Rigidity of Graphs and Frameworks

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Bar-and-Joint Frameworks

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- We consider the framework to be a straight line **realization** of G in $\mathbb{R}^d$ in which the *length* of an edge $uv \in E$ is given by the Euclidean distance $\|p(u) - p(v)\|$ between the points $p(u)$ and $p(v)$.

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A framework (G, p) is:

- **globally rigid** if every framework which is equivalent to (G, p) is congruent to (G, p);
- **rigid** if there exists an $\epsilon > 0$ such that every framework (G, q) which is equivalent to (G, p) and satisfies $\|p(v) - q(v)\| < \epsilon$ for all $v \in V$, is congruent to (G, p). (This is equivalent to saying that every continuous motion of the vertices of (G, p) which preserves the lengths of all edges of (G, p), also preserves the distances between all pairs of vertices of (G, p).)

Example

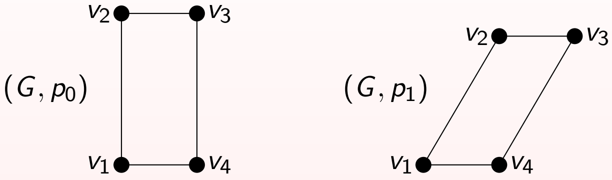

Figure : A 2-dimensional example. The framework (G, p_1) can be obtained from (G, p_0) by a continuous motion which preserves all edge lengths, but changes the distance between v_1 and v_3. Thus (G, p_0) is not rigid.

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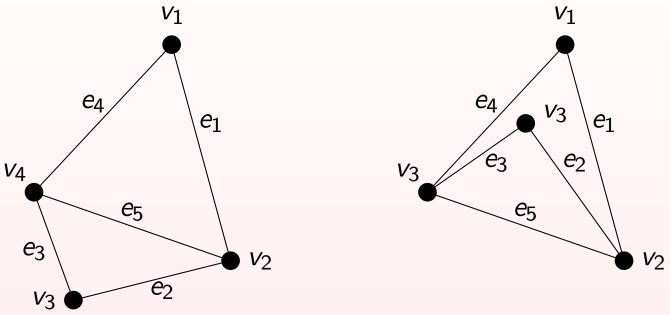

Figure : A rigid 2-dimensional framework which is not globally rigid. All edges in both frameworks have the same length, but the distance from v_1 to v_3 is different.

Complexity

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- It is NP-hard to determine whether a given d-dimensional framework (G, p) is globally rigid for $d \geq 1$ (J. B. Saxe), or rigid for $d \geq 2$ (Abbot).
- These problems becomes more tractable if we restrict attention to *'generic'* frameworks (those for which the set of coordinates of all points $p(v)$, $v \in V$, is algebraically independent over $\mathbb{Q}$).

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The entries in the row corresponding to an edge $e \in E$ and columns corresponding to a vertex $u \in V$ are given by the vector $p(u) - p(v)$ if $e = uv$ is incident to u and is the zero vector if e is not incident to u.

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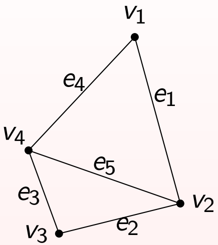
The rigidity matrix is the Jacobean matrix of the **rigidity map** $f_G : \mathbb{R}^{dn} \to \mathbb{R}^m$ defined by

$$f_G(p) = (\ell_p(e_1), \ell_p(e_2), \ldots, \ell_p(e_m))$$

where $\ell_p(e_i)$ is the squared length of edge e_i in (G, p).

Rigidity matrix: Example



$$\begin{pmatrix} p(v_1)-p(v_2) & p(v_2)-p(v_1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & p(v_2)-p(v_3) & p(v_3)-p(v_2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & p(v_3)-p(v_4) & p(v_4)-p(v_3) \\ p(v_1)-p(v_4) & \mathbf{0} & \mathbf{0} & p(v_4)-p(v_1) \\ \mathbf{0} & p(v_2)-p(v_4) & \mathbf{0} & p(v_4)-p(v_2) \end{pmatrix}$$

The Rigidity Matrix: Theorem

Theorem [Asimow and Roth, 1979]

Let (G, p) be a d-dimensional framework with $n \geq d + 1$ vertices. Then:

- rank $R(G, p) \leq nd - \binom{d+1}{2}$.
- If rank $R(G, p) = nd - \binom{d+1}{2}$ then (G, p) is rigid.
- When (G, p) is generic, (G, p) is rigid if and only if

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It follows that the rigidity of a generic framework (G, p) depends only on the graph G.

Independent graphs

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- If we can determine when G is independent in $\mathbb{R}^d$ then we can decide if G is rigid.
- A necessary condition for independence in $\mathbb{R}^d$ is that

$$i(X) \leq d|X| - \binom{d+1}{2}$$

for all $X \subseteq V$ with $|X| \geq d+1$ (where $i(X)$ denotes the number of edges of G joining vertices in X.)

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- This necessary condition is sufficient to imply independence when $d = 1$ and when $d = 2$ (Laman 1970). It is not sufficient when $d \geq 3$.

The Stress Matrix

A *stress* for a framework (G, p) is a map $w : E \to \mathbb{R}^m$ such that, for all $v \in V$,

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The associated **stress matrix** $S(G, p, w)$ is the $n \times n$ matrix with rows and columns indexed by V in which the entry corresponding to an edge $uv \in E$ is w_e, all other off-diagonal entries are zero, and the diagonal entries are chosen to give zero row and column sums.

The Stress Matrix: Theorem

Theorem [Connelly (2005); Gortler, Healy and Thurston (2010)]

Let (G, p) be a generic d-dimensional framework with $n \geq d + 1$ vertices. Then

- rank $S(G, p, w) \leq n - d - 1$ for all stresses w for (G, p).
- (G, p) is globally rigid if and only if (G, p) has a stress w such that rank $S(G, p, w) = n - d - 1$.

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This implies that the global rigidity of a generic framework (G, p) depends only on the graph G.

Theorem [Hendrickson (1992)]

If G is globally rigid in $\mathbb{R}^d$ and $n \geq d+1$ then G is $d+1$-connected and redundantly rigid i.e. $G-e$ is rigid for all $e \in E$.

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These necessary conditions for global rigidity are sufficient when $d = 1$ and when $d = 2$ (Connelly, 2005; Jackson and Jordán, 2005). They are not sufficient for $d \geq 3$.

Point-Line Frameworks

Let $G = (P \cup L, E)$ be a graph with two types of vertices representing points and lines in $\mathbb{R}^2$. A **point-line framework** is defined by a map $p : P \cup L \to \mathbb{R}^2$, where $p(v)$ gives the coordinates of v for $v \in P$ and $p(l)$ gives the cartesian equation for l when $l \in L$.

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Problem [John Owen]

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Problem [John Owen]

Determine when a generic point-line framework is rigid.

Two necessary conditions for generic independence are that:
$i(X) \leq 2|X| - 3$ for all $X \subseteq P \cup L$ with $|X| \geq 2$;
$i(X) \leq |X| - 1$ for all $X \subseteq L$ with $|X| \geq 1$.
These conditions are not sufficient.

Scaler-Product Rigidity

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Two necessary conditions for generic independence of G are that:

- $i(X) \leq d|X| - \binom{d}{2}$ for all $X \subseteq V$ with $|X| \geq d$;
- $|E(H)| \leq d|V(H)| - d^2$ for all bipartite subgraphs $H \subseteq G$ with at least d vertices on each side of their bipartition.

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The necessary conditions for generic independence are not sufficient when $d \geq 2$.