Incidentor coloring: methods and results

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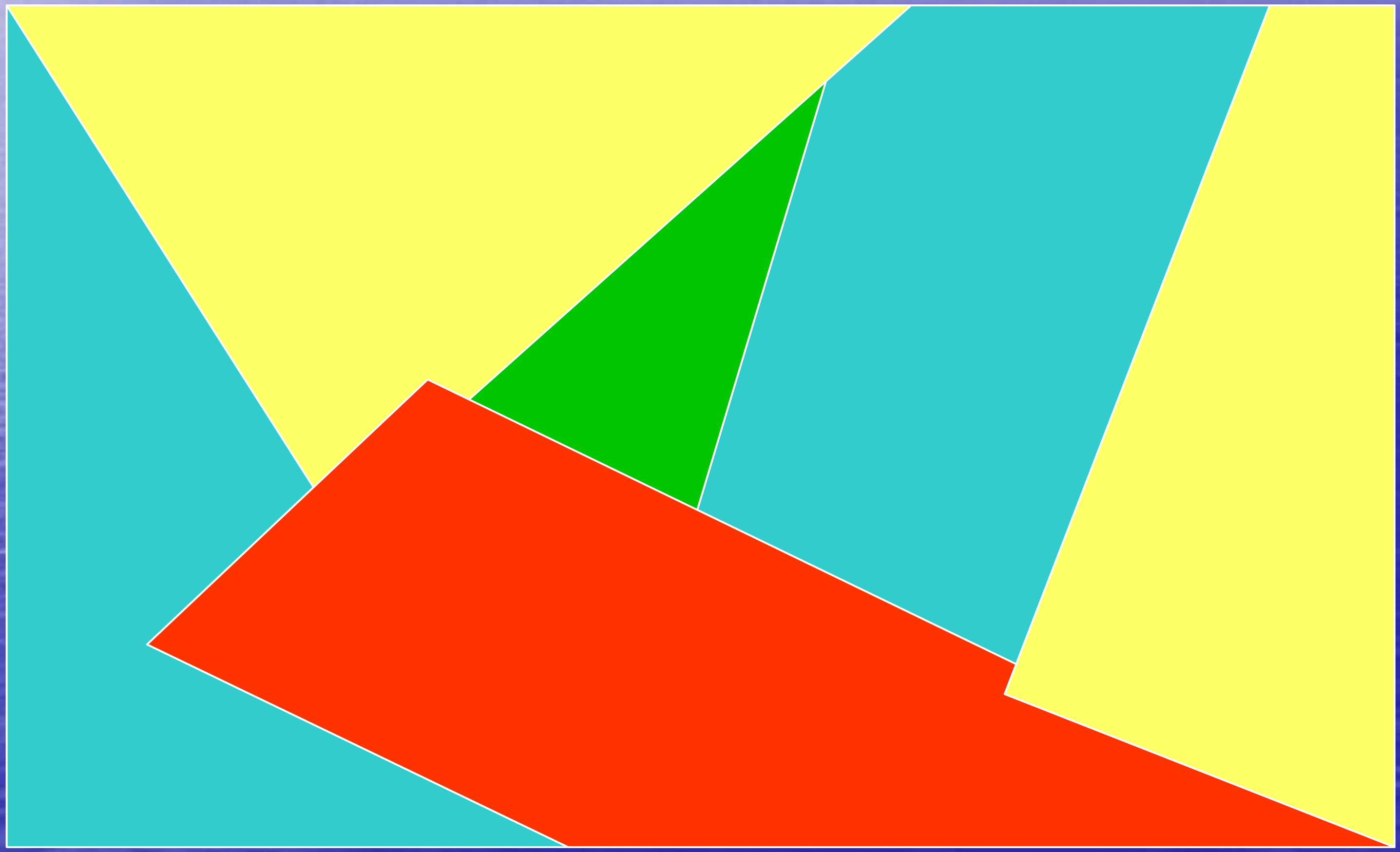
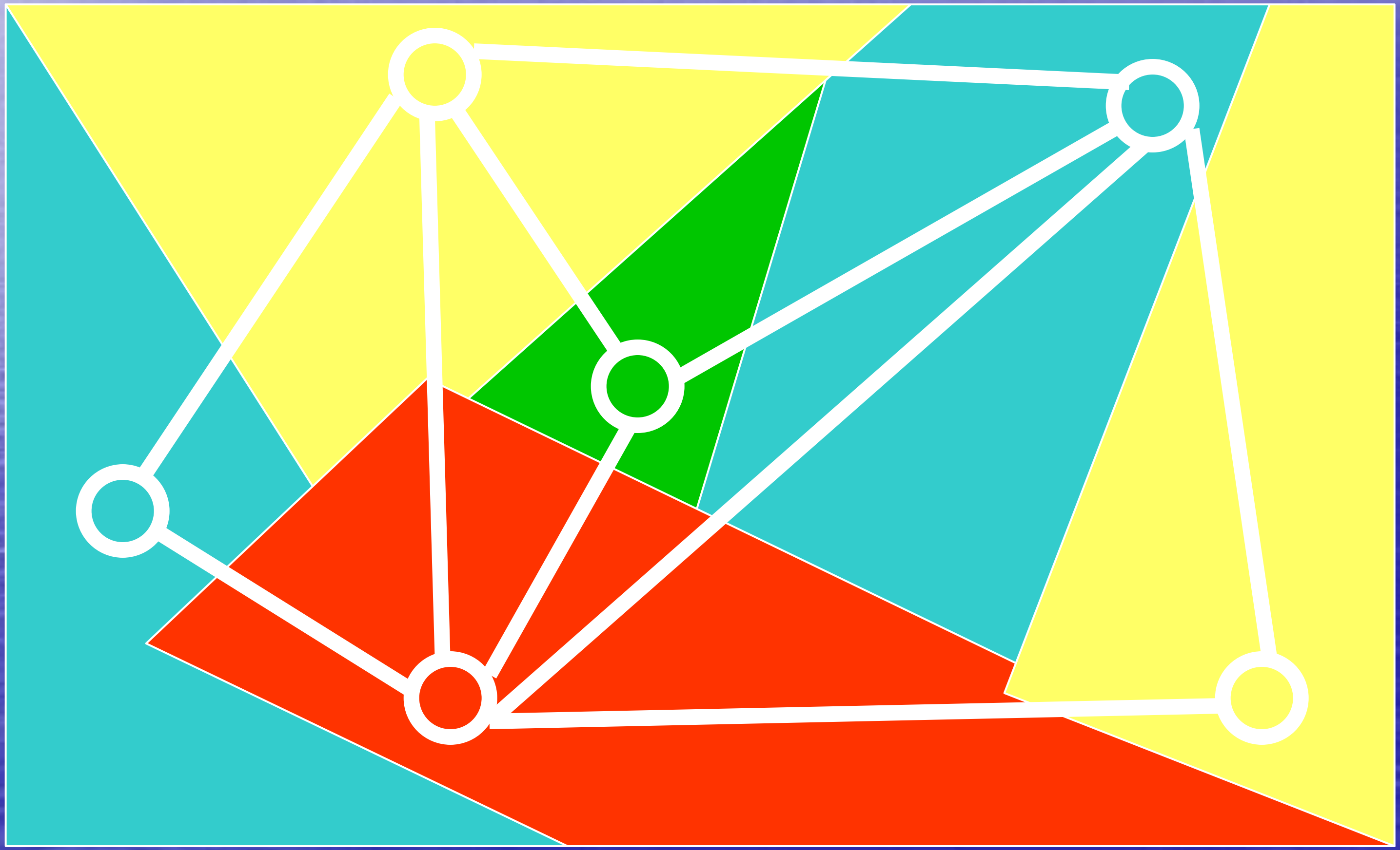
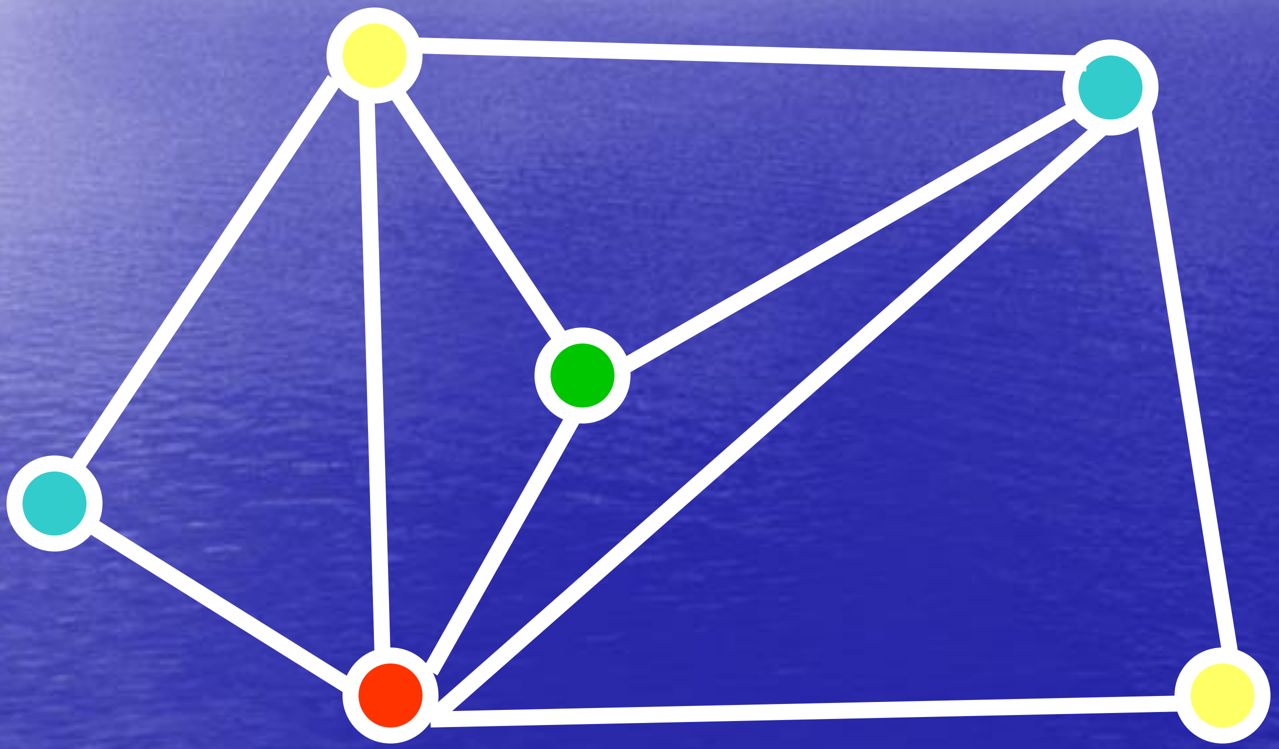
4-color problem



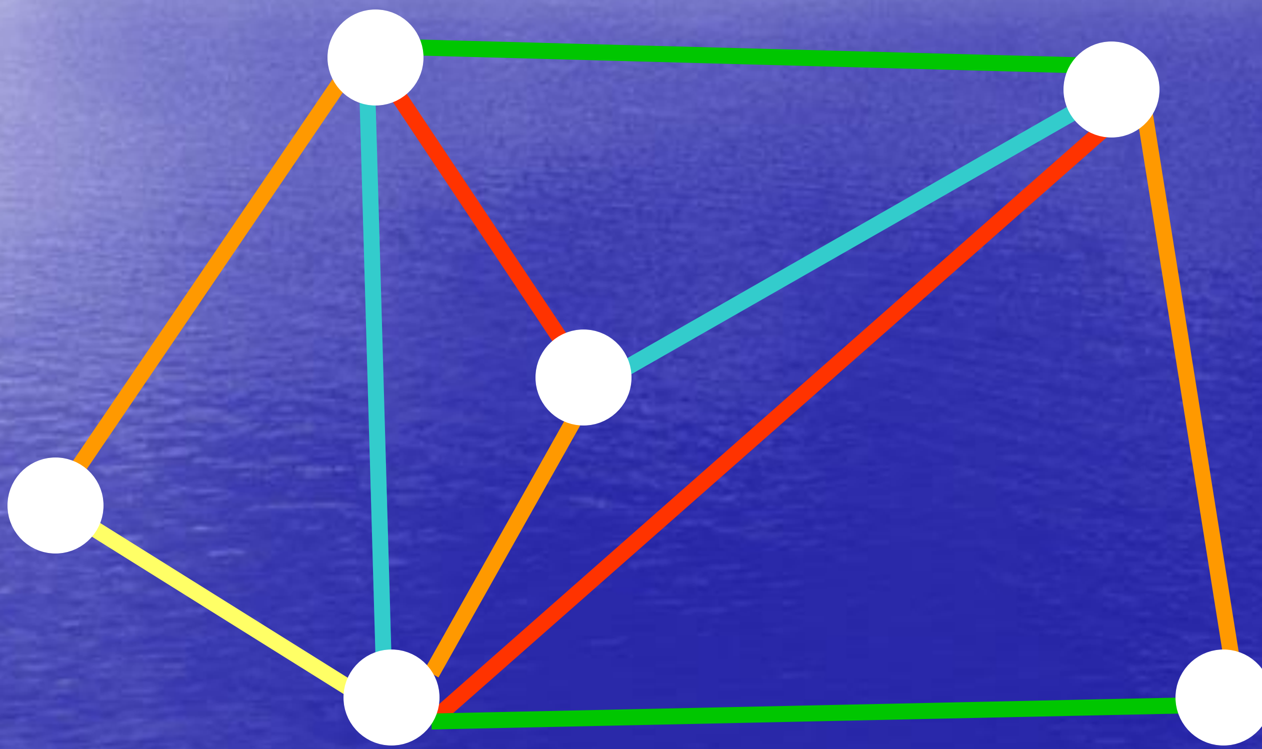
Reduction to the vertex coloring



Vertex coloring problem



Edge coloring problem



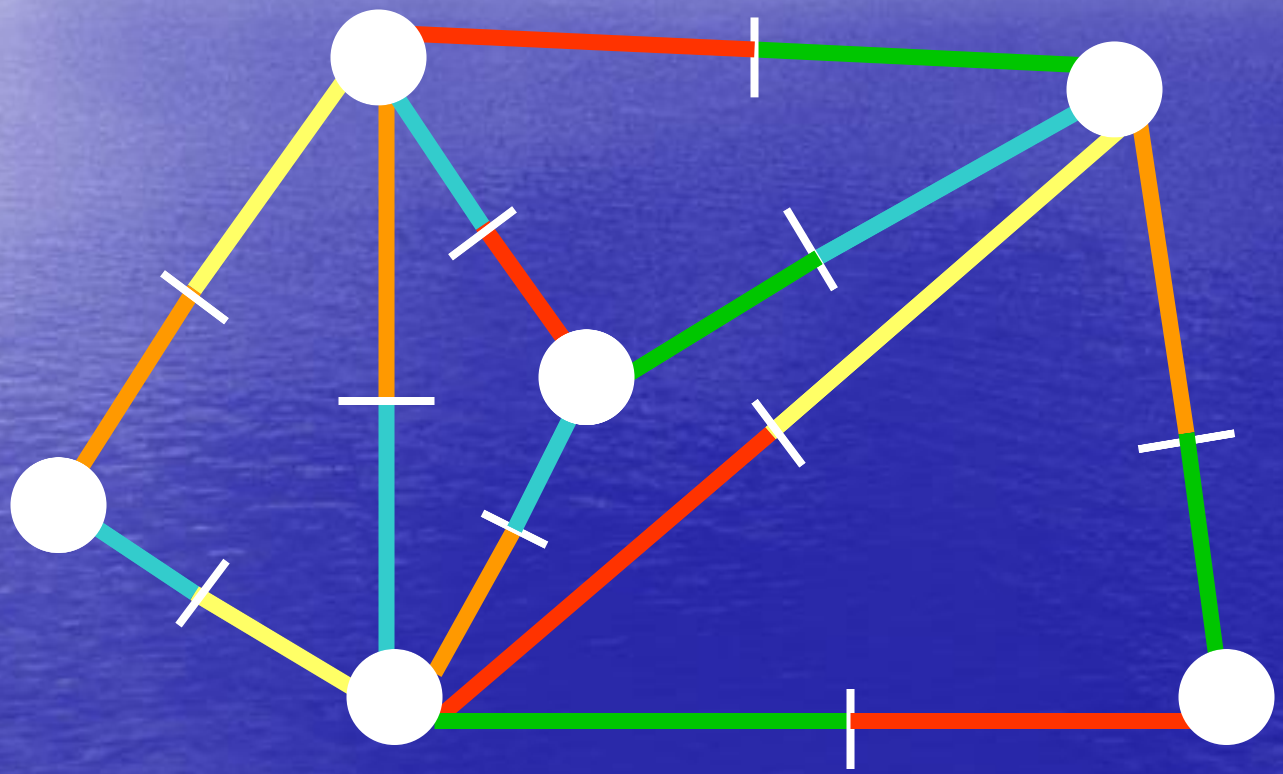
Incidentor coloring

- *Incidentor* is a pair (v,e) of a vertex v and an arc (edge) e, incident with it.
- It is a half of an arc (edge) adjoining to a given vertex.
- 
- initial final
- (mated incidentors)

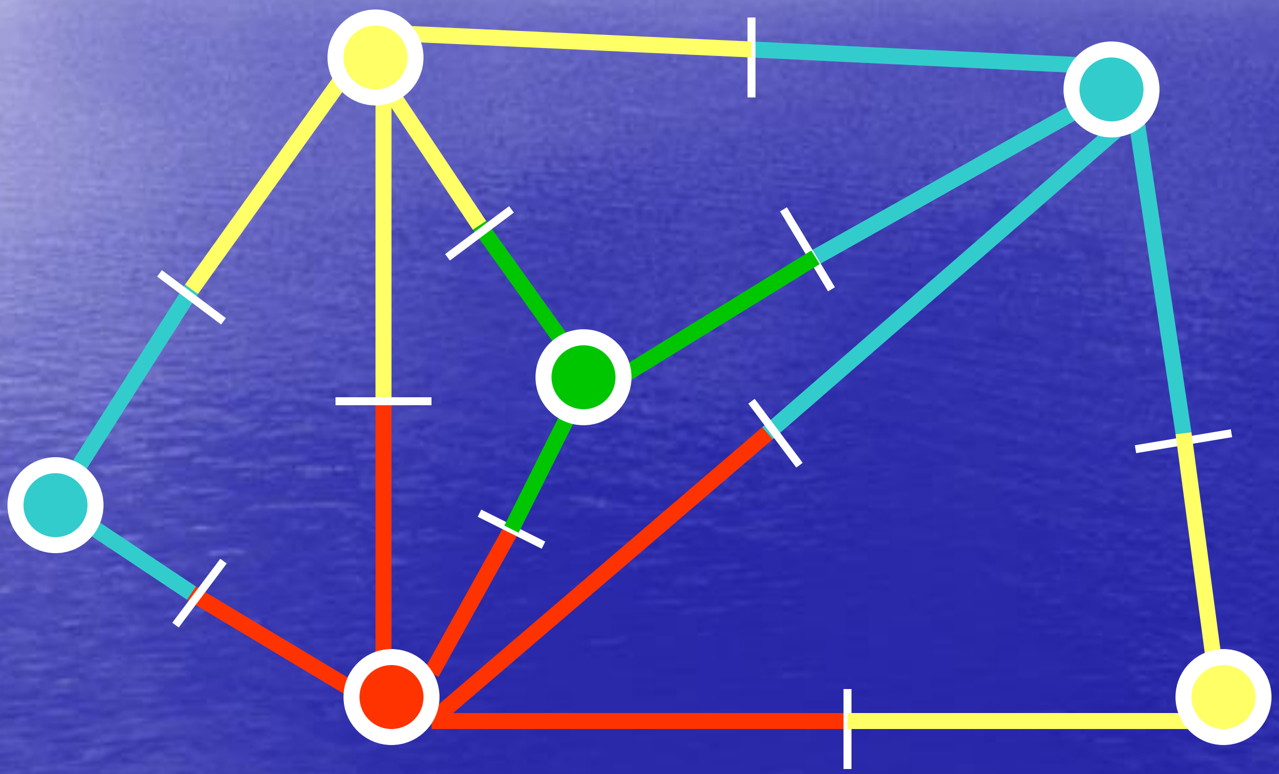
Incidentor coloring problem

- Color all incidentors of a given multigraph by the minimum number of colors in such a way that the given restrictions on the colors of *adjacent* (having a joint vertex) and *mated* (having a joint arc) incidentors would be stisfied

An example of incidentor coloring



Incidentor coloring generalizes both vertex and edge coloring



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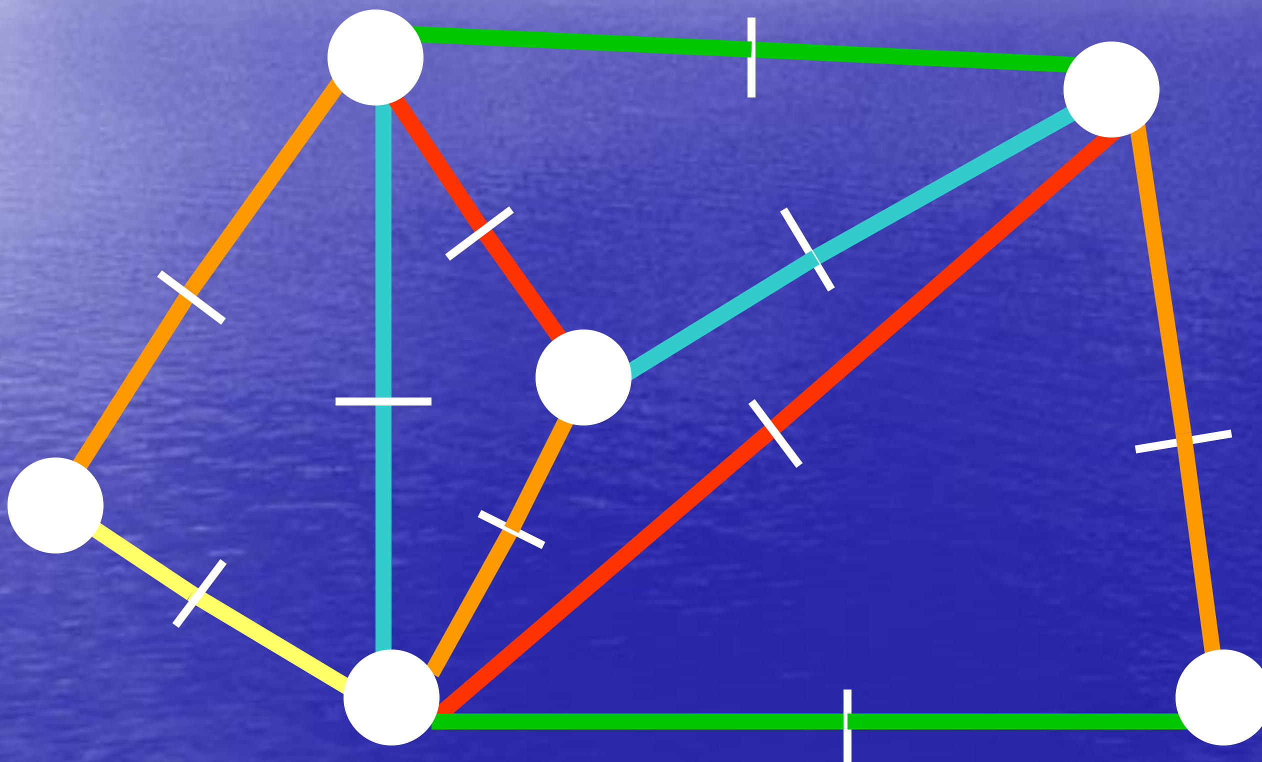
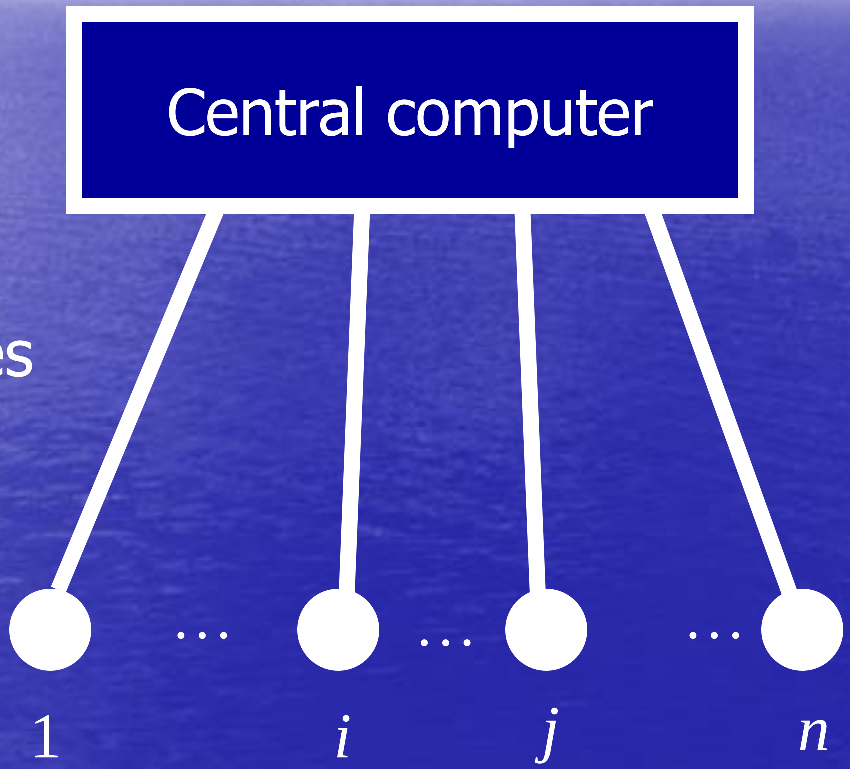


Motivation



All links capacities are equal to 1

i-th object must send to j-th one d_{ij} units of information

There are two ways of information transmission:

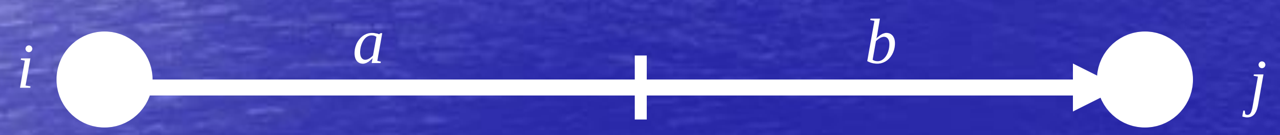
- 1) Directly from i-th object to j-th one (during one time unit);
- 2) With memorizing in the central computer (receive the message from i-th object, memorize it, and transmit to j-th one later).

Reduction to the incidentor coloring

- Each object corresponds to a vertex of the multigraph (n vertices).
- Each unit of information to transmit from i-th object to j-th one corresponds to the arc ij of the multigraph (there are d_{ij} arcs going from a vertex i to the a vertex j).
- The maximum degree Δ equals the maximum load of the link.

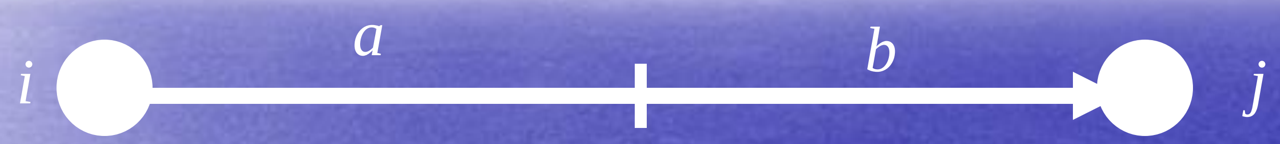
Scheduling

- To each information unit two time moments should be assigned – when it goes via *i*-th and *j*-th links.



- These moments could be interpreted as the colors of the incidentors of the arc *ij*.

Restrictions



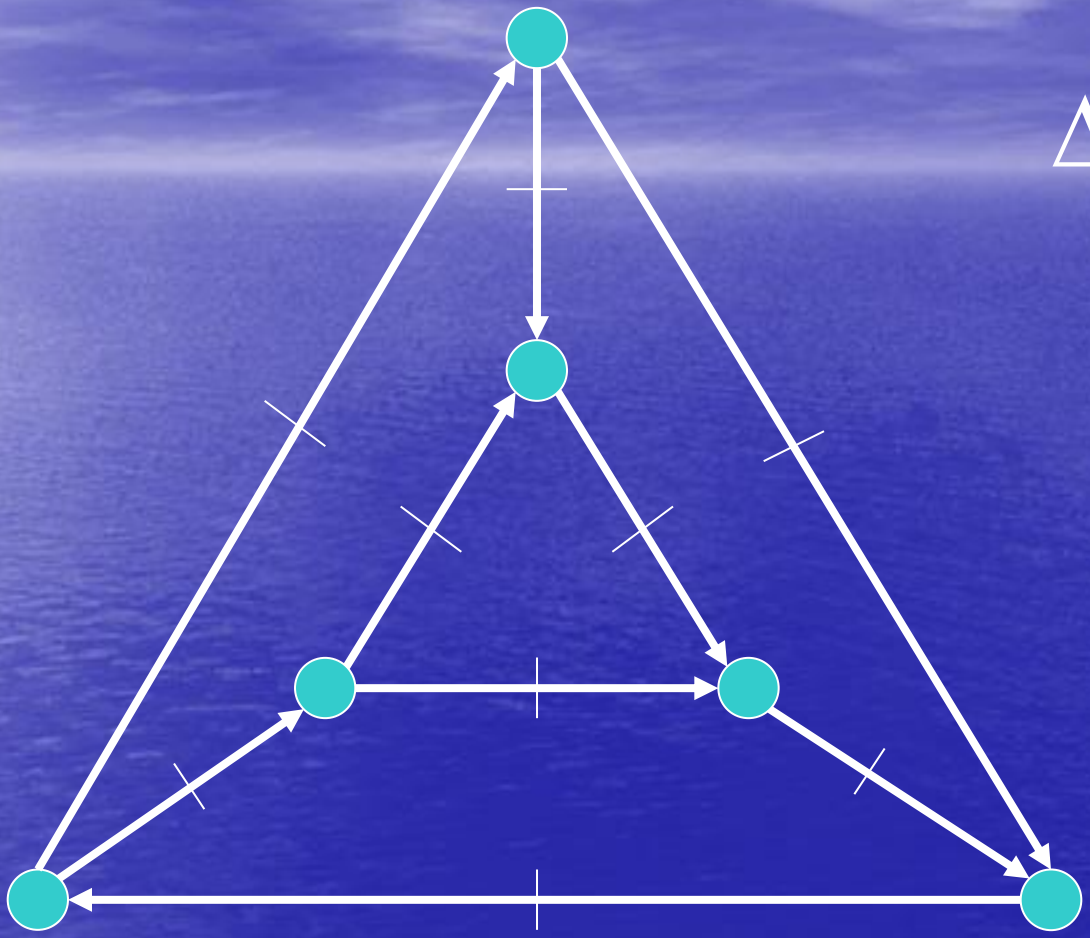
- The colors of adjacent incidentors must be distinct.
- For every arc, the color of its initial incidentor is at most the color of the final incidentor, i.e. $a \leq b$.

- It is required to color all incidentors by the minimum number of colors χ satisfying all the restrictions (the length of the schedule is χ).

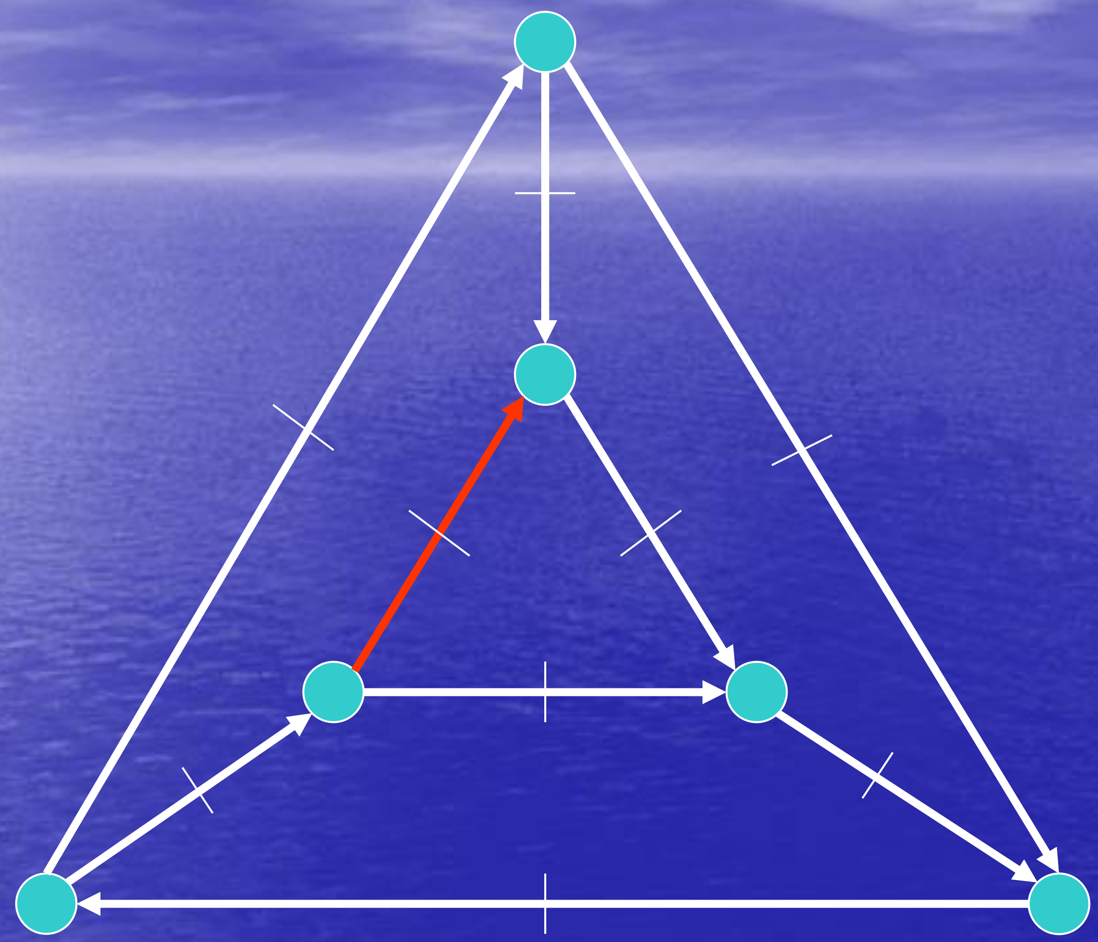
- For this problem $\chi = \Delta$. Such coloring can be found in $O(n^2\Delta^2)$ time.
- (P., 1995)

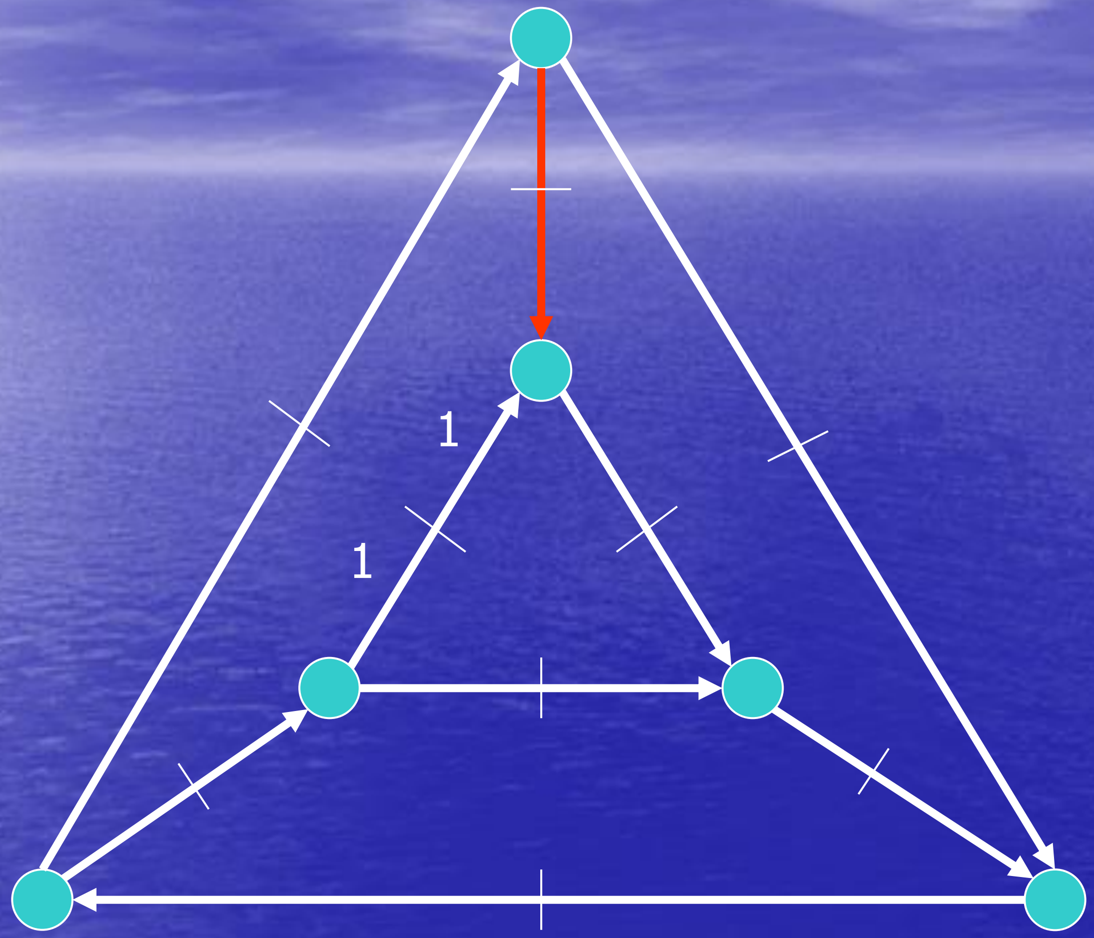
Sketch of the algorithm

- Consider an arc that is not colored yet
- Try to color its incidentors:
- 1. In such a way that $a=b$
- 2. In such a way that $a<b$
- Otherwise, modify the coloring (consider bicolored chains)



$\Delta=3$

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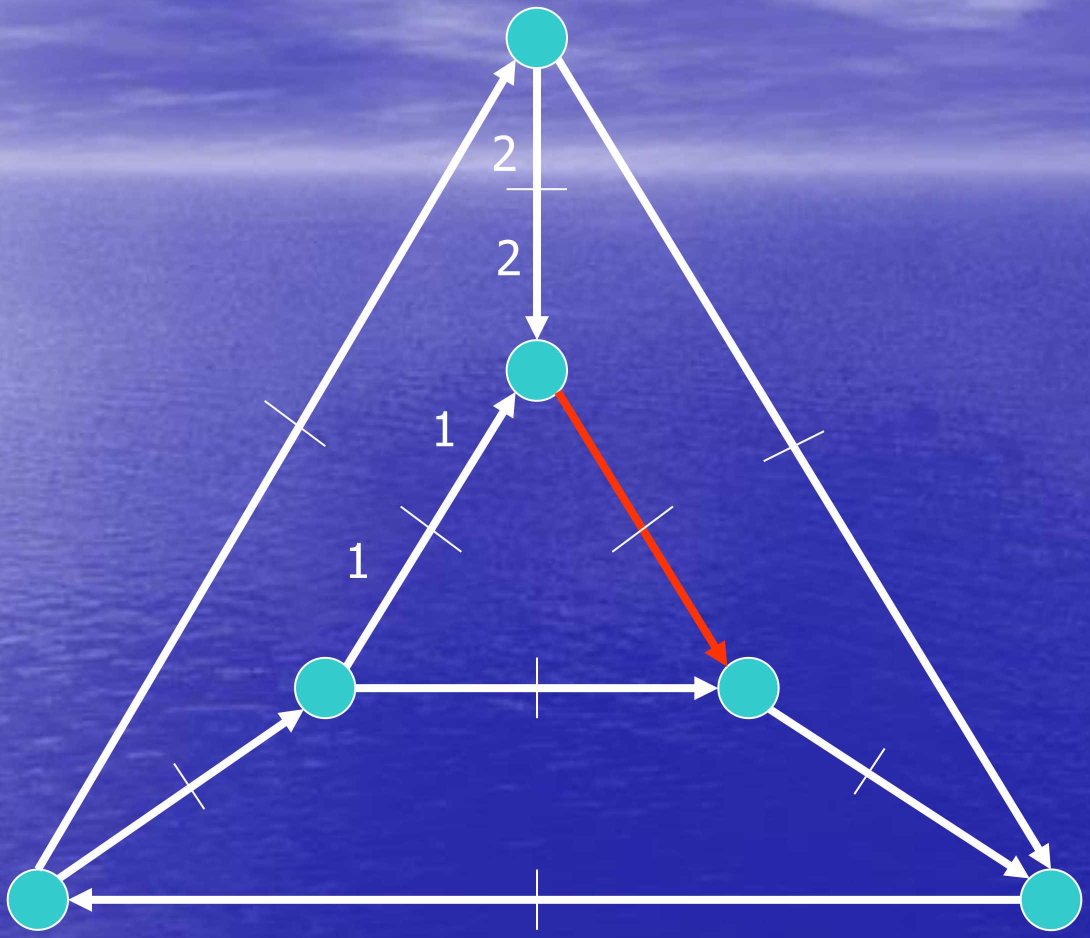
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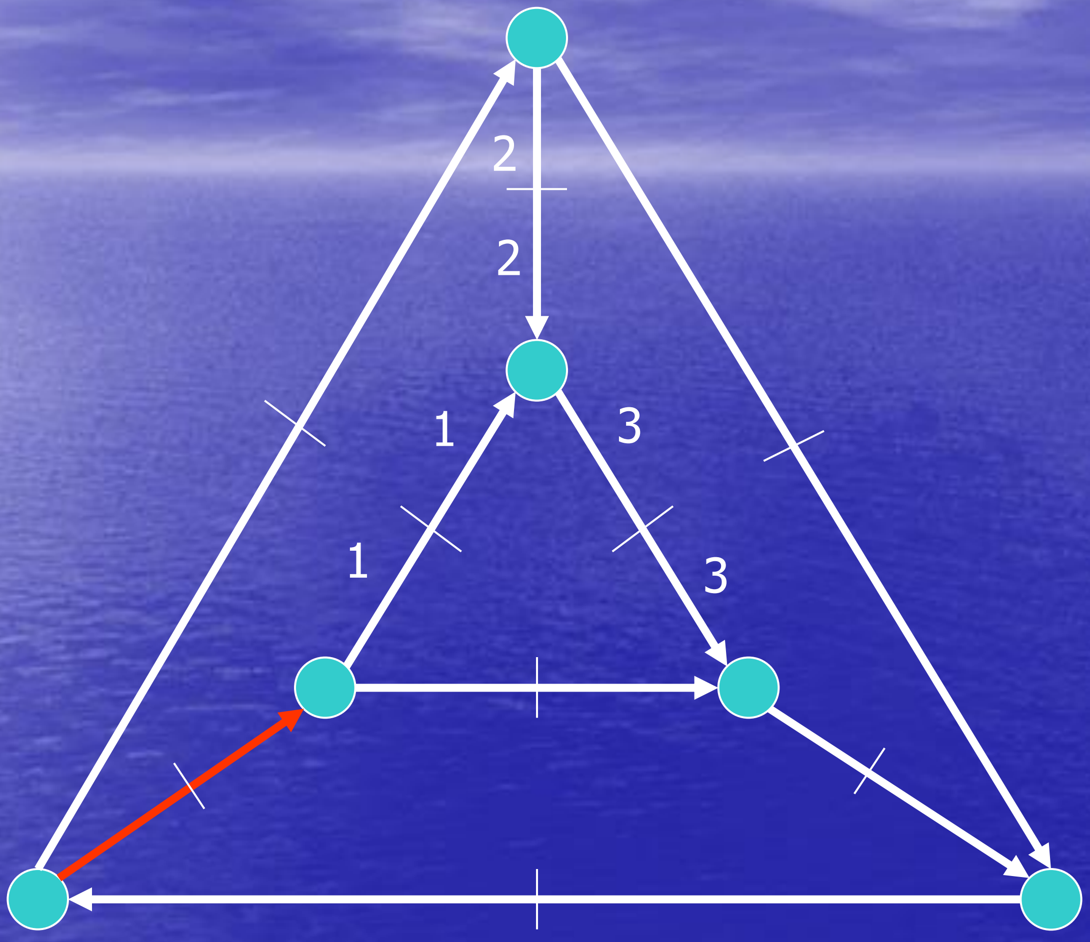
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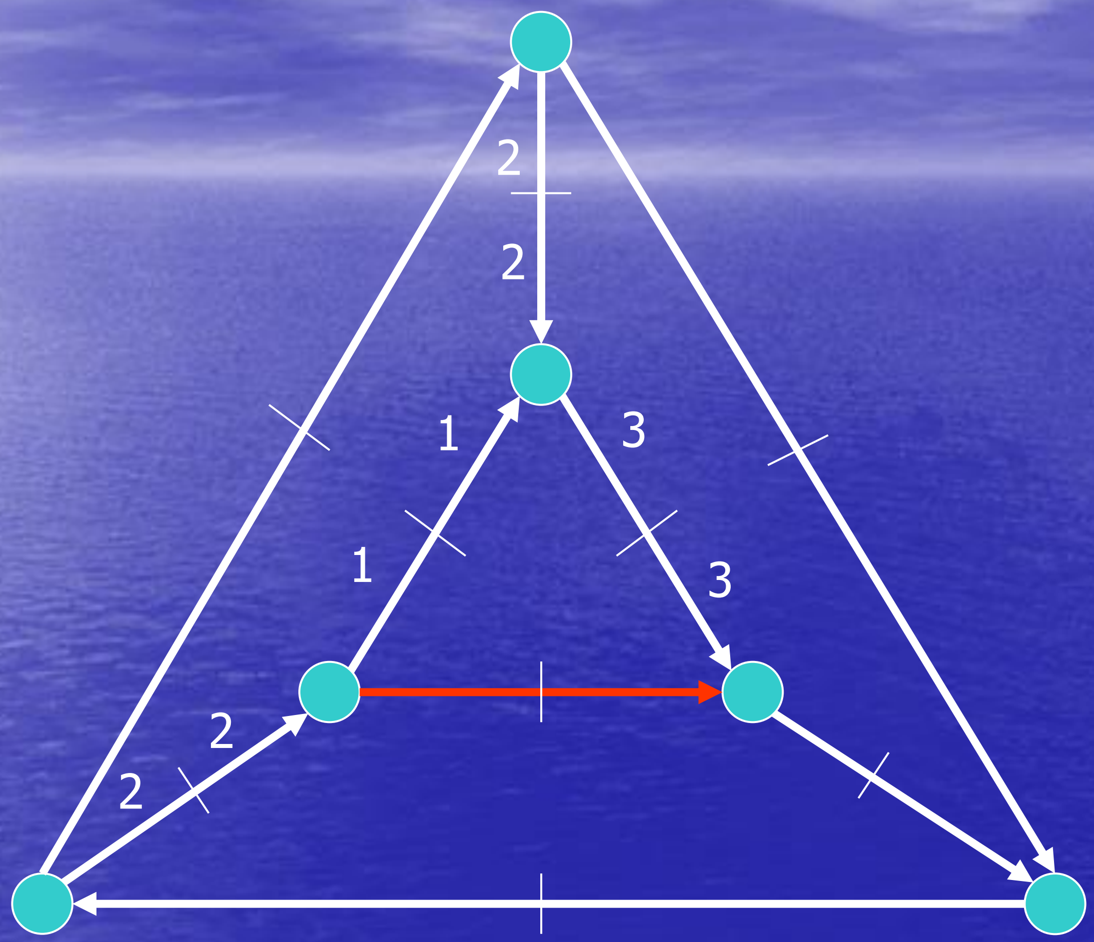
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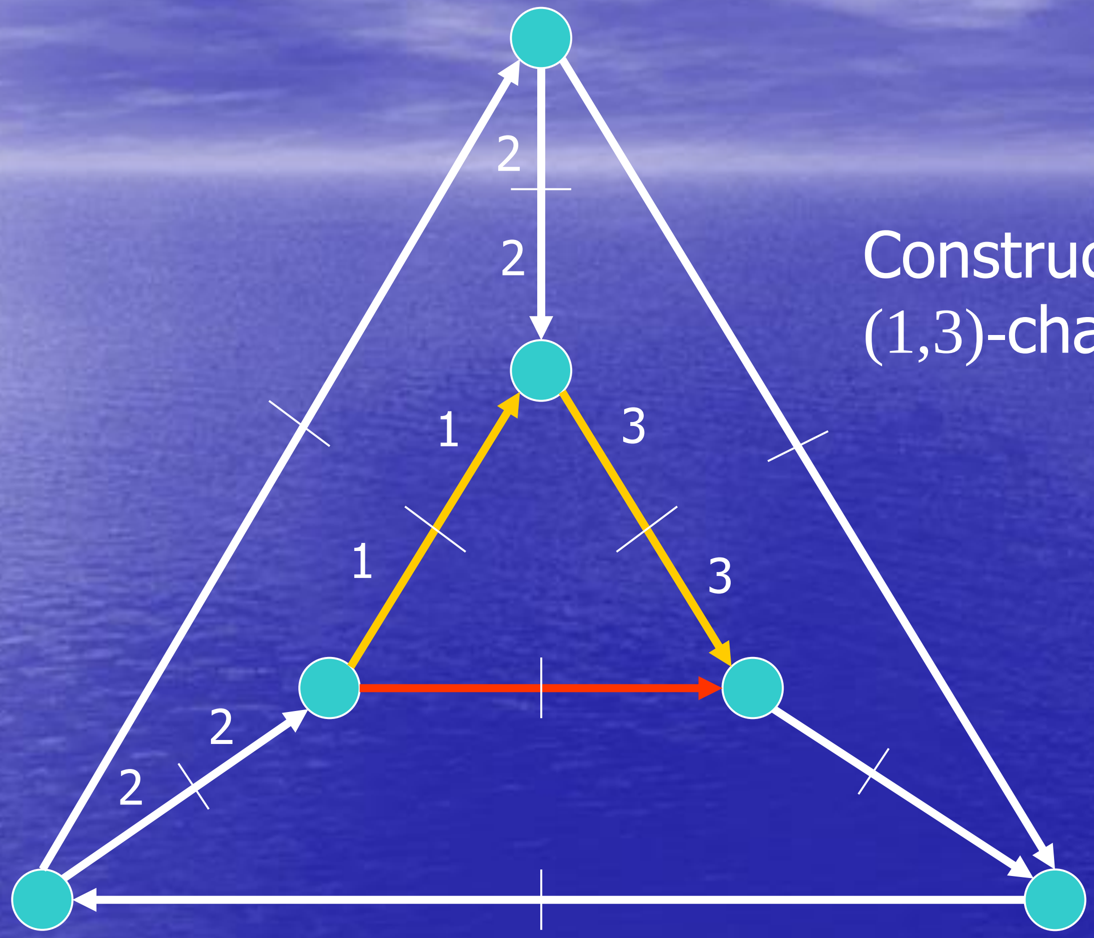
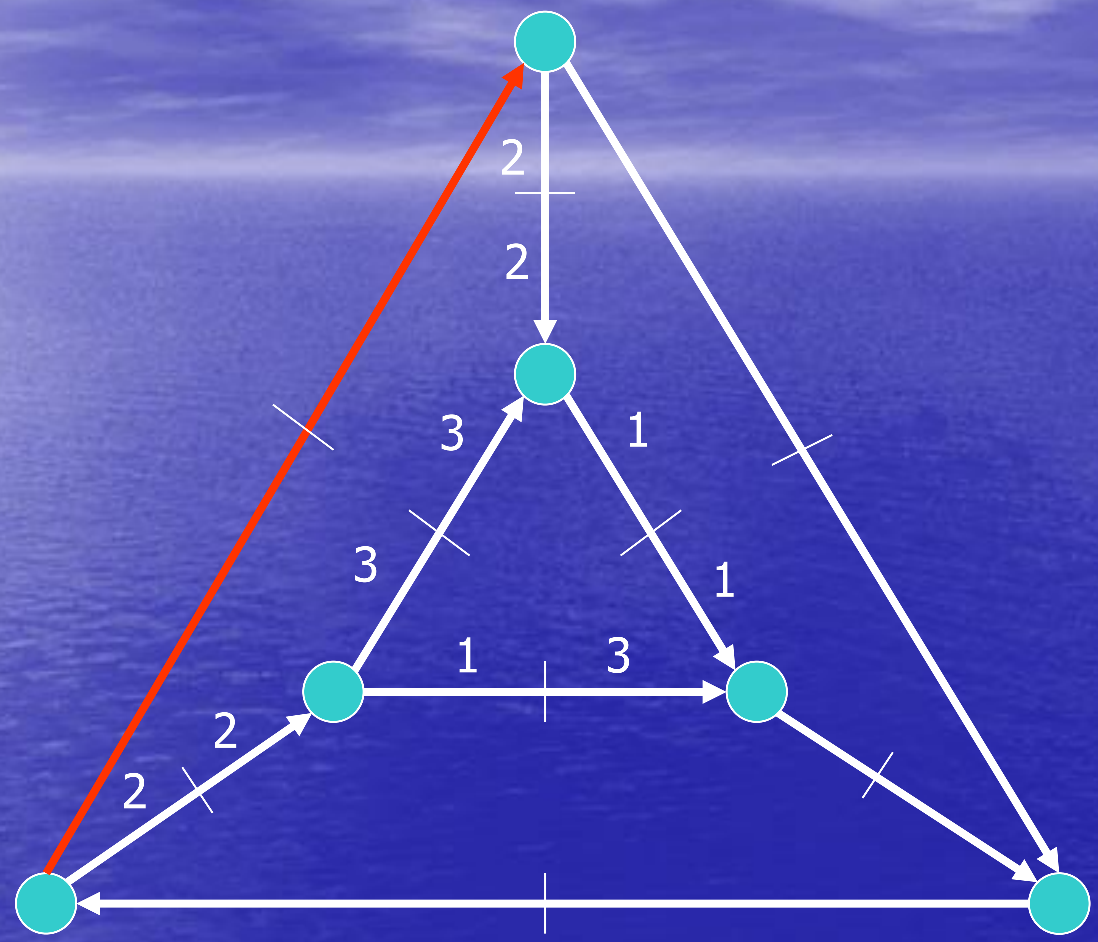


Construct a (1,3)-chain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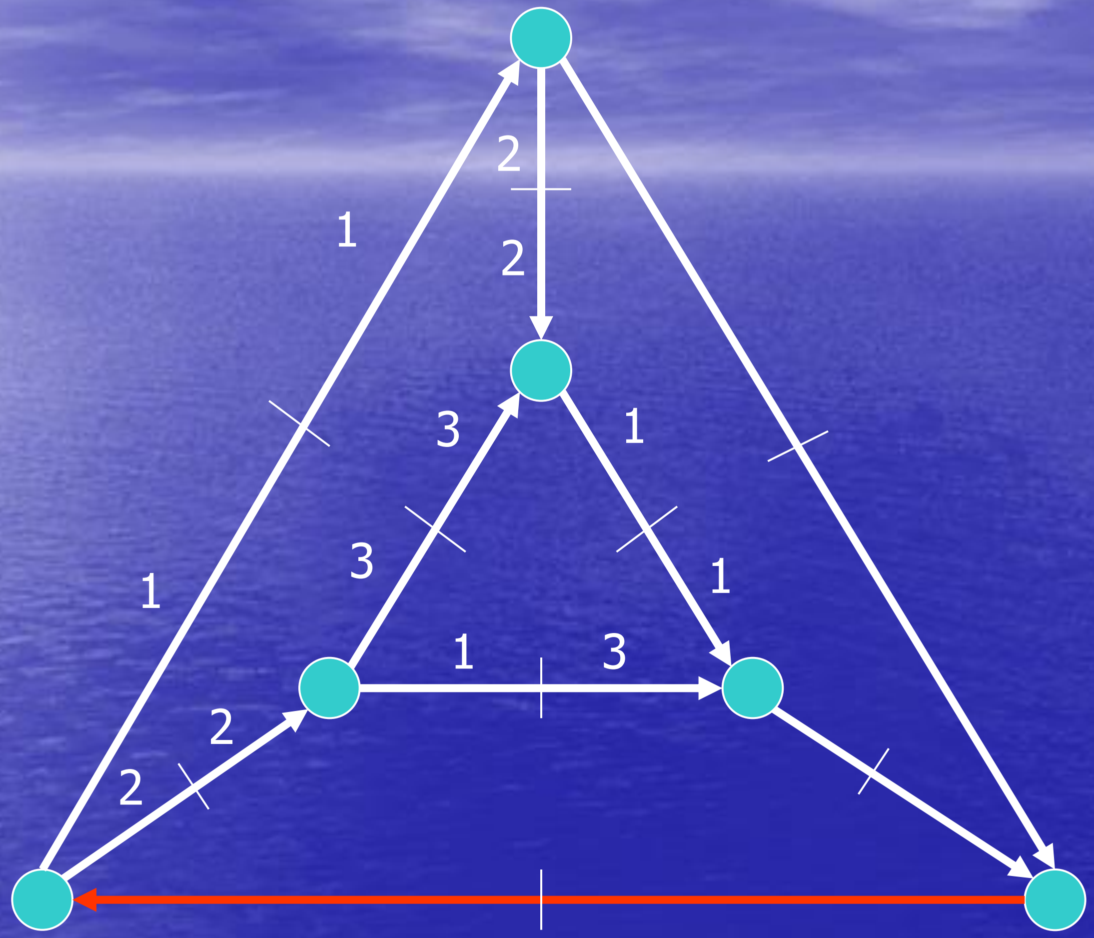

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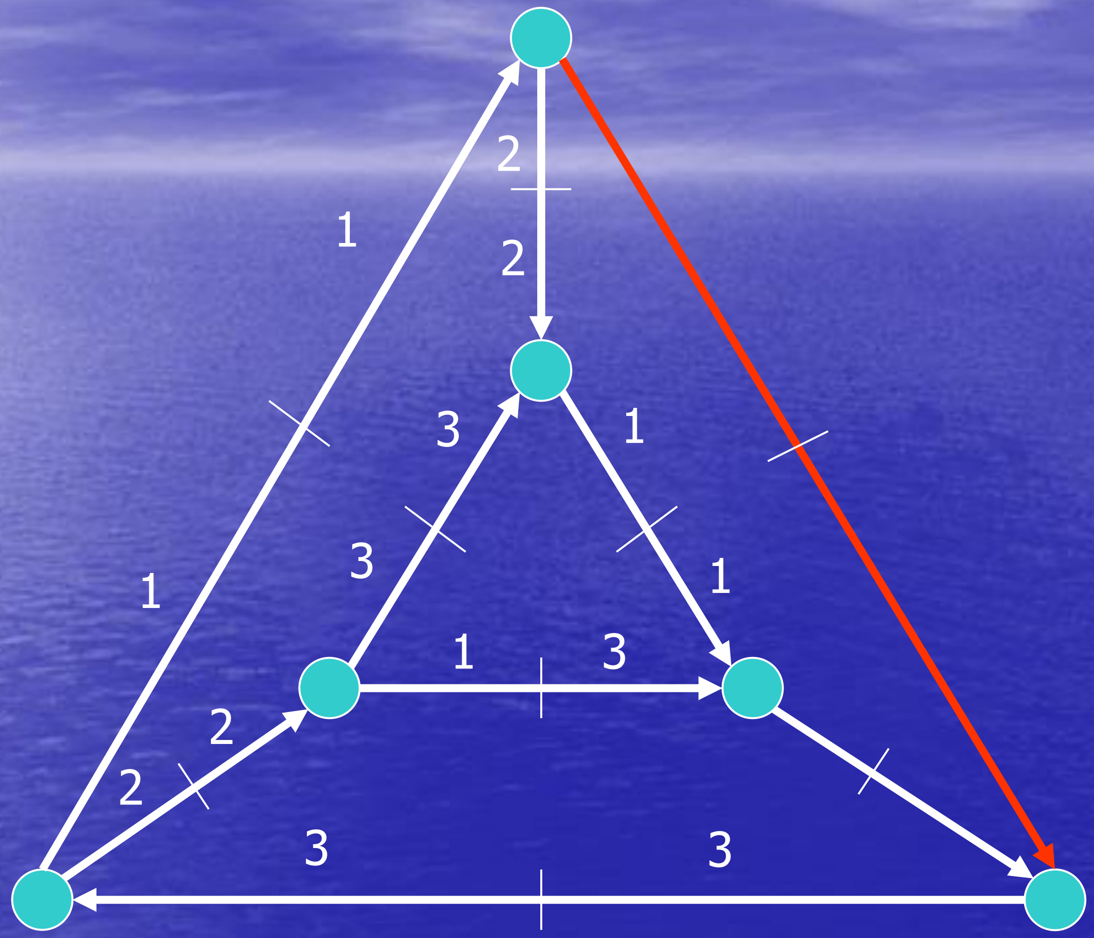
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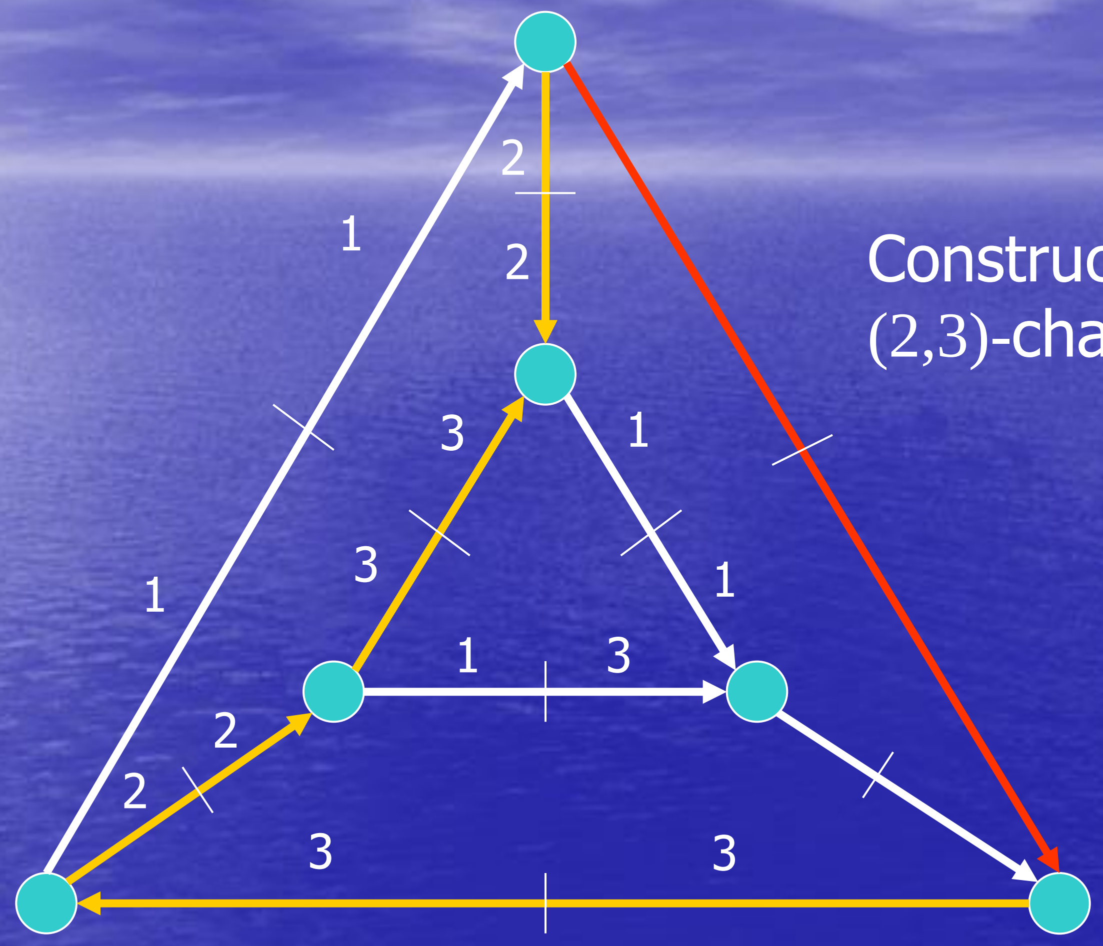
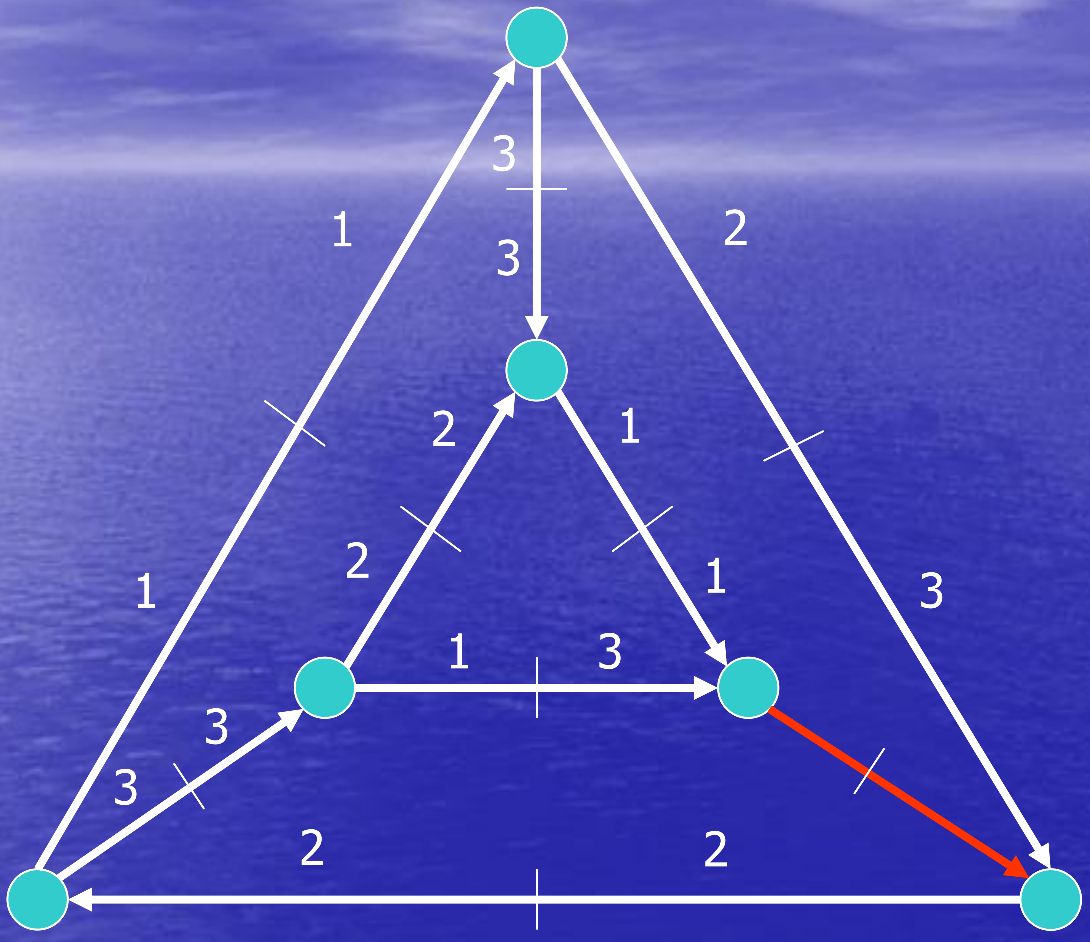
Construct a (2,3)-chain

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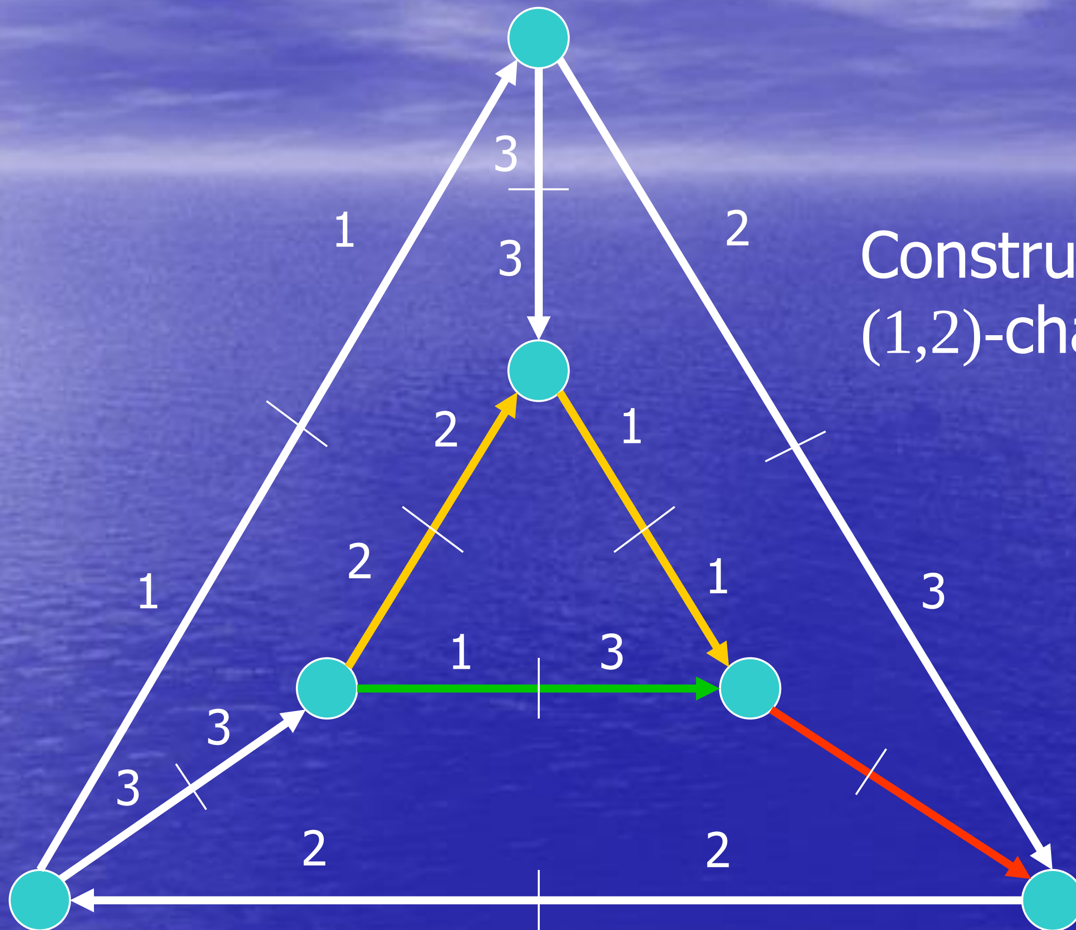
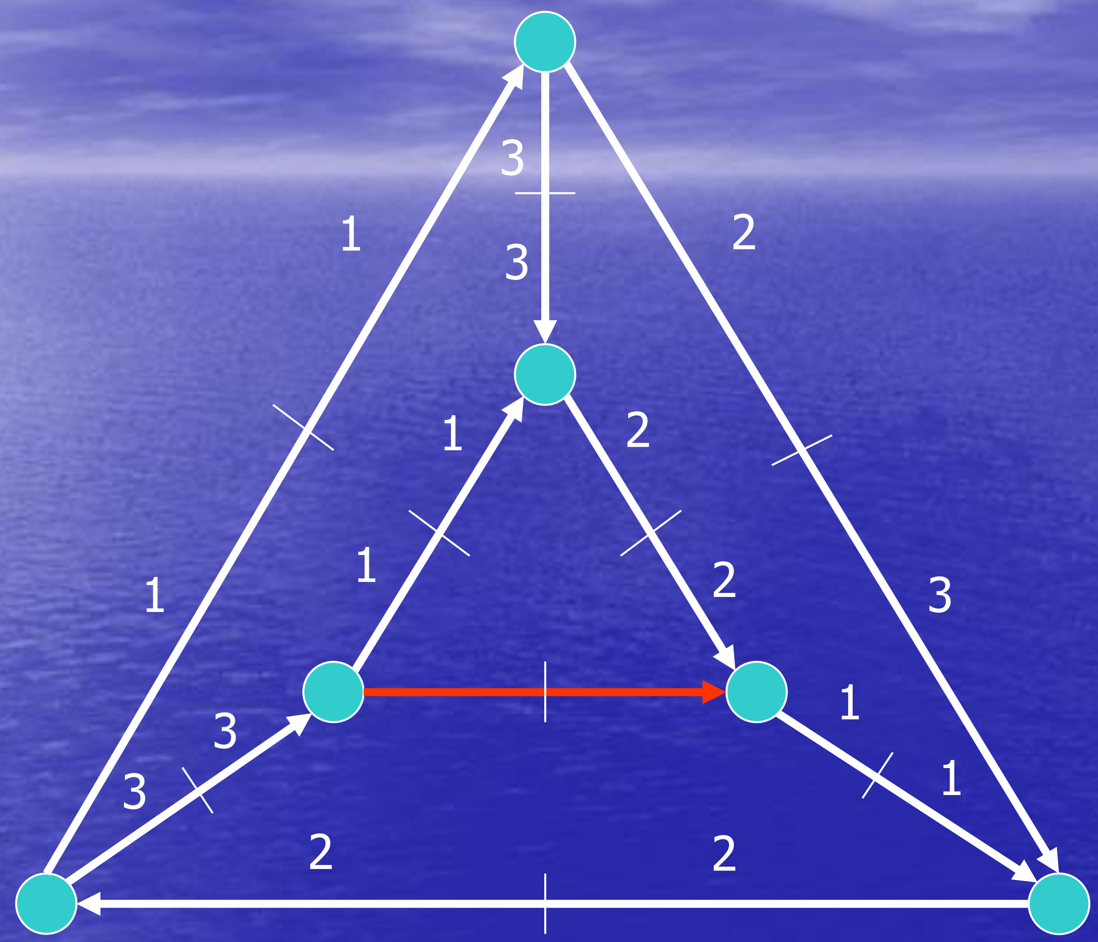
Construct a (1,2)-chain





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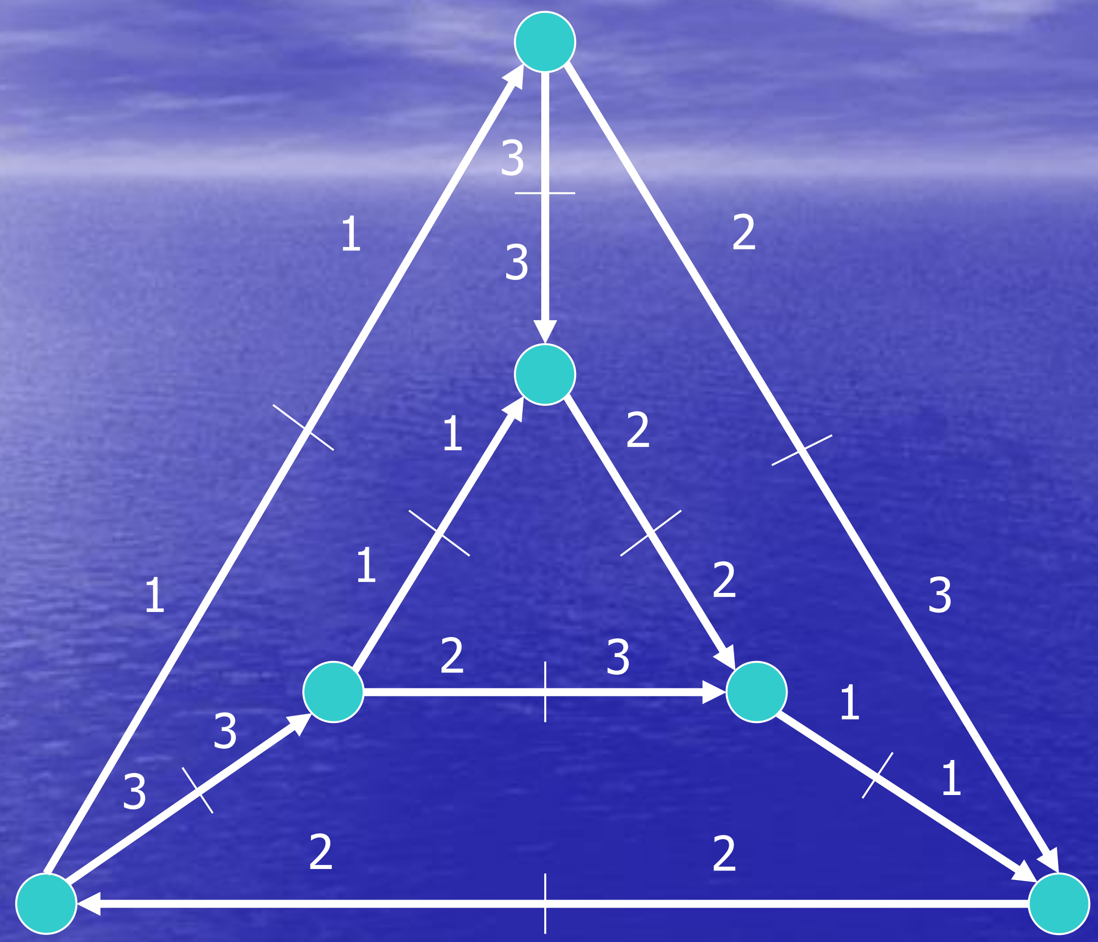
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Further investigations

- 1) Modifications of initial problem
- 2) Investigation of the incidentor coloring itself

Modifications of initial problem

- 1) Arbitrary capacities
- 2) Two sessions of message transmission
- 3) Memory restrictions
- 4) Problem of Melnikov & Vizing
- 5) Bilevel network

Memory restriction

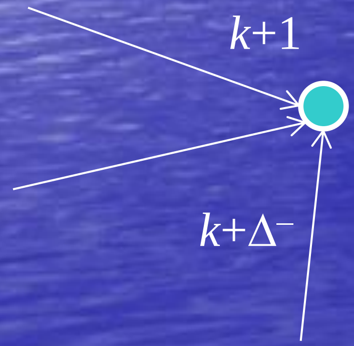
- The memory of the central computer is at most Q
- If $Q=0$ then second way of transmission is impossible and we have the edge coloring problem

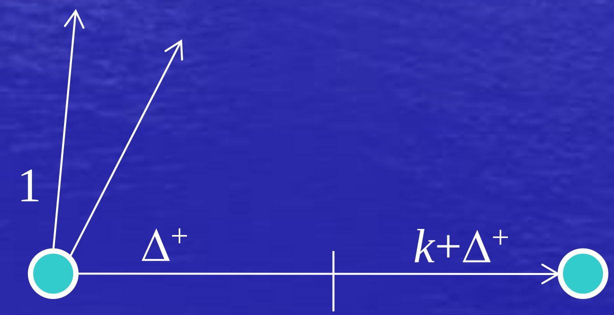
- If $Q \geq n$ then we can store each message in the central computer during 1 unit of time. Incidentor coloring problem with the following restriction on mated incidentors colors appears:
- $b - 1 \leq a \leq b$
- In this case $\chi = \Delta$
- (Melnikov, Vizing, P.; 2000).

(k,l)-coloring of incidentors

- Let $0 \le k \le l \le \infty$. Restrictions:
- 1) adjacent incidentors have distinct colors;
- 2) mated incidentors colors satisfy
- $k \le b - a \le l.$
- Denote the minimum number of colors by $\chi_{k,l}(G)$.

- Case $k=0$ is solved:
- $\chi_{0,0}(G)$ is an edge chromatic number
- $\chi_{0,1}(G) = \chi_{0,\infty}(G) = \Delta$ (Melnikov, P., Vizing, 2000)
- Another solved case is $l=\infty$:
- $\chi_{k,\infty}(G) = \max\{\Delta, k+\Delta^{+}, k+\Delta^{-}\}$ (P.,1999)





Vizing's proof

- Let $t = \max\{\Delta, k + \Delta^{+}, k + \Delta^{-}\}$

- 1. Construct a bipartite interpretation H of the graph G:

- $v \in V(G)$ corresponds to $v^{+}, v^{-} \in V(H)$
- $vu \in E(G)$ corresponds to $v^{+}u^{-} \in E(H)$

Vizing's proof

- 2. Color the edges of H by $\Delta(H)$ colors. Clearly, $\Delta(H) = \max\{\Delta^{+}(G), \Delta^{-}(G)\}$
- 3. If $v^{+}u^{-} \in E(H)$ is colored a, color a the initial incidentor of the arc $vu \in E(G)$ and color $a+k$ its final incidentor

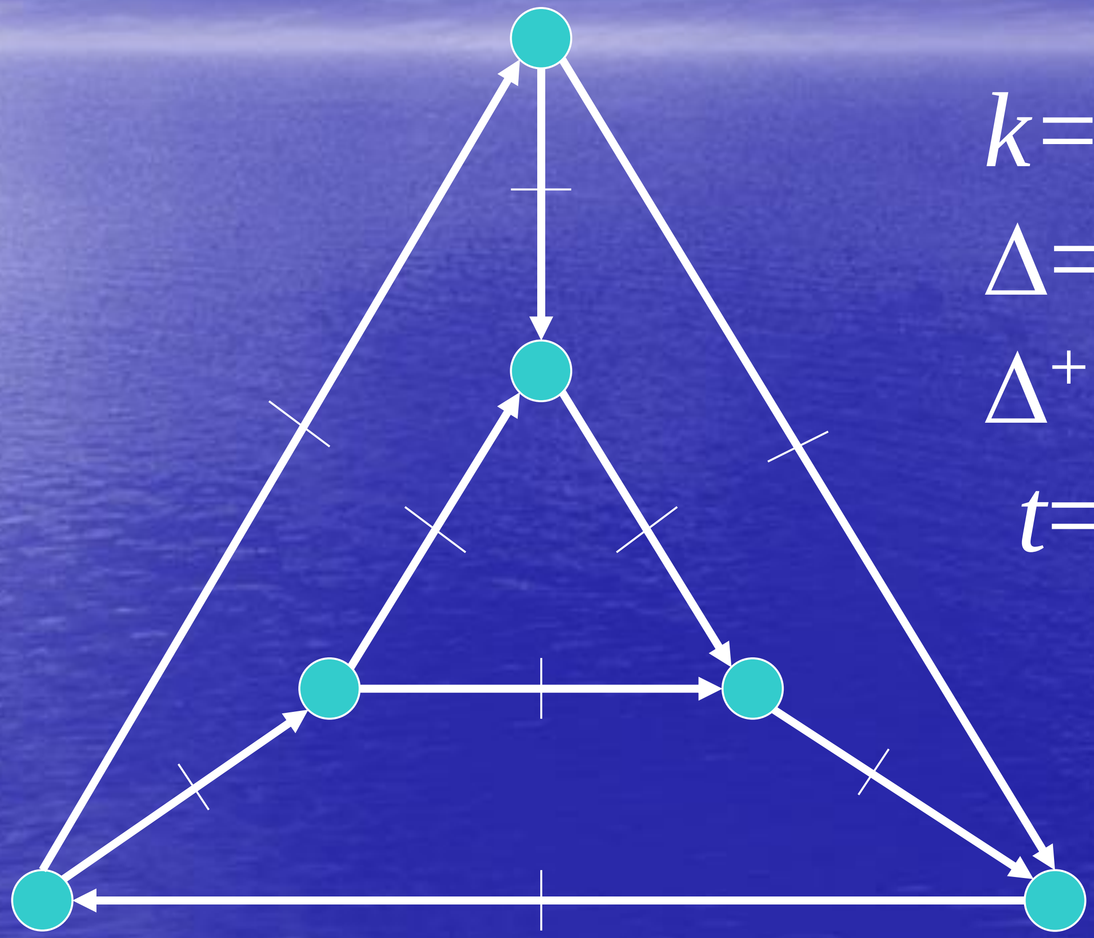
Vizing's proof

- 4. Shift colors at every vertex

- Initial: turn $a_1 < a_2 < \ldots < a_p$ into $1, 2, \ldots, p$
- Final: turn $b_1 > b_2 > \ldots > b_q$ into $t, t-1, \ldots, t-q+1$

- We get a required incidentor coloring of G by t colors

Example



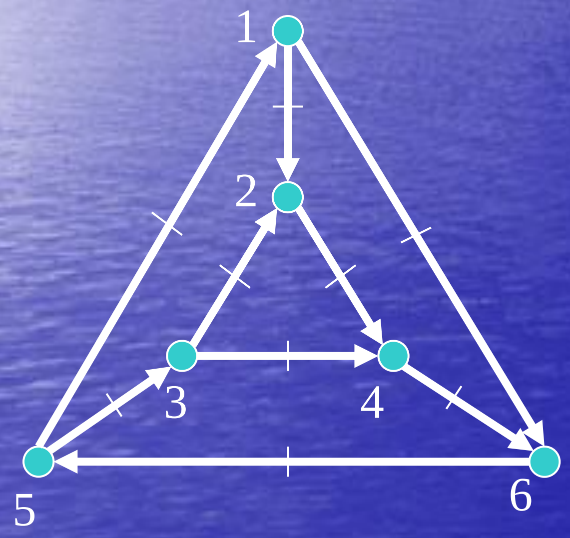
$k=1$

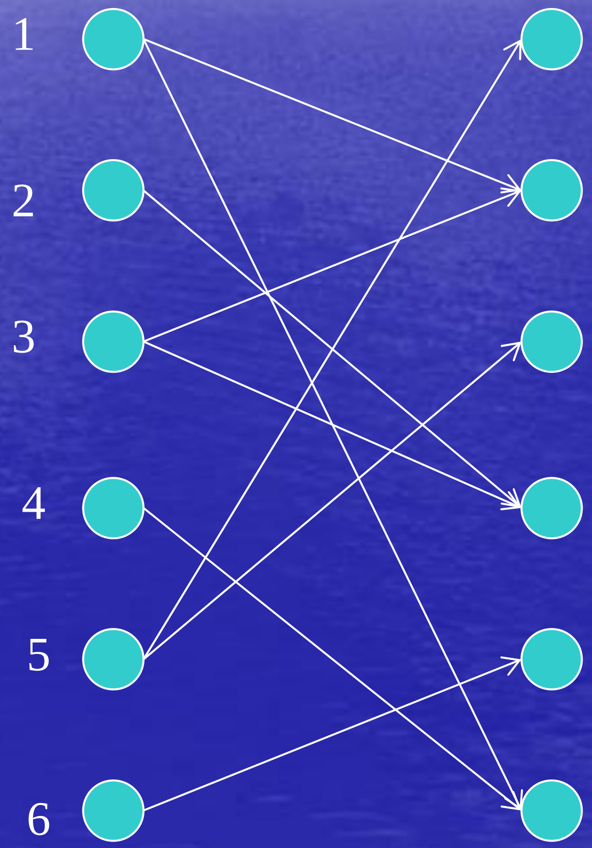
$\Delta=3$

$\Delta^{+}=\Delta^{-}=2$

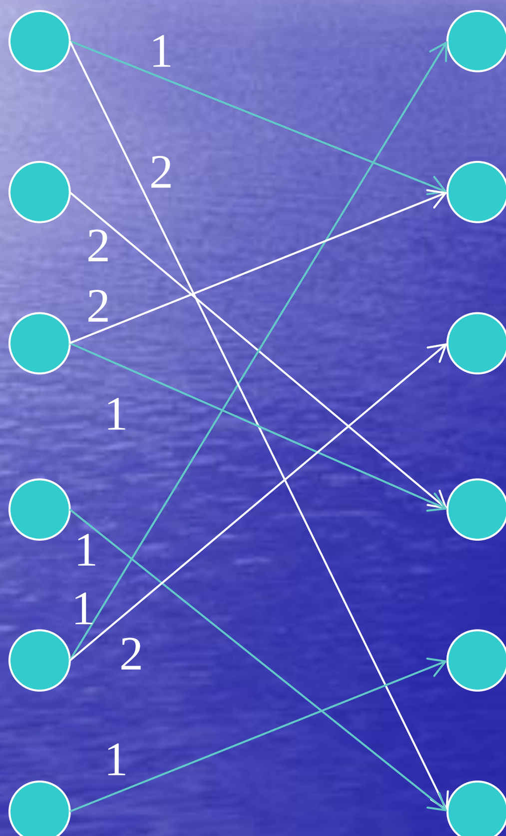
$t=3$

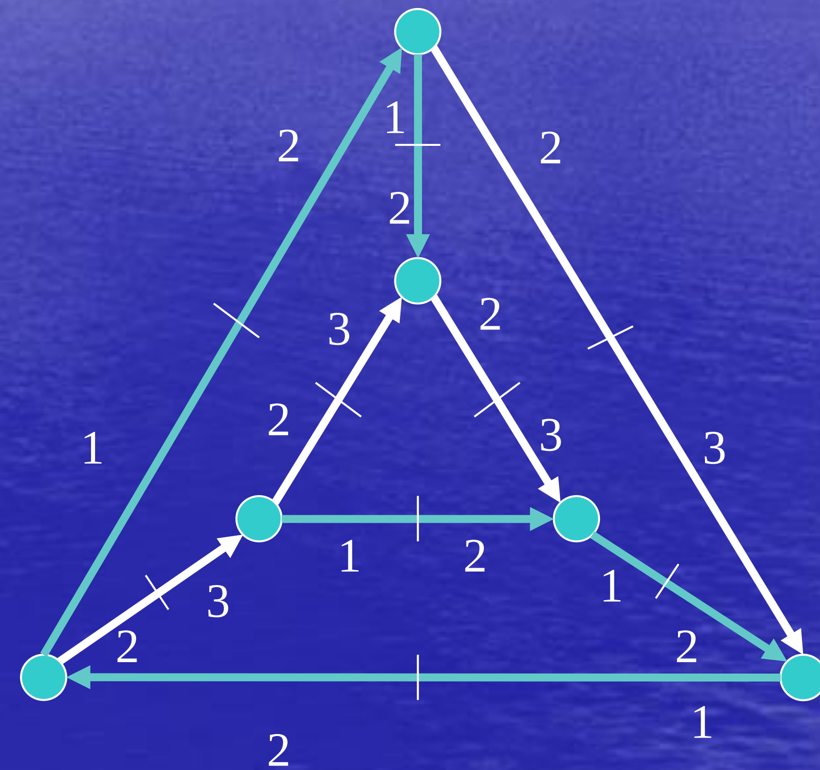
Bipartite interpretation



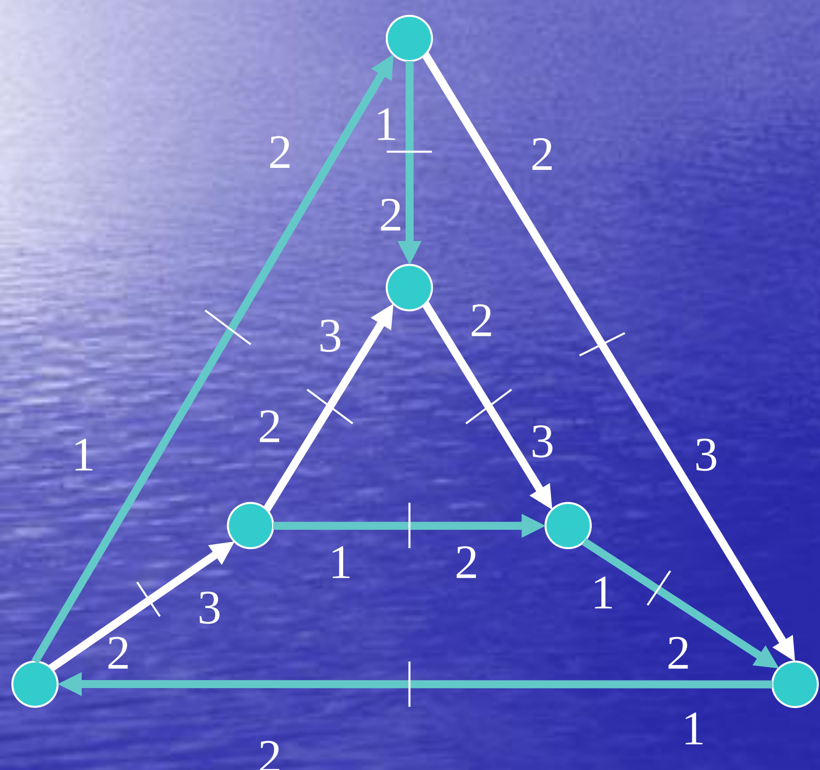


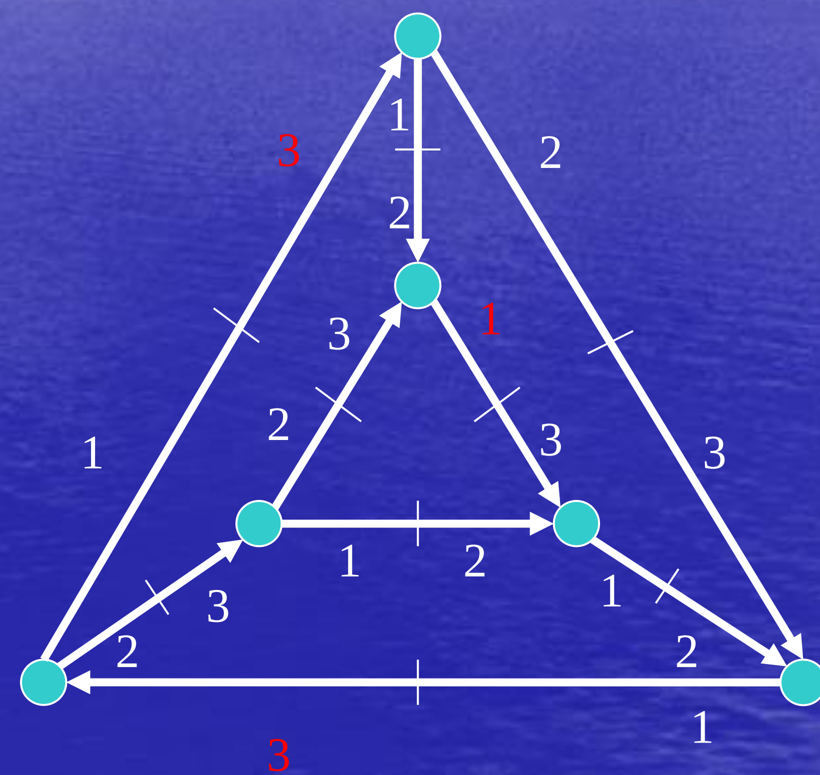
Edge coloring





Shifting the colors





Equivalent problem in scheduling theory

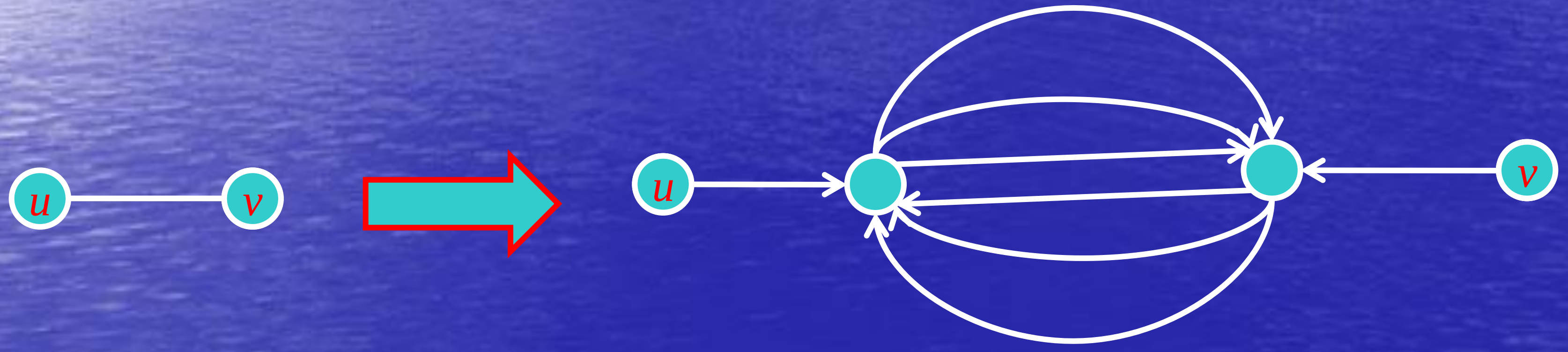
- Job Shop with n machines and m jobs, each of which has two unit operations (at different machines), and there must be a delay at least k and at most l between the end of the first operation and the beginning of the second one.

- It is NP-complete to find out whether there is a $(1,1)$-coloring of a multigraph by Δ colors even for $\Delta=7$ (Bansal, Mahdian, Sviridenko, 2006).

- Reduction from 3-edge-coloring of a 3-regular graph

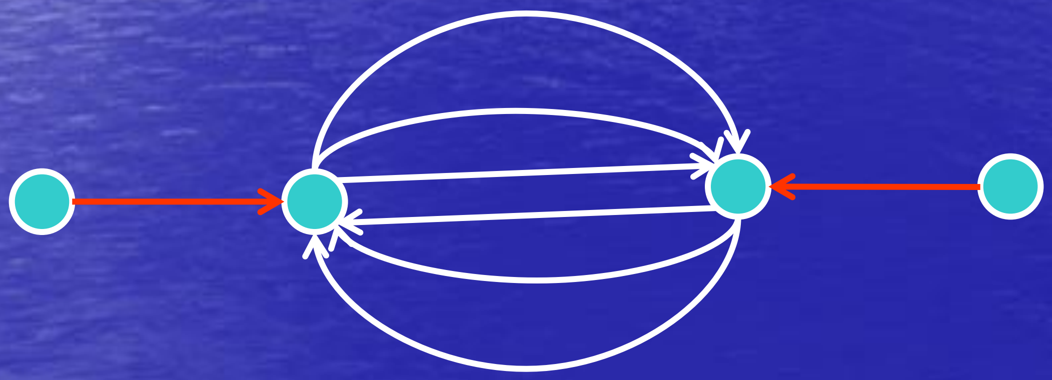
Reduction from 3-edge-coloring

- Substitute each edge by the following gadget:



Reduction from 3-edge-coloring

- It can be verified that in any (1,1)-coloring by 7 colors the incidentors of the initial incidentors of the red edges must be colored by the same even color



Results on (k,l)-coloring

- $\chi_{k,k}(G) = \chi_{k,\infty}(G)$ for $k \geq \Delta(G) - 1$
- $\chi_{k,\Delta(G)-1}(G) = \chi_{k,\infty}(G)$ (Vizing, 2003)

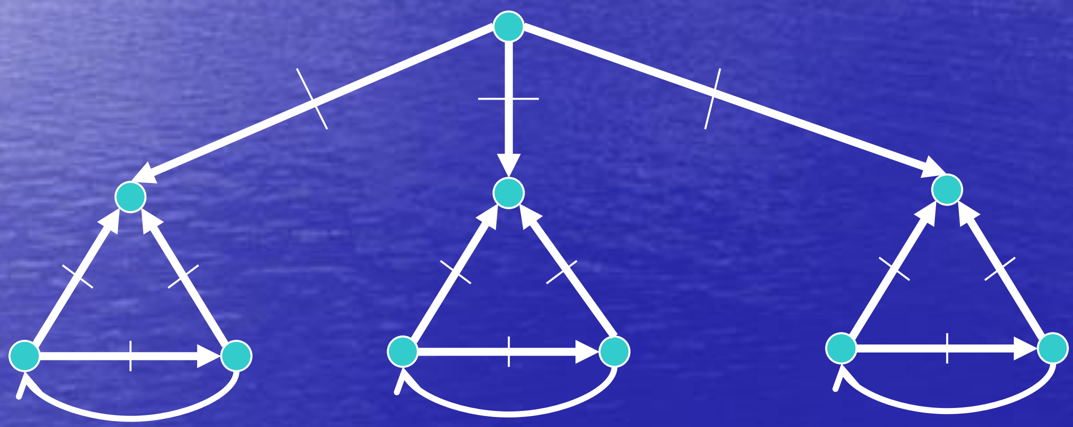
- Let $\chi_{k,l}(\Delta) = \max\{\chi_{k,l}(G) \mid \deg(G) = \Delta\}$
- $\chi_{k,\infty}(\Delta) = k + \Delta$
- $\chi_{k,k}(\Delta) \geq \chi_{k,l}(\Delta) \geq k + \Delta$
- $\chi_{0,1}(\Delta) = \Delta$

Results on (k,l)-coloring

- $\chi_{k,l}(2) = k+2$ except $k = l = 0$
- (Melnikov, P.,Vizing, 2000)
- $\chi_{k,l}(3) = k+3$ except $k = l = 0$ and $k = l = 1$ (P., 2003)
- $\chi_{k,l}(4) = k+4$ except $k = l = 0$
- For $l \geq \lceil \Delta/2 \rceil$, $\chi_{k,l}(\Delta) = k + \Delta$
- (P., 2004)

Results on (1,1)-coloring

- For odd Δ, $\chi_{1,1}(\Delta) > \Delta+1$ (P., 2004)



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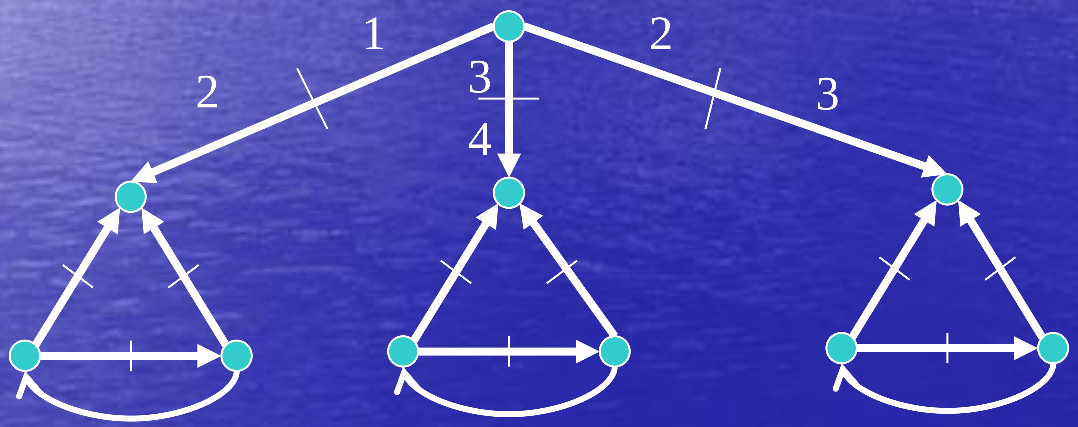

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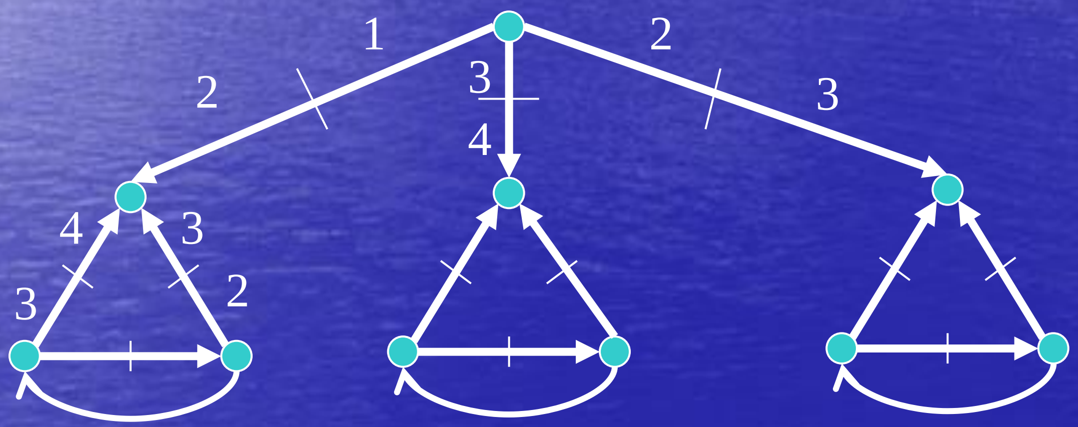

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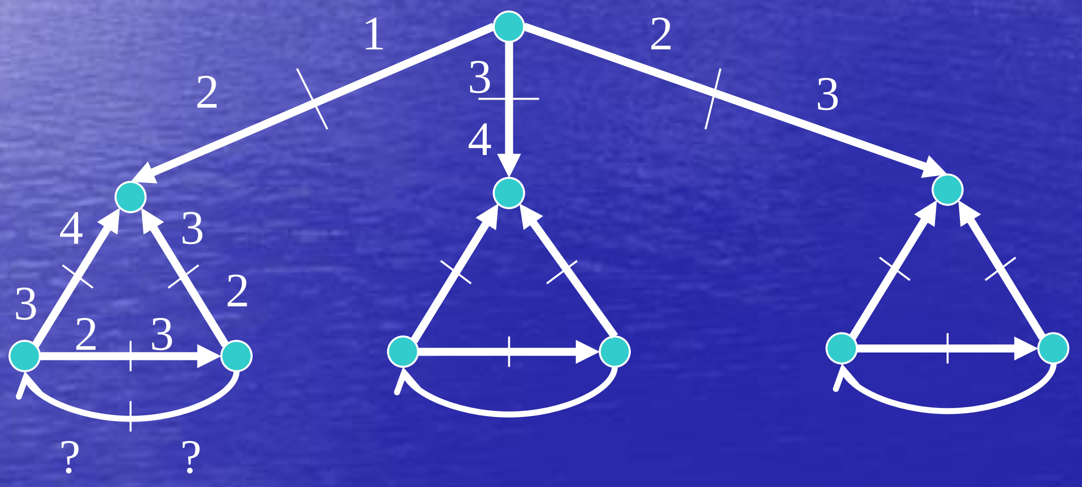


Results on (1,1)-coloring

- For even Δ, it is unknown whether there is a graph G of degree Δ such that $\chi_{1,1}(G) > \Delta + 1$. If such G exists, then it has degree at least 6.

- Theorem. $\chi_{1,1}(4)=5$ (P., 2004)

Proof

- Consider an Eulerian route in a given 4-regular multigraph
- Say that an edge is red, if its orientation is the same as in the route and blue otherwise
- Construct a bipartite interpretation according to this route (it consists of the even cycles)

- Find an edge coloring f:$E \rightarrow \{1,2\}$ such that:

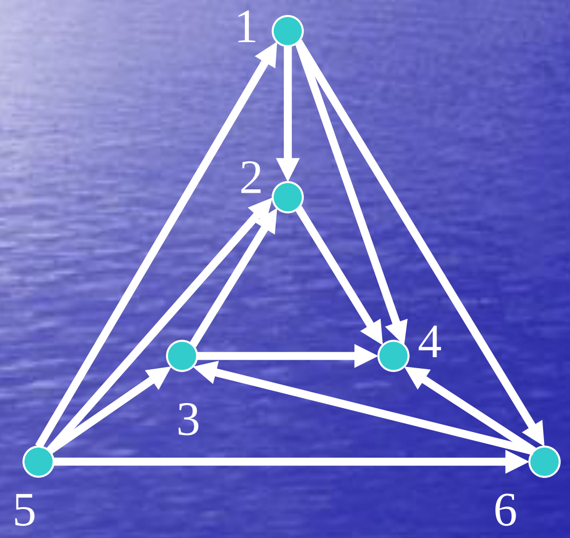
- 1) any two edges adjacent at the right side have distinct colors;
- 2) any two blue or red edges adjacent at the left side have distinct colors;
- 3) If a red edge e meets a blue one e' at the left side, then $f(e) \neq f(e') + 1$

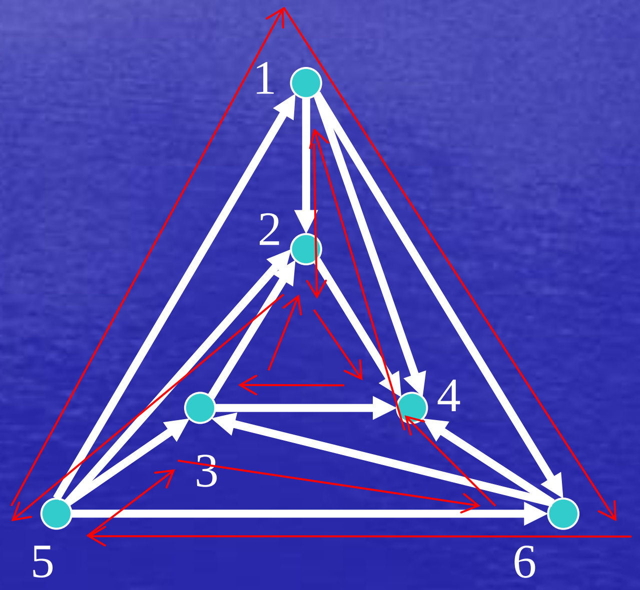
- Construct an incidentor coloring g in the following way:

- 1) For the right incidentor let $g=2f$
- 2) For the left red incidentor let $g=2f-1$
- 3) For the left blue incidentor let $g=2f+1$

- We obtain an incidentor $(1,1)$-coloring of the initial multigraph by 5 colors

Example





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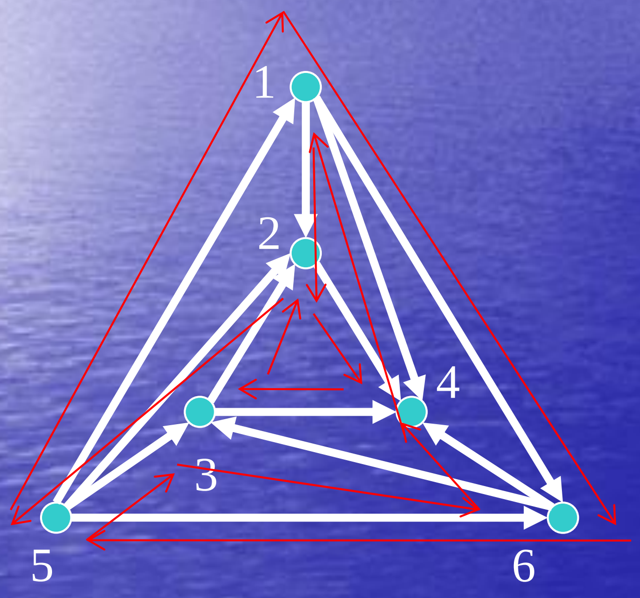
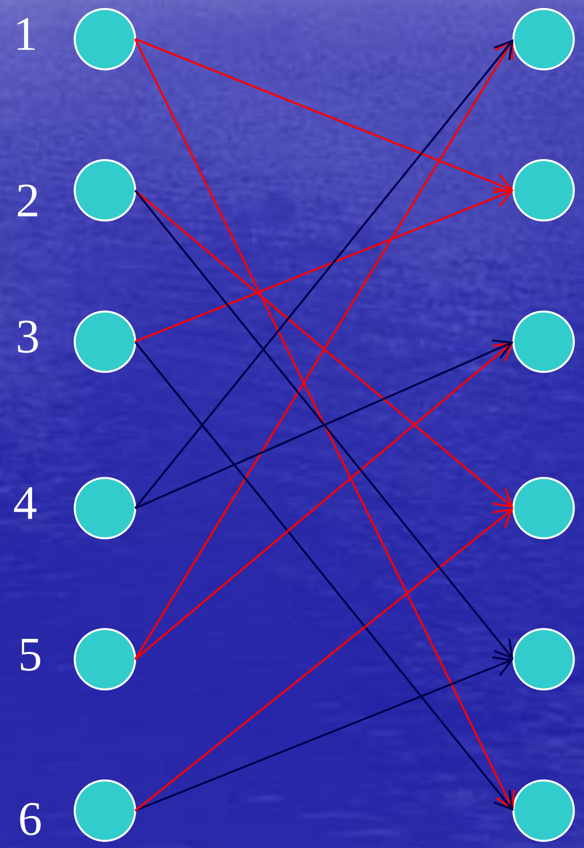
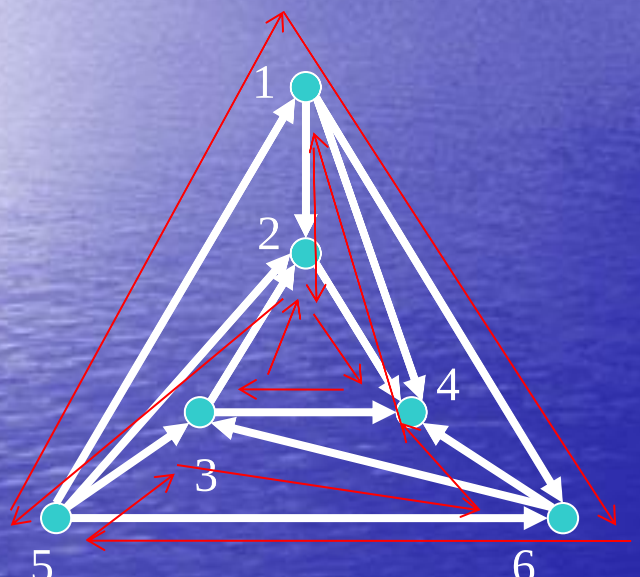
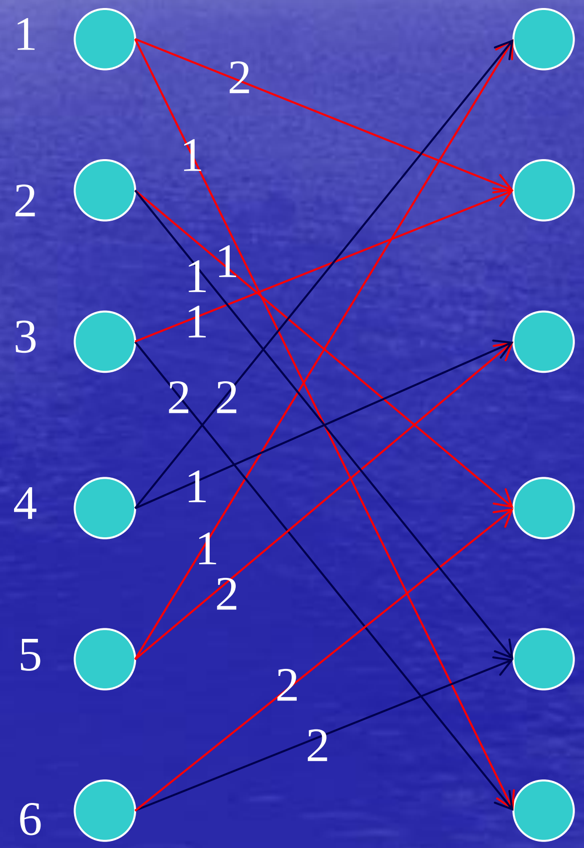


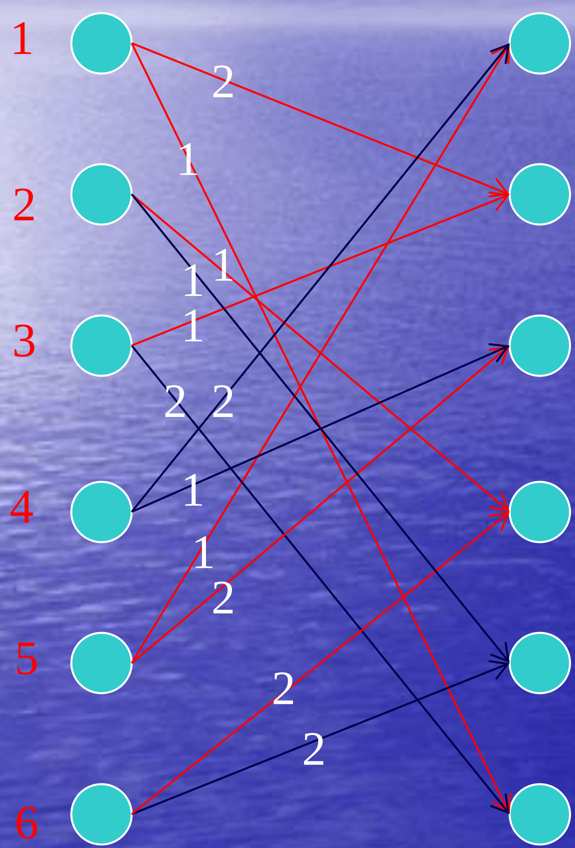


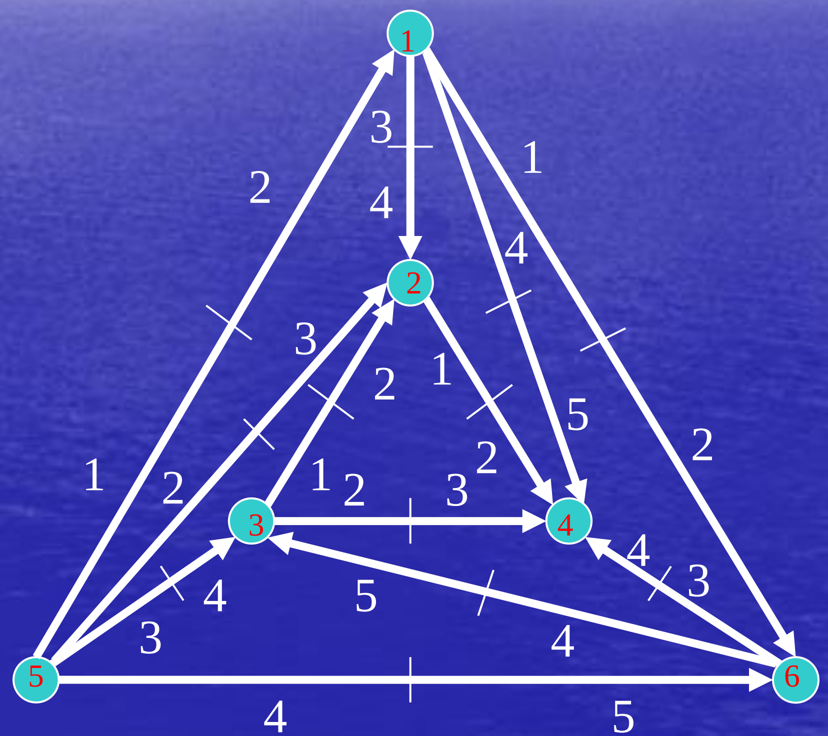
Example





Example





Incidentor coloring of weighted multigraph

- Each arc e has weight $w(e)$
- Coloring restrictions:
- 1) adjacent incidentors have distinct colors;
- 2) For every arc e, $w(e) \le b - a$

Results on weighted coloring

- Problem is NP-hard in a strong sense for $\chi = \Delta$ (P., Vizing; 2005)
- For $\chi > \Delta$ the problem is NP-hard in a strong sense even for multigraphs on two vertices (Lenstra, Hoogevan,Yu; 2004)
- It can be solved approximately with a relative error $3/2$ (Vizing, 2006)

List incidentor coloring

- A weighted incidentor coloring where each arc e has a list $L(e)$ of allowed colors for its incidentors

- Conjecture. If $|L(e)| \geq w(e)+\Delta$ for every arc e then an incidentor coloring exists

- True for $|L(e)| \geq w(e)+\Delta+1$. Proved for even Δ (Vizing, 2001) and for $\Delta=3$ (P., 2007)

Total incidentor coloring

- Color incidentors and vertices in such a way that vertex coloring is correct and a color of each vertex is distinct from the color of all incidentors adjoining this vertex

- $\chi^T_{k,\infty}(G) \leq \chi_{k+1,\infty}(G)+1 \leq \chi_{k,\infty}(G)+2;$
- $\chi^T_{0,\infty}(G)= \Delta+1$ (Vizing, 2000)

- Conjecture. $\chi^T_{k,\infty}(G) \leq \chi_{k,\infty}(G)+1$

Interval incidentor coloring

- The colors of adjacent incidentors must form an interval

- $\chi^I{}_{0,\infty}(G) \le \max\{\Delta, \Delta^+ + \Delta^- - 1\}$
- $\chi^I{}_{1,\infty}(G) \le \Delta^+ + \Delta^-$

- For $k \ge 2$ there could be no interval incidentor (k,∞)-coloring (e.g. directed cycle) (Vizing, 2001)

Undirected case

- Instead of $b–a$ use $|b–a|$ for colors of mated incidentors
- Undirected incidentor chromatic number is equal to the best directed ones taken among all orientations

Undirected case

- $\chi_{k,\infty}(G) = \max\{\Delta, \lceil \Delta/2 \rceil + k\}$
- $\chi^{T}_{k,\infty}(G) \leq \chi_{k,\infty}(G) + 1$ (Vizing,Toft, 2001)

- If $k \geq \Delta/2$ then $\chi_{k,k}(G) = \lceil \Delta/2 \rceil + k$
- If $\Delta = 2kr$ then $\chi_{k,k}(G) = \Delta$
- If $\Delta = 2kr + s$ then $\chi_{k,k}(G) \leq \Delta + k - \lfloor s/2 \rfloor$
- (Vizing, 2005)

Undirected case

- For every regular multigraph G with $\Delta \geq 2k$ $\chi_{k,l}(G) \in \{\Delta, \chi_{k,k}(G)\}$ depending only on l; in particular, $\chi_{k,l}(G)=\chi_{k,l}(H)$ for every two regular multigraphs G and H of degree Δ (Vizing,2005)

Undirected case

- Interval incidentor coloring of undirected multigraphs always exists
- $\chi^I_{0,\infty}(G) = \chi^I_{1,\infty}(G) = \Delta$
- For $k \geq 2$, $\chi^I_{k,\infty}(G) \geq \max\{\Delta, \min\{2k, \Delta + k\}\}$ and
- $\chi^I_{k,\infty}(G) \leq 2\Delta + k(k-1)/2$
- (Vizing,2003)

Undirected case

- The incidentor coloring of weighted undirected multigraph is NP-hard in a strong sense even for $\chi=\Delta$
- It can be solved approximately with a relative error 5/4
- (Vizing, P., 2008)

Open problems

- 1. Is it true that for every k there is l such that $\chi_{k,l}(\Delta) = \chi_{k,\infty}(\Delta)=k+\Delta$?
- Proved for $k=0$ $(l=1)$. Incorrect for $\chi_{k,l}(G)$
- 2. What are the values of $\chi_{1,2}(5)$ and $\chi_{2,2}(5)$?

Open problems

- 3. Given a Δ-regular bipartite graph with red and blue edges is there an edge coloring $f: E \to \{1,2,\ldots,\Delta\}$ such that:
- 1) any two edges adjacent at the right side have distinct colors;
- 2) any two blue or red edges adjacent at the left side have distinct colors;
- 3) If a red edge e meets a blue one e' at the left side, then $f(e) \neq f(e')+1$?

Open problems

- 4. Is it true that if $|L(e)|\geq w(e)+\Delta$ for every arc e then a list incidentor coloring exists?
- 5. Is it true that $\chi^{T}{}_{k,\infty}(G) \leq \chi_{k,\infty}(G)+1$?

Thanks for your attention!!!