begin{aligned} -\nabla \cdot (a\nabla(x_d + \phi_d^{\mathbb{H}_D})) &= 0 \quad \text{in } \mathbb{H}^d_+, \\ \phi_d^{\mathbb{H}_D} &= 0 \quad \text{on } \partial\mathbb{H}^d_+. \end{aligned}$$

- The plan: Prove a large-scale $C^{1,\alpha}$-excess decay for the Dirichlet half-space excess

$$\mathrm{Exc}^{\mathbb{H}_D}(r) = \inf_{b\in\mathbb{R}} \fint_{B_r^+} |\nabla(u - b(x_d + \phi_d^{\mathbb{H}_D}))|^2 \, dx.$$

Need: A sublinear pair $(\phi_d^{\mathbb{H}_D}, \sigma_d^{\mathbb{H}_D})$.

$$\boxed{\nabla_\bullet \cdot \sigma_{dj\bullet}^{\mathbb{H}_D} = e_j \cdot (a\nabla(x_d + \phi_d^{\mathbb{H}_D}) - a_{hom}e_d) \text{ on } \mathbb{H}^d_+}$$

This choice of boundary data is helpful in the proof of the excess-decay because... in our setting the homogenization error on $B_{R'}^+$ is of the form

$$w^{\mathbb{H}_D} = u - u_{hom} - \eta\partial_i u_{hom}\phi_i^{\mathbb{H}_D},$$

which has Dirichlet boundary data on $B_{R'}^+$.

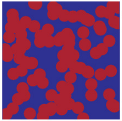
Construction of the Dirichlet half-space corrector

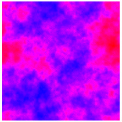
- Existence of a sublinear $(\phi_d^{\mathbb{H}_D}, \sigma_d^{\mathbb{H}_D})$, (Fischer and R., (2017)) : Assuming that there exists a whole-space pair (ϕ, σ) satisfying

$$\sum_{m=0}^{\infty} m \cdot \left(\frac{1}{2^m} \left(\fint_{B_{2^m}} |(\phi,\sigma)|^2 \, dx \right)^{1/2} \right)^{1/3} < \infty, \qquad \text{(quant. sublin.)}$$

we may construct a pair $(\phi_d^{\mathbb{H}_D}, \sigma_d^{\mathbb{H}_D})$ that is sublinear.

- For (ϕ, σ) to satisfy the quantified sublinearity condition it suffices that $\delta_r \lesssim \frac{1}{|\log r|^{6+\epsilon}}$ for large r.
  $\rightarrow$ To obtain this relation a.s. one can impose various quantified versions of the ergodicity assumption:





1) (Gloria and Otto, (2015)) Stationary ensembles with a finite range of dependence.

2) (Fischer and Otto, (2016) or GNO, (2014)) Let $a(x) = \psi(\tilde{a}(x))$, where:

- $\tilde{a}(x)$ = matrix valued stationary Gaussian random field satisfying a decorrelation estimate
- $\psi : \mathbb{R}^{d \times d} \rightarrow \Omega$ is Lipschitz.

Construction of the Dirichlet half-space corrector

- We construct the Dirichlet half-space corrector pair inductively:

  1) We construct a sublinear intermediate Dirichlet half-space corrector pair up to a certain scale.
  2) We obtain a large-scale $C^{1,\alpha}$- excess decay up to that scale.
  3) We use this to construct another sublinear intermediate Dirichlet half-space corrector pair on a larger scale.
  4) We then pass to the limit in this construction.

$\rightarrow$ This strategy mimics the construction used to build *higher order correctors* (Fischer and Otto, (2016)).

Previously...

$\rightarrow$ In their '87 work Avellaneda and Lin have already used Dirichlet boundary correctors, but theirs are adapted locally and differently for every scale.

$\rightarrow$ In the almost periodic case Armstrong and Shen (2016) have shown a $C^{0,1}$- regularity theory up the boundary for both the Dirichlet and Neumann case.

Construction of the Dirichlet half-space corrector

- The main idea is to "correct" the whole-space corrector: $\phi_d^{\mathbb{H}_D} = \phi_d - \varphi$.

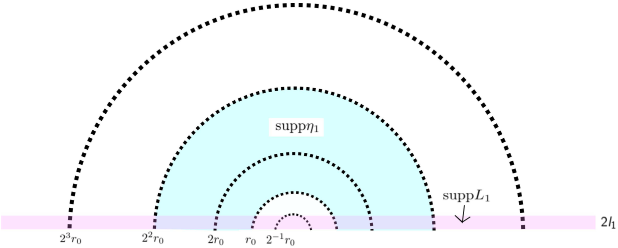
- We must construct the correction φ such that it solves:

$$\begin{aligned} -\nabla \cdot (a\nabla\varphi) &= 0 && \text{in} \quad \mathbb{H}^d_+ \\ \varphi &= \phi_d && \text{on} \quad \partial\mathbb{H}^d_+ \end{aligned}$$

and is sublinear.

$\rightarrow$ But $\nabla\phi_d \notin L^2(\mathbb{H}^d_+)$

- Let $\{\eta_m\}$ be a radial dyadic partition of unity and $\{L_m\}$ be vertical cut-offs of height $2l_m$:



- Consider the solutions to

$$
\begin{aligned}
-\nabla \cdot (a \nabla \varphi_m) &= \nabla \cdot (a \nabla (\eta_m L_m \phi_d)) && \text{in } \mathbb{H}^d_+, \\
\varphi_m &= 0 && \text{on } \partial \mathbb{H}^d_+.
\end{aligned}
$$

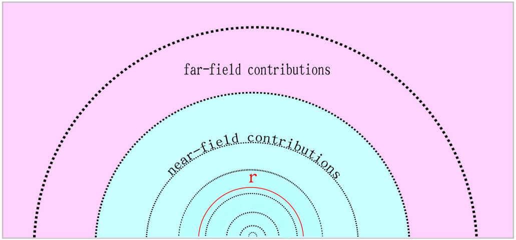
- The correction is then given by $\varphi = \sum_m (\varphi_m + \eta_m L_m \phi_d)$

- The crucial step: There exists $r_0 > 0$ such that for all m the bound

$$\left(\fint_{B_r^+} |\nabla \varphi_m|^2 \, dx \right)^{1/2} \leq 8^d C_1(d,\lambda) C_{Mean}(d,\lambda) \min \left\{ 1, \left(\frac{r_0 2^{m+1}}{r} \right)^{d/2} \right\} \times \left(\frac{1}{r_0 2^{m+1}} \left(\fint_{B^+_{r_0 2^{m+1}}} |(\phi,\sigma)|^2 \, dx \right)^{1/2} \right)^{1/3} \quad (**)$$

holds for all $r \geq r_0$.

- For any $r \geq r_0$ we split the φ_m into two groups: near-field and far-field terms.



near-field contributions: φ_m with $m \in \{-1, ..., 5\}$
far-field contributions: φ_m with $m \geq 5$

- After optimizing the heights l_m the energy estimate for the φ_m gives that for any $r > 0$:

$$\left(\fint_{B_r^+} |\nabla \varphi_m|^2 \, dx \right)^{1/2} \leq C_1(d,\lambda) \left(\frac{r_0 2^{m+1}}{r} \right)^{d/2} \times \left(\frac{1}{r_0 2^{m+1}} \left(\fint_{B^+_{r_0 2^{m+1}}} |(\phi,\sigma)|^2 \, dx \right)^{1/2} \right)^{1/3}$$

- For the near-field contributions (of $r \geq r_0$) we have that $r \geq r_0 2^{m-3}$, which turns the above energy estimate into:

$$\left(\fint_{B_r^+} |\nabla \varphi_m|^2 \, dx \right)^{1/2} \leq 8^d C_1(d,\lambda) \min \left\{ 1, \left(\frac{r_0 2^{m+1}}{r} \right)^{d/2} \right\} \times \left(\frac{1}{r_0 2^{m+1}} \left(\fint_{B^+_{r_0 2^{m+1}}} |(\phi,\sigma)|^2 \, dx \right)^{1/2} \right)^{1/3}$$

$\rightarrow$ So, (**) holds for φ_m for all $r \geq r_0$ for which it is a near-field contribution.

Treating the far-field contributions

An inductive procedure:

1) Notice that (**) holds for φ_m for $m \in \{-1, 0, 1, 2, 3\}$ for any $r \geq r_0$.

2) Obtain a intermediate half-space corrector pair, which gives us access to the mean-value property up to the scale $r_0 2^2$:
$\rightarrow$ *i.e.* if u is a-harmonic function on $B^+_{r_0 2^2}$ and vanishes on $\partial\mathbb{H}^d_+ \cap B_{r_0 2^2}$ then for $r_0 2^2 \geq R \geq r \geq r_0$:

$$\fint_{B_r^+} |\nabla u|^2 \, dx \leq C_{Mean}(d, \lambda) \fint_{B_R^+} |\nabla u|^2 \, dx.$$

3) φ_4 is a near-field term unless $r \in [r_0, 2r_0)$.

4) When $r \in [r_0, 2r_0)$ then φ_4 is a far-field term. For this case, notice that φ_4 is a-harmonic on $B^+_{r_0 2^3}$ and vanishes on the flat part of the boundary. Therefore, the mean value property from Step 2 applied to φ_4 gives

$$\fint_{B_r^+} |\nabla \varphi_4|^2 \, dx \leq C_{Mean}(d, \lambda) \fint_{B^+_{r_0 2^2}} |\nabla \varphi_4|^2 \, dx$$

and (**) follows from the energy estimate.

5) So, (**) holds for φ_m for $m \in \{-1, 0, 1, 2, 3, 4\}$ for any $r \geq r_0$.

$\rightarrow$ Must choose r_0 large enough for Step 2.

Half-space case with homogeneous Neumann boundary data

- We expect that $\langle\cdot\rangle$-almost surely: If u is a-harmonic on $\mathbb{H}^d_+$ with no-flux boundary data and satisfies the growth condition $|u(x)| \le C(1+|x|^{1+\alpha})$, then $u = b\cdot x + \phi_b^{\mathbb{H}_N} + c$ for some $c\in\mathbb{R}$ and $b\in B$, where

$$B := \left\{ b\in\mathbb{R}^d \,|\, e_d\cdot a_{hom} b = 0 \right\}.$$

- Let $\{b_1, ..., b_{d-1}\}$ be a basis for B. For $i=1,...,d-1$ we construct $\phi_{b_i}^{\mathbb{H}_N}$, the Neumann half-space corrector in the direction b_i, which solves

$$\begin{aligned} -\nabla\cdot(a\nabla(b_i\cdot x+\phi_{b_i}^{\mathbb{H}_N})) &= 0 && \text{in} \quad \mathbb{H}^d_+, \\ e_d\cdot a\nabla(b_i\cdot x+\phi_{b_i}^{\mathbb{H}_N}) &= 0 && \text{on} \quad \partial\mathbb{H}^d_+. \end{aligned}$$

- The plan: Prove a large-scale $C^{1,\alpha}$-excess decay for the Neumann half-space excess

$$\mathrm{Exc}^{\mathbb{H}_N}(r) = \inf_{b\in B} \fint_{B_r^+} |\nabla(u - b\cdot x + \phi_b^{\mathbb{H}_N}))|^2\,dx.$$

Need: For $i=1,...,d-1$ a sublinear pair $(\phi_{b_i}^{\mathbb{H}_N}, \sigma_{b_i}^{\mathbb{H}_N})$.

$$\boxed{\nabla_\bullet\cdot\sigma_{b_i j\bullet}^{\mathbb{H}_N} = e_j\cdot(a\nabla(x+\phi_{b_i}^{\mathbb{H}_N}) - a_{hom}b_i) \text{ on } \mathbb{H}^d_+}$$

Construction of the Neumann half-space corrector

- Existence of sublinear $(\phi_{b_i}^{\mathbb{H}_N}, \sigma_{b_i}^{\mathbb{H}_N})$ for $i = 1, ..., d-1$, (R., (2017)): Assuming that there exists a whole-space pair (ϕ, σ) satisfying the same quantified sublinearity condition as in the Dirichlet case we may construct pairs $(\phi_{b_i}^{\mathbb{H}_N}, \sigma_{b_i}^{\mathbb{H}_N})$ that are sublinear.

- The general idea is essentially the same as in the Dirichlet case. In particular, one constructs a sublinear correction φ_{b_i} to the whole-space corrector such that

$$
\begin{aligned}
-\nabla \cdot (a\nabla\varphi_{b_i}) &= 0 && \text{in} \quad \mathbb{H}^d_+, \\
e_d \cdot a\nabla\varphi_{b_i} &= -e_d \cdot a(\phi_{b_i} + b_i) && \text{on} \quad \partial\mathbb{H}^d_+
\end{aligned}
$$

  and then defines $\phi_{b_i}^{\mathbb{H}_N} = \phi_{b_i} + \varphi_{b_i}$.

- This construction of φ_{b_i} relies on the same inductive procedure used for the Dirichlet case. The only difference is in the treatment of the near-field terms, which in the Neumann case relies on the identity

$$\nabla_\bullet \cdot \sigma_{b_i d \bullet} = e_d \cdot a\nabla(\phi_{b_i} + b_i \cdot x),$$

  which holds distributionally on $\mathbb{H}^d_+$.

Thanks for your attention!

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