Birational Arakelov Geometry
— Durham LMS Symposium —

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Problem:

For a real number $\lambda > 1$, find an asymptotic estimate of

$$\log \# \{(a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \leq \lambda^{2n}\}$$

with respect to $n$. 
How many lattice points in the ellipse?

\[ a^2 + 2b^2 \leq \lambda^{2n} \]
Considering a shrinking map \((x, y) \mapsto (\lambda^{-n}x, \lambda^{-n}y)\),

\[
\# \{(a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \leq \lambda^{2n}\} = \# \left\{(a', b') \in (\mathbb{Z}\lambda^{-n})^2 \mid a'^2 + 2b'^2 \leq 1\right\}.
\]

We assign a square

\[
\left[a' - \frac{\lambda^{-n}}{2}, a' + \frac{\lambda^{-n}}{2}\right] \times \left[b' - \frac{\lambda^{-n}}{2}, b' + \frac{\lambda^{-n}}{2}\right]
\]

to each element of

\[
\left\{(a', b') \in (\mathbb{Z}\lambda^{-n})^2 \mid a'^2 + 2b'^2 \leq 1\right\}.
\]
\[ x^2 + 2y^2 \leq 1 \]

\[ \sum \text{(the volume of each square)} \sim \text{the volume of the ellipse} \]
Thus

\[ \# \left\{ (a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \leq \lambda^{2n} \right\} \times (\lambda^{-n})^2 \sim \text{the volume of } \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + 2y^2 \leq 1 \right\} = \frac{\pi}{\sqrt{2}}. \]

Therefore,

\[ \log \# \left\{ (a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \leq \lambda^{2n} \right\} \sim (2 \log \lambda)n. \]
Let $K$ be a number field (i.e. a finite extension of $\mathbb{Q}$) and let $K(\mathbb{C})$ be the set of all embeddings $K \hookrightarrow \mathbb{C}$. Note that 
$$\#(K(\mathbb{C})) = [K : \mathbb{Q}]$$ and $K(\mathbb{C})$ is the set of $\mathbb{C}$-valued points of $\text{Spec}(K)$. Let $O_K$ be the ring of integers in $K$, that is,

$$O_K = \{ x \in K \mid x \text{ is integral over } \mathbb{Z} \}.$$ 

We set $X = \text{Spec}(O_K)$. Let $\text{Div}(X)$ be the group of divisors on $X$, that is,

$$\text{Div}(X) := \bigoplus_{P \in X \setminus \{0\}} \mathbb{Z}[P].$$

For $D = \sum_P a_P [P]$, $\deg(D)$ is defined by

$$\deg(D) := \sum_P a_P \log \#(O_K / P).$$
\( \hat{\text{Div}}(X) \) is defined by

\[
\hat{\text{Div}}(X) = \text{Div}(X) \times \{ \xi \in \mathbb{R}^K(\mathbb{C}) \mid \xi_\sigma = \xi_{\bar{\sigma}} \ (\forall \sigma \in K(\mathbb{C})) \},
\]

where \( \bar{\sigma} \) is the composition of \( \sigma : K \hookrightarrow \mathbb{C} \) and the complex conjugation \( \mathbb{C} \overset{\sim}{\to} \mathbb{C} \). An element of \( \hat{\text{Div}}(X) \) is called an arithmetic divisor on \( X \). For simplicity, an element \( \xi \in \mathbb{R}^K(\mathbb{C}) \) is denoted by \( \sum_\sigma \xi_\sigma[\sigma] \). For example, if we set

\[
\left( \hat{x} \right) := \left( \sum_P \text{ord}_P(x)[P], \sum_\sigma - \log |\sigma(x)|^2[\sigma] \right)
\]

for \( x \in K^\times \), then \( \left( \hat{x} \right) \in \hat{\text{Div}}(X) \), which is called an arithmetic principal divisor.
The arithmetic degree $\hat{\deg}(\overline{D})$ for $\overline{D} = (D, \xi)$ is defined by

$$\hat{\deg}(\overline{D}) := \deg(D) + \frac{1}{2} \sum_{\sigma} \xi_{\sigma}.$$ 

Note that $\hat{\deg}(\hat{x}) = 0$ by the product formula. For

$$\overline{D} = \left( \sum_{P} n_{P}[P], \sum_{\sigma} \xi_{\sigma}[\sigma] \right),$$

$$\overline{D} \geq 0 \iff n_{P} \geq 0 \text{ and } \xi_{\sigma} \geq 0 \text{ for all } P \text{ and } \sigma.$$ 

We set

$$\hat{H}^{0}(X, \overline{D}) := \{ x \in K^{\times} \mid \overline{D} + \hat{x} \geq 0 \} \cup \{0\}.$$
Set $K = \mathbb{Q}(\sqrt{-2})$. Then $O_K = \mathbb{Z} + \mathbb{Z}\sqrt{-2}$ and $K(\mathbb{C}) = \{\sigma_1, \sigma_2\}$ given by $\sigma_1(\sqrt{-2}) = \sqrt{-2}$ and $\sigma_2(\sqrt{-2}) = -\sqrt{-2}$. We set $\overline{D} = (0, \log(\lambda^2)[\sigma_1] + \log(\lambda^2)[\sigma_2])$. Then $\deg(\overline{D}) = 2 \log(\lambda)$. Note that, for $x = a + b\sqrt{-2} \in \mathbb{Q}(\sqrt{-2}) \setminus \{0\}$,

$$n\overline{D} + \widehat{(x)} \geq 0 \iff \begin{cases} n \log(\lambda^2) - \log(a^2 + 2b^2) \geq 0 \\ a, b \in \mathbb{Z} \end{cases}$$

$$\iff \begin{cases} a^2 + 2b^2 \leq \lambda^{2n} \\ a, b \in \mathbb{Z} \end{cases}$$
Therefore,

\[ \hat{H}^0(X, nD) = \left\{ x \in K^\times \mid nD + \hat{(x)} \geq 0 \right\} \cup \{0\} \]

\[ = \{ a + b\sqrt{-2} \in O_K \mid a^2 + 2b^2 \leq \lambda^{2n} \}. \]

Thus the previous observation means that

\[ \log \#\hat{H}^0(X, nD) \sim \hat{\deg}(D)n. \]
Theorem (Arithmetic Hilbert-Samuel formula for Spec($O_K$))

If $\hat{\deg} (D) > 0$, then $\log \# \hat{H}^0 (nD) = n \hat{\deg} (D) + O(1)$. In particular, if $n \gg 1$, then there is $x \in K^\times$ with $nD + (x) \geq 0$. Moreover, $\lim_{n \to \infty} \log \# \hat{H}^0 (nD)/n = \hat{\deg} (D)$.

Remark

Let $r_2$ be the number of complex embeddings $K$ into $\mathbb{C}$ and let $D_K$ be the discriminant of $K$ over $\mathbb{Q}$. Then, if

$$\hat{\deg} (D) \geq \log((\pi/2)^{r_2} \sqrt{|D_K|}),$$

then $\hat{H}^0 (D) \neq \{0\}$. 

Atsushi MORIWAKI
Birational Arakelov Geometry — Durham LMS Symposium —
\[
\begin{align*}
\text{Div}(X)_{\mathbb{R}} & := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}, \\
\widehat{\text{Div}}(X)_{\mathbb{R}} & := \text{Div}(X)_{\mathbb{R}} \times \{ \xi \in \mathbb{R}^{K(\mathbb{C})} \mid \xi_{\sigma} = \xi_{\bar{\sigma}} \ (\forall \sigma \in K(\mathbb{C})) \}, \\
K^\times_{\mathbb{R}} & := (K^\times, \times) \otimes_{\mathbb{Z}} \mathbb{R}
\end{align*}
\]

For \( \bar{D} = (\sum_P x_P[P], \xi) \in \widehat{\text{Div}}(X)_{\mathbb{R}} \), \( \widehat{\deg}(D) \) is defined by
\[
\widehat{\deg}(\bar{D}) := \sum_P x_P \log \#(O_K/P) + \frac{1}{2} \sum_{\sigma \in K(\mathbb{C})} \xi_{\sigma}.
\]

For \( x = x_1^{a_1} \cdots x_r^{a_r} \in K^\times_{\mathbb{R}} \) \( (x_1, \ldots, x_r \in K^\times, a_1, \ldots, a_r \in \mathbb{R}) \),
\[
\widehat{(x)}_{\mathbb{R}} := \sum a_i(x_i).
\]

For \( \bar{D} = (\sum_P x_P[P], \xi) \in \widehat{\text{Div}}(X)_{\mathbb{R}} \),
\[
\bar{D} \geq 0 \ \overset{\text{def}}{\iff} \ x_P \geq 0 \text{ and } \xi_{\sigma} \geq 0 \text{ for all } P \text{ and } \sigma.
\]
**Theorem (Dirichlet’s unit theorem)**

If \( \widehat{\deg(D)} \geq 0 \), then there is \( x \in K^\times_R \) such that \( D + (\hat{x})_R \geq 0 \).

**Remark**

The above theorem implies the classical Dirichlet’s unit theorem, that is, for any \( \xi \in R^{K(C)} \) with \( \sum_\sigma \xi_\sigma = 0 \) and \( \xi_\sigma = \xi_{\bar{\sigma}} \), there are \( x_1, \ldots, x_r \in O^\times_K \) and \( a_1, \ldots, a_r \in R \) such that \( \xi_\sigma = \sum_i a_i \log |\sigma(x_i)| \) for all \( \sigma \).
Let $M$ be an $n$-equidimensional smooth projective variety over $\mathbb{C}$. Let $\text{Div}(M)$ be the group of Cartier divisors on $M$ and let $\text{Div}(M)_{\mathbb{R}} := \text{Div}(M) \otimes_{\mathbb{Z}} \mathbb{R}$, whose element is called an $\mathbb{R}$-divisor.

Let us fix $D \in \text{Div}(M)_{\mathbb{R}}$. We set $D = a_1D_1 + \cdots + a_lD_l$, where $a_1, \ldots, a_l \in \mathbb{R}$ and $D_i$'s are prime divisors on $M$.

Let $g : M \to \mathbb{R} \cup \{\pm \infty\}$ be a locally integrable function on $M$. We say $g$ is a $D$-Green function of $C^\infty$-type (resp. $C^0$-type) if, for each point $x \in M$, there are an open neighborhood $U_x$ of $x$, local equations $f_1, \ldots, f_l$ of $D_1, \ldots, D_l$ respectively and a $C^\infty$ (resp. $C^0$) function $u_x$ over $U_x$ such that

$$g = u_x + \sum_{i=1}^l (-a_i) \log |f_i|^2 \quad (a.e.)$$

over $U_x$. The above equation is called a local expression of $g$. 
Let $g$ be a $D$-Green function of $C^0$-type on $M$. Let
\[ g = u + \sum (-a_i) \log |f_i|^2 = u' + \sum (-a_i) \log |f'_i|^2 \quad (a.e.) \]
be two local expressions of $g$. Then, as $\sum (-a_i) \log |f_i/f'_i|^2$ is $dd^c$-closed, we have $dd^c(u) = dd^c(u')$ as currents, so that it can be defined globally. We denote it by $c_1(D, g)$. Note that $c_1(D, g)$ is a closed $(1, 1)$-current on $M$. If $g$ is of $C^\infty$-type, then $c_1(D, g)$ is represented by a $C^\infty$-form.
Let $X$ be a $d$-dimensional, generically smooth normal projective arithmetic variety, that is,

1. $X$ is projective flat integral scheme over $\mathbb{Z}$.

2. If $X_{\mathbb{Q}} = X \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q})$ is the generic fiber of $X \to \text{Spec}(\mathbb{Z})$, then $X_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$.

3. The Krull dimension of $X$ is $d$, that is, $\dim X_{\mathbb{Q}} = d - 1$.

4. $X$ is normal.

Let $\text{Div}(X)$ be the group of Cartier divisors on $X$ and $\text{Div}(X)_\mathbb{R} = \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, whose element is called an $\mathbb{R}$-divisor on $X$. For $D \in \text{Div}(X)_\mathbb{R}$, we set $D = \sum_i a_i D_i$, where $a_i \in \mathbb{R}$ and $D_i$'s are reduced and irreducible subschemes of codimension one. We say $D$ is effective if $a_i \geq 0$ for all $i$. Moreover, for $D, E \in \text{Div}(X)_\mathbb{R}$,

$$D \leq E \text{ (or } E \geq D) \iff E - D \text{ is effective}$$
Let $D$ be an $\mathbb{R}$-divisor on $X$ and let $g$ be a locally integrable function on $X(\mathbb{C})$. We say a pair $\overline{D} = (D, g)$ is an arithmetic $\mathbb{R}$-divisor on $X$ if $F^*_\infty(g) = g$ (a.e.), where $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ is the complex conjugation map, i.e. for $x \in X(\mathbb{C})$, $F_\infty(x)$ is given by the composition $\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(\mathbb{C}) \rightarrow X$. Moreover, we say $\overline{D}$ is of $C^\infty$-type (resp. $C^0$-type) if $g$ is a $D$-Green function of $C^\infty$-type (resp. $C^0$-type). For arithmetic divisors $\overline{D}_1 = (D_1, g_1)$ and $\overline{D}_2 = (D_2, g_2)$, we define $\overline{D}_1 = \overline{D}_2$ and $\overline{D}_1 \leq \overline{D}_2$ to be

$$
\overline{D}_1 = \overline{D}_2 \iff D_1 = D_2 \text{ and } g_1 = g_2 \text{ (a.e.)},
$$

$$
\overline{D}_1 \leq \overline{D}_2 \iff D_1 \leq D_2 \text{ and } g_1 \leq g_2 \text{ (a.e.)}.
$$

We say $\overline{D}$ is effective if $\overline{D} \geq (0, 0)$. 
Let $\text{Rat}(X)$ be the field of rational functions on $X$. For $\phi \in \text{Rat}(X)^\times$, we set

$$(\phi) := \sum_{\Gamma} \text{ord}_\Gamma(\phi)\Gamma \quad \text{and} \quad \widehat{\phi} := ((\phi), -\log |\phi|^2).$$

Note that $\widehat{\phi}$ is an arithmetic divisor of $C^\infty$-type.

Let $K$ be either $\mathbb{Q}$ or $\mathbb{R}$. Let $\text{Rat}(X)_K^\times := \text{Rat}(X)^\times \otimes_{\mathbb{Z}} K$ and let

$$\widehat{\bigotimes}_K : \text{Rat}(X)_K^\times \to \widehat{\text{Div}}_{C^0}(X)_\mathbb{R}$$

be the natural extension of the homomorphism

$$\text{Rat}(X)^\times \to \widehat{\text{Div}}_{C^0}(X)$$

given by $\phi \mapsto \widehat{\phi}$. 

Atsushi MORIWAKI
Birational Arakelov Geometry — Durham LMS Symposium
Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-divisor of $C^0$-type on $X$.

- $H^0(X, D) := \{ \phi \in \text{Rat}(X)^\times \mid D + (\phi) \geq 0 \} \cup \{ 0 \}$. Note that $H^0(X, D)$ is finitely generated $\mathbb{Z}$-module.

- $H^0_K(X, D) := \{ \phi \in \text{Rat}(X)^\times_K \mid D + (\phi)_K \geq 0 \} \cup \{ 0 \}$.

- $\hat{H}^0(X, \overline{D}) := \{ \phi \in \text{Rat}(X)^\times \mid \overline{D} + (\hat{\phi}) \geq (0, 0) \} \cup \{ 0 \}$. Note that $\hat{H}^0(X, \overline{D})$ is a finite set.

- $\hat{H}^0_K(X, \overline{D}) := \{ \phi \in \text{Rat}(X)^\times_K \mid \overline{D} + (\hat{\phi})_K \geq (0, 0) \} \cup \{ 0 \}$.

- $\hat{h}^0(X, \overline{D}) := \log \# \hat{H}^0(X, \overline{D})$.

- $\hat{\text{vol}}(\overline{D}) := \limsup_{n \to \infty} \frac{\log \# \hat{H}^0(X, n\overline{D})}{n^d / d!}$. 


Theorem

1. \( \hat{\text{vol}}(\overline{D}) < \infty. \)

2. (H. Chen) \( \hat{\text{vol}}(\overline{D}) := \lim_{n \to \infty} \frac{\log \# \hat{H}^0(X, n\overline{D})}{n^d/d!} \).

3. \( \hat{\text{vol}}(a\overline{D}) = a^d\hat{\text{vol}}(\overline{D}) \) for \( a \in \mathbb{R}_{\geq 0}. \)

4. (M.) \( \hat{\text{vol}} : \hat{\text{Div}}_{C^0(X)}(X)_{\mathbb{R}} \to \mathbb{R} \) is continuous in the following sense: Let \( \overline{D}_1, \ldots, \overline{D}_r, \overline{A}_1, \ldots, \overline{A}_s \) be arithmetic \( \mathbb{R} \)-divisors of \( C^0 \)-type on \( X \). For a compact subset \( B \) in \( \mathbb{R}^r \) and a positive number \( \epsilon \), there are positive numbers \( \delta \) and \( \delta' \) such that

\[
\left| \hat{\text{vol}} \left( \sum a_i \overline{D}_i + \sum \delta_j \overline{A}_j + (0, \phi) \right) - \hat{\text{vol}} \left( \sum a_i \overline{D}_i \right) \right| \leq \epsilon
\]

for all \( a_1, \ldots, a_r, \delta_1, \ldots, \delta_s \in \mathbb{R} \) and \( \phi \in C^0(X) \) with \( (a_1, \ldots, a_r) \in B, |\delta_1| + \cdots + |\delta_s| \leq \delta \) and \( \|\phi\|_{\text{sup}} \leq \delta' \).
Let $C$ be a reduced and irreducible 1-dimensional closed subscheme of $X$. We would like to define $\hat{\deg}(D|_C)$. It is characterized by the following properties:

1. $\hat{\deg}(D|_C)$ is linear with respect to $D$.

2. If $\phi \in \text{Rat}(X)^\times_{\mathbb{R}}$, then $\hat{\deg}((\phi)_{\mathbb{R}}|_C) = 0$.

3. If $C \not\subseteq \text{Supp}(D)$ and $C$ is vertical, then $\hat{\deg}(D|_C) = \log(p) \deg(D|_C)$, where $C$ is contained in the fiber over a prime $p$.

4. If $C \not\subseteq \text{Supp}(D)$ and $C$ is horizontal, then $\hat{\deg}(D|_C) = \hat{\deg}(\tilde{D}|_{\tilde{C}})$, where $\tilde{C}$ is the normalization of $C$. Note that $\tilde{C} = \text{Spec}(O_K)$ for some number field $K$. 

• $\overline{D}$ is big $\iff \hat{\text{vol}}(\overline{D}) > 0$.
• $\overline{D}$ is pseudo-effective $\iff \overline{D} + \overline{A}$ is big for any big arithmetic $\mathbb{R}$-divisor $\overline{A}$ of $C^0$-type.
• $\overline{D} = (D, g)$ is nef $\iff$
  1. $\hat{\text{deg}}(\overline{D}|_C) \geq 0$ for all reduced and irreducible 1-dimensional closed subschemes $C$ of $X$.
  2. $c_1(D, g)$ is a positive current.
• $\overline{D} = (D, g)$ is relatively nef $\iff$
  1. $\hat{\text{deg}}(\overline{D}|_C) \geq 0$ for all vertical reduced and irreducible 1-dimensional closed subschemes $C$ of $X$.
  2. $c_1(D, g)$ is a positive current.
• $\overline{D} = (D, g)$ is integrable $\iff \overline{D} = \overline{P} - \overline{Q}$ for some nef arithmetic $\mathbb{R}$-divisors $\overline{P}$ and $\overline{Q}$. 
Theorem (Arithmetic Hilbert-Samuel formula)

(Gillet-Soulé-Abbes-Bouche-Zhang-M.) If $\overline{D}$ is nef, then

$$\hat{h}^0(X, n\overline{D}) = \frac{\hat{\deg}(\overline{D}^d)}{d!} n^d + o(n^d).$$

In other words, $\hat{\text{vol}}(\overline{D}) = \hat{\deg}(\overline{D}^d)$. 
Remark

The above theorem suggests that $\hat{\deg}(\overline{D}^d)$ can be defined by $\hat{\text{vol}}(\overline{D})$. Note that

$$d!X_1 \cdots X_d = \sum_{I \subseteq \{1, \ldots, d\}} (-1)^{d-\#(I)} \left( \sum_{i \in I} X_i \right)^d$$

in $\mathbb{Z}[X_1, \ldots, X_d]$. Thus, for nef arithmetic $\mathbb{R}$-divisors $\overline{D}_1, \ldots, \overline{D}_d$,

$$d!\hat{\deg}(\overline{D}_1 \cdots \overline{D}_d) = \sum_{I \subseteq \{1, \ldots, d\}} (-1)^{d-\#(I)} \hat{\text{vol}} \left( \sum_{i \in I} \overline{D}_i \right).$$

In general, for integrable arithmetic $\mathbb{R}$-divisors $\overline{D}_1, \ldots, \overline{D}_d$, we can define $\hat{\deg}(\overline{D}_1 \cdots \overline{D}_d)$ by linearity.
Theorem (Generalized Hodge index theorem)

(M.) If $\overline{D}$ is relatively nef, then $\hat{\text{vol}}(\overline{D}) \geq \hat{\text{deg}}(\overline{D}^d)$.

Corollary (The existence of small sections)

(Faltings-Gillet-Soulé-Zhang-M.) If $\overline{D}$ is a relatively nef and $\hat{\text{deg}}(\overline{D}^d) > 0$, then there are $n$ and $\phi \in \text{Rat}(X)^\times$ such that $n\overline{D} + (\phi) \geq 0$. 

Atsushi MORIWAKI

Birational Arakelov Geometry — Durham LMS Symposium
Corollary (Arithmetic Bogomolov’s inequality)  

(Miyaoka-Soulé-M.) We assume $d = 2$ and $X$ is regular. Let $(E, h)$ be a $C^\infty$-hermitian locally free sheaf on $X$. If $E$ is semistable on the generic fiber, then

$$\widehat{\deg} \left( \widehat{c}_2(E) - \frac{r - 1}{2r} \widehat{c}_1(E)^2 \right) \geq 0,$$

where $r = \text{rk} E$.

Let $\pi : Y = \text{Proj} \left( \bigoplus_{n \geq 0} \text{Sym}^n(E) \right) \to X$ and $D$ the tautological divisor on $Y$ (i.e. $\mathcal{O}_Y(D) = \mathcal{O}(1)$). Roughly speaking, if we give a suitable Green function $g$ to $D$, then $(D, g) - (1/r) \pi^* (\widehat{c}_1(E))$ is relatively nef and its volume is zero, so that

$$\widehat{\deg} \left( ((D, g) - (1/r) \pi^* (\widehat{c}_1(E)))^{r+1} \right) \leq 0$$

by the Generalized Hodge index theorem, which gives the above inequality.
Theorem (Arithmetic Fujita’s approximation theorem)

(Chen-Yuan) We assume that $\overline{D}$ is big. For a given $\epsilon > 0$, there are a birational morphism $\nu_\epsilon : Y_\epsilon \to X$ of generically smooth, normal projective arithmetic varieties and a nef and big arithmetic $\mathbb{Q}$-divisor $\overline{P}$ of $C^\infty$-type such that $\nu_\epsilon^* (\overline{D}) \geq \overline{P}$ and $\hat{\text{vol}}(P) \geq \hat{\text{vol}}(D) - \epsilon$. 
Let $S$ be a non-singular projective surface over an algebraically closed field. Let $D$ be an effective divisor on $S$. By virtue of Bauer, the positive part of the Zariski decomposition of $D$ is characterized by the greatest element of

$$\{ M \mid M \text{ is a nef } \mathbb{R}\text{-divisor on } S \text{ and } M \leq D \}$$
Theorem (Zariski decomposition on arithmetic surfaces)

(M.) We assume that $d = 2$ and $X$ is regular. Let $\overline{D}$ be an arithmetic $\mathbb{R}$-divisor of $C^0$-type on $X$ such that the set

$$\Upsilon(\overline{D}) = \{ \overline{M} \mid \overline{M} \text{ is a nef arithmetic } \mathbb{R}-\text{divisor on } X \text{ and } \overline{M} \leq \overline{D} \}$$

is not empty. Then there is a nef arithmetic $\mathbb{R}$-divisor $\overline{P}$ such that $\overline{P}$ gives the greatest element of $\Upsilon(\overline{D})$, that is, $\overline{P} \in \Upsilon(\overline{D})$ and $\overline{M} \leq \overline{P}$ for all $\overline{M} \in \Upsilon(\overline{D})$. Moreover, if we set $\overline{N} = \overline{D} - \overline{P}$, then the following properties hold:

1. $\hat{H}^0(X, n\overline{P}) = \hat{H}^0(X, n\overline{D})$ for all $n \geq 0$.
2. $\hat{\text{vol}}(\overline{D}) = \hat{\text{vol}}(\overline{P}) = \hat{\text{deg}}(\overline{P}^2)$.
3. $\hat{\text{deg}}(\overline{P} \cdot \overline{N}) = 0$.
4. If $\overline{B}$ is an integrable arithmetic $\mathbb{R}$-divisor of $C^0$-type with $(0, 0) \preceq \overline{B} \leq \overline{N}$, then $\hat{\text{deg}}(\overline{B}^2) < 0$. 
For the proof of the property (3), the following characterization of nef arithmetic $\mathbb{R}$-Cartier is used:

**Theorem (Generalized Hodge index theorem on arithmetic surfaces)**

(M.) We assume that $d = 2$ and $\overline{D}$ is integrable. If $\deg(D_{\mathbb{Q}}) \geq 0$, then $\widehat{\deg(D^2)} \leq \widehat{\text{vol}}(D)$. Moreover, we have the following:

1. We assume that $\deg(D_{\mathbb{Q}}) = 0$. The equality holds if and only if there are $\lambda \in \mathbb{R}$ and $\phi \in \text{Rat}(X)_{\mathbb{R}}$ such that $\overline{D} = (\hat{\phi})_{\mathbb{R}} + (0, \lambda)$.

2. We assume that $\deg(D_{\mathbb{Q}}) > 0$. The equality holds if and only if $\overline{D}$ is nef.
Let $X$ be a $d$-dimensional, generically smooth normal projective arithmetic variety and let $\overline{D}$ be a big arithmetic $\mathbb{R}$-divisor of $C^0$-type on $X$. By the above theorem, a decomposition $\overline{D} = \overline{P} + \overline{N}$ is called a *Zariski decomposition of $\overline{D}$* if

1. $\overline{P}$ is a nef arithmetic $\mathbb{R}$-divisor on $X$.
2. $\overline{N}$ is an effective arithmetic $\mathbb{R}$-divisor of $C^0$-type on $X$.
3. $\hat{\text{vol}}(\overline{D}) = \hat{\text{vol}}(\overline{P})$. 
Let $\mathbb{P}_\mathbb{Z}^n = \text{Proj}(\mathbb{Z}[T_0, T_1, \ldots, T_n])$, $D = \{T_0 = 0\}$ and $z_i = T_i / T_0$ for $i = 1, \ldots, n$. Let us fix a sequence $a = (a_0, a_1, \ldots, a_n)$ of positive numbers. We define a $D$-Green function $g_a$ of $C^\infty$-type on $\mathbb{P}_\mathbb{Z}^n(\mathbb{C})$ and an arithmetic divisor $\overline{D}_a$ of $C^\infty$-type on $\mathbb{P}_\mathbb{Z}^n$ to be

$$g_a := \log(a_0 + a_1|z_1|^2 + \cdots + a_n|z_n|^2) \quad \text{and} \quad \overline{D}_a := (D, g_a).$$

Note that $c_1(\overline{D}_a)$ is positive. Let $\vartheta_a : \mathbb{R}_{\geq 0}^{n+1} \to \mathbb{R}$ be a function given by

$$\vartheta_a(x_0, x_1, \ldots, x_n) := \frac{1}{2} \left( - \sum_{i=0}^n x_i \log x_i + \sum_{i=0}^n x_i \log a_i \right),$$

and let

$$\Theta_a := \{(x_1, \ldots, x_n) \in \Delta_n \mid \vartheta_a(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n) \geq 0\},$$

where $\Delta_n := \{(x_1, \ldots, x_n) \in \mathbb{R}_{\geq 0}^n \mid x_1 + \cdots + x_n \leq 1\}$. 
$a=(0.4,0.4,0.4)$
The following properties (1) – (6) hold for $D_a$:

1. $D_a$ is ample $\iff a_0 > 1, a_1 > 1, \ldots, a_n > 1$.
2. $D_a$ is nef $\iff a_0 \geq 1, a_1 \geq 1, \ldots, a_n \geq 1$.
3. $D_a$ is big $\iff a_0 + a_1 + \cdots + a_n > 1$.
4. $D_a$ is pseudo-effective $\iff a_0 + a_1 + \cdots + a_n \geq 1$. 
Figure: Geography of $\overline{D_a}$ on $\mathbb{P}^1_{\mathbb{Z}}$
(5) (Integral formula) The following formulae hold:

\[ \hat{\text{vol}}(D_a) = (n + 1)! \int_{\Theta_a} \vartheta_a(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n) \, dx_1 \cdots dx_n \]

and

\[ \hat{\text{deg}}(D_a^{n+1}) = (n + 1)! \int_{\Delta_n} \vartheta_a(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n) \, dx_1 \cdots dx_n. \]

Boucksom and H. Chen generalized the above formulae to a general situation by using Okounkov bodies.
(6) (Zariski decomposition for \( n = 1 \)) We assume \( n = 1 \). The Zariski decomposition of \( \overline{D_a} \) exists if and only if \( a_0 + a_1 \geq 1 \). Moreover, the positive part of \( \overline{D_a} \) is given by \((\theta_a H_0 - \theta'_a H_1, p_a)\), where \( H_0 = D = \{ T_0 = 0 \} \), \( H_1 = \{ T_1 = 0 \} \), \( \theta'_a = \inf \Theta_a \), \( \theta_a = \sup \Theta_a \) and

\[
p_a(z_1) = \begin{cases} 
\theta'_a \log |z_1|^2 & \text{if } |z_1| < \sqrt{\frac{a_0 \theta'_a}{a_1(1-\theta'_a)}}, \\
\log(a_0 + a_1|z_1|^2) & \text{if } \sqrt{\frac{a_0 \theta'_a}{a_1(1-\theta'_a)}} \leq |z_1| \leq \sqrt{\frac{a_0 \theta_a}{a_1(1-\theta_a)}}, \\
\theta_a \log |z_1|^2 & \text{if } |z_1| > \sqrt{\frac{a_0 \theta_a}{a_1(1-\theta_a)}}.
\end{cases}
\]
Let $\overline{D}_g = (H_0, g)$ be a big arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $\mathbb{P}^n_{\mathbb{Z}}$. We assume that

$$g(\exp(2\pi \sqrt{-1}\theta_1)z_1, \ldots, \exp(2\pi \sqrt{-1}\theta_n)z_n) = g(z_1, \ldots, z_n)$$

for all $\theta_1, \ldots, \theta_n \in [0, 1]$. We set

$$\xi_g(y_1, \ldots, y_n) := \frac{1}{2}g(\exp(y_1), \ldots, \exp(y_n))$$

for $(y_1, \ldots, y_n) \in \mathbb{R}^n$. Let $\vartheta_g$ be the Legendre transform of $\xi_g$, that is,

$$\vartheta_g(x_1, \ldots, x_n) := \sup\{x_1y_1 + \cdots + x_ny_n - \xi_g(y_1, \ldots, y_n) \mid (y_1, \ldots, y_n) \in \mathbb{R}^n\}$$

for $(x_1, \ldots, x_n) \in \Delta_n.$
Note that if $g = \log(a_0 + a_1|z_1|^2 + \cdots + a_n|z_n|^2)$, then

$$\vartheta_g = \varphi_a = \frac{1}{2} \left( - \sum_{i=0}^{n} x_i \log x_i + \sum_{i=0}^{n} x_i \log a_i \right),$$

where $x_0 = 1 - x_1 - \cdots - x_n$. 
Theorem (Burgos Gil, M., Philippon and Sombra)

There is a Zariski decomposition of $f^*(\overline{D}_g)$ for some birational morphism $f : Y \to \mathbb{P}^n_\mathbb{Z}$ of generically smooth and projective normal arithmetic varieties if and only if

$$\Theta_g := \{(x_1, \ldots, x_n) \in \Delta_n \mid \vartheta_g(x_1, \ldots, x_n) \geq 0\}$$

is a quasi-rational convex polyhedron, that is, there are $\gamma_1, \ldots, \gamma_l \in \mathbb{Q}^n$ and $b_1, \ldots, b_l \in \mathbb{R}$ such that

$$\Theta_g = \{x \in \mathbb{R}^n \mid \langle x, \gamma_i \rangle \geq b_i \ \forall i = 1, \ldots, l\},$$

where $\langle , \rangle$ is the standard inner product of $\mathbb{R}^n$.

The above theorem holds for toric varieties.
For example, if $g = \log \max\{a_0, a_1|z_1|^2, a_2|z_2|^2\}$, then $\overline{D}_g$ is big if and only if $\max\{a_0, a_1, a_2\} > 1$. Moreover,

$$\Theta_g = \left\{ (x_1, x_2) \in \Delta_2 \mid \log \left( \frac{a_1}{a_0} \right) x_1 + \log \left( \frac{a_2}{a_0} \right) x_2 + \log(a_0) \geq 0 \right\}.$$

Thus there is a Zariski decomposition of $f^*(\overline{D}_g)$ for some birational morphism $f : Y \to \mathbb{P}^2_{\mathbb{Z}}$ of generically smooth and projective normal arithmetic varieties if and only if there is $\lambda \in \mathbb{R}_{>0}$ such that

$$\lambda \left( \log \left( \frac{a_1}{a_0} \right), \log \left( \frac{a_2}{a_0} \right) \right) \in \mathbb{Q}^2.$$
Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-divisor of $C^0$-type on $X$.

**Fundamental question**

*Are the following conditions (1) and (2) equivalent?*

1. $\overline{D}$ is pseudo-effective.
2. $\overline{D} + (\widehat{\varphi})_{\mathbb{R}}$ is effective for some $\varphi \in \text{Rat}(X)^{\times}_{\mathbb{R}}$.

Obviously (2) implies (1). Moreover, if $\hat{H}^0(X, a\overline{D}) \neq \{0\}$ for some $a \in \mathbb{R}_{>0}$, then (2) holds. Indeed, as we can choose $\phi \in \text{Rat}(X)^{\times}$ with $a\overline{D} + (\widehat{\phi}) \geq 0$, we have $\phi^{1/a} \in \text{Rat}(X)^{\times}_{\mathbb{R}}$ and $\overline{D} + (\widehat{\phi^{1/a}})_{\mathbb{R}} \geq 0$.

As we remarked, it is nothing more than Dirichlet's unit theorem in the case where $d = 1$. Moreover, in the geometric case, (1) does not necessarily imply (2).
We have partial answers to the above question.

**Theorem**

*If $\overline{D}$ is pseudo-effective and $D$ is numerically trivial on $X_\mathbb{Q}$, then there exists $\varphi \in \text{Rat}(X) \times \mathbb{R}$ such that $\overline{D} + (\widehat{\varphi})_\mathbb{R}$ is effective.*

This theorem is a consequence of arithmetic Hodge index theorem and a kind of compactness theorem.
Thank you for your attention.