A Weil-Petersson metric for graphs

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Finite graphs

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Then $\pi_1 G$ is a free group of rank ≥ 2.

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The pair (G, ℓ) may be thought of as a toy analogue of a compact hyperbolic surface, i.e. a compact smooth surface S of genus ≥ 2, equipped with a Riemannian metric of constant Gaussian curvature -1.

A Moduli Space

Just as the Teichmüller space of a smooth surface Teich(S) parametrizes hyperbolic metrics on S, we can consider a space of lengths (or, equivalently, a space of metrics) on a fixed graph G.

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Define

$$\mathcal{M}_G = \{\ell : E \to \mathbb{R}^{>0}\}$$

and a space of normalised lengths

$$\mathcal{M}_G^1 = \{\ell \in \mathcal{M}_G : h(G,\ell) = 1\},$$

where

$$h(G,\ell) = \lim_{t\to\infty} \frac{1}{t} \log \#\{\text{cycles } \gamma : \ell(\gamma) \le t\}.$$

Entropy

We call the number

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From a dynamical point of view, it is the topological entropy of a certain flow ($\mathbb{R}$-action) but we shall not use that description here.

Teichmüller Space

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$\mathrm{Teich}(S)$ is a smooth manifold diffeomorphic to $\mathbb{R}^{6k-6}$.

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The Weil-Petersson metric has the desirable property of making Teich(S) negatively curved.

Theorem (Ahlfors, 1961)

Teich(S) *is negatively curved with respect to* $\|\cdot\|_{\mathrm{WP}}$.

Thurston's definition

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Then we can expand

$$g_\lambda = g_0 + \lambda \dot{g}_0 + \frac{\lambda^2}{2} \ddot{g}_0 + \cdots ,$$

where $\dot{g}_0 \in T_{g_0}(\mathrm{Teich}(S))$.

Thurston's definition

Let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence of closed geodesics on (S, g_0) which are equidistributed with respect to the g_0-area measure: for all $f \in C(S, \mathbb{R})$,

$$\lim_{n\to\infty} \frac{1}{\text{length}_{g_0}(\gamma_n)} \int_{\gamma_n} f = \int_S f \, d\text{area}_{g_0}.$$

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Then

$$\|\dot{g}_0\|^2_{\text{Thurston}} := \lim_{n\to\infty} \frac{\partial^2}{\partial\lambda^2} \frac{\text{length}_{g_\lambda}(\gamma_n)}{\text{length}_{g_0}(\gamma_n)}\bigg|_{\lambda=0}.$$

Wolpert's Theorem

Theorem (Wolpert, 1980s)

- *Thurston's metric* $\| \cdot \|_{\text{Thurston}}$ *is equal to the Weil-Petersson metric* $\| \cdot \|_{\text{WP}}$*.*
- Teich(S) *is incomplete with respect to* $\| \cdot \|_{\text{WP}}$*.*

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Define $f : T^1(S, g_0) \to \mathbb{R}$ by $f(v) = \dot{g}_0(v, v)$ and

$$\sigma^2(\dot{g}_0) := \lim_{t\to\infty} \frac{1}{t} \int_{T^1(S,g_0)} \left(\int_0^t f(\phi_u v)\, du \right)^2 d\mu_{g_0}(v),$$

where μ_{g_0} is the Liouville measure on $T^1(S, g_0)$ (the product of the area measure on (S, g_0) and Lebesgue measure on the fibres).

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Theorem (McMullen, 2007)

$$\sigma^2(\dot{g}_0) = \frac{4}{3} \frac{\|\dot{g}_0\|^2_{\mathrm{WP}}}{\mathrm{area}(S, g_0)} = \frac{\|\dot{g}_0\|^2_{\mathrm{WP}}}{3\pi(k-1)}.$$

Outer Space

The natural analogue of Teichmüller space in the Culler-Vogtmann outer space X_k. This parametrizes lengths on all (marked) graphs with rank k fundamental group.

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We will give a definition of a Riemannian metric on $\mathcal{M}_G^1$ which is analogous to McMullen's definition.

Oriented edges

Consider again the graph $G = (V, E)$. Let E^o denote the oriented edges of G. (So $|E^o| = 2|E|$.)

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If $e \in E^o$ then $\overline{e} \in E^o$ will denote the edge with the reversed orientation.

The length ℓ defines a function $\ell : E^o \to \mathbb{R}^{>0}$ satisfying $\ell(e) = \ell(\overline{e})$.

The incidence matrix for oriented edges

Define a matrix A indexed by $E^o \times E^o$ by

$$A(e, e') = \begin{cases} 1 & \text{if } e' \text{ follows } e, \\ 0 & \text{otherwise.} \end{cases}$$

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In face, A is aperiodic (A^n has positive entries for some n) unless G is bipartite.

A subshift of finite type

Define a space

$$\Sigma = \{\underline{e} = (e_n)_{n=0}^{\infty} : A(e_n, e_{n+1}) = 1 \; \forall n \geq 0\},$$

i.e. Σ is the space of infinite paths in G.

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Σ can be made into a compact metric space by setting

$$d(\underline{e}, \underline{e}') = 2^{-n},$$

where

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It supports the shift map $T : \Sigma \to \Sigma$ defined by $(T\underline{e})_n = e_{n+1}$.

Pressure

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By the Perron-Frobenius Theorem, A_f has a simple positive eigenvalue equal to its spectral radius. We denote this eigenvalue by $e^{P(f)}$ and call $P(f)$ the *pressure* of f.

The Parry measure

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The matrix

$$P(e, e') = \frac{A_{-h\ell}(e, e') q_{e'}}{q_e}$$

is row stochastic.

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We can define a measure μ_P on Σ in the following way. Let $[e_0, \ldots, e_n]$ denote the set of infinite paths in G starting with a fixed finite path $(e_0, \ldots, e_n)$. Then

$$\mu_P([e_0, \ldots, e_n]) = p_{e_0} P(e_0, e_1) \cdots P(e_{n-1}, e_n).$$

This extends to a probability measure on Σ called the Parry measure, which is invariant under the shift map T: for integrable $f : \Sigma \to \mathbb{R}$,

$$\int_\Sigma f \circ T \, d\mu_P = \int_\Sigma f \, d\mu_P.$$

The Parry measure

For $f : E^o \to \mathbb{R}$ (identified with a function on Σ),

$$\int_{\Sigma} f \, d\mu_P = \sum_{e \in E^o} p_e f(e).$$

Differentiating pressure

Lemma

Suppose that $(-\epsilon, \epsilon) \to \mathbb{R}^{E^o} : \lambda \mapsto \phi_\lambda$ *is analytic with* $\phi_0 = -h\ell$*. Then the function* $\lambda \mapsto P(\phi_\lambda)$ *is analytic and*

$$\frac{d}{dt} P(\phi_\lambda)\Big|_{\lambda=0} = \int_\Sigma \dot{\phi}_0 \, d\mu_P = \sum_{e \in E^o} p_e \dot{\phi}_0(e).$$

Proof of Lemma

We have an eigenvalue equation

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$$\psi_\lambda(e, e') = \phi_\lambda(e) + \log w_\lambda(e') - \log w_\lambda(e).$$

Then $P(\psi_\lambda) = P(\phi_\lambda)$ and

$$A_{\psi_\lambda} 1 = e^{P(\phi_\lambda)} 1,$$

where $1 = (1, \ldots, 1)^T$.

Proof of Lemma

Differentiating and evaluating at $\lambda = 0$ (using $P(\phi_0) = 0$ and $A_{\psi_0} = P$) we obtain

$$\left.\frac{dP(\phi_\lambda)}{d\lambda}\right|_{\lambda=0} = \sum_{e' \in E^o} \left(\dot{\phi}_0(e) + \dot{w}_0(e') - \dot{w}_0(e)\right) P(e, e').$$

Proof of Lemma

Multiplying by p_e and summing over $e \in E^o$ we get (using $\sum_{e\in E^o} p_e = 1$)

$$\left.\frac{dP(\phi_\lambda)}{d\lambda}\right|_{\lambda=0}$$
$$= \sum_{e,e' \in E^o} p_e \dot{\phi}_0(e) P(e,e') + \sum_{e,e' \in E^o} \left(\dot{w}_0(e') - \dot{w}_0(e)\right) p_e P(e,e')$$
$$= \sum_{e \in E^o} p_e \dot{\phi}_0(e),$$

as required, using the fact that P is row stochastic and that $pP = p$. □

The tangent space to $\mathcal{M}^1_G$

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Then we can expand

$$\ell_\lambda = \ell_0 + \lambda \dot{\ell}_0 + \frac{\lambda^2}{2} \ddot{\ell}_0 + \cdots \, ,$$

where $\dot{\ell}_0 \in T_{\ell_0}(\mathcal{M}_G^1)$.

The tangent space to $\mathcal{M}_G^1$

Since $\ell_\lambda \in \mathcal{M}_G^1$, we have

$$P(-\ell_\lambda) = 0.$$

By the lemma above, we have

$$0 = \left.\frac{dP(-\ell_\lambda)}{d\lambda}\right|_{\lambda=0} = -\int_\Sigma \dot{\ell}_0 \, d\mu_P.$$

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Remark

This parallels the fact that in the surface case

$$\int_{T^1(S,g_0)} \dot{g}_0(v,v) \, d\mu_{g_0}(v) = 0.$$

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we thus have

$$T_\ell \mathcal{M}_G^1 \subset \left\{ f : E^o \to \mathbb{R} : \ f(e) = f(\overline{e}) \text{ and } \sum_{e \in E^o} p_e f(e) = 0 \right\}.$$

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Therefore

$$T_\ell \mathcal{M}_G^1 = \left\{ f : E^o \to \mathbb{R} : \; f(e) = f(\bar{e}) \text{ and } \sum_{e \in E^o} p_e f(e) = 0 \right\}.$$

A Weil-Petersson metric on $\mathcal{M}_G^1$

By analogy with McMullen's definition, for $f \in T_\ell \mathcal{M}_G^1$ we set

$$\sigma^2(f) = \lim_{n\to\infty} \frac{1}{n} \int_\Sigma \left(f(\underline{e}) + f(T(\underline{e})) + \cdots f(T^{n-1}(\underline{e})) \right)^2 \, d\mu_P.$$

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In fact, one can calculate that

$$\sigma^2(f) = \sum_{e\in E^o} p_e (f(e))^2.$$

Finally, we use this to define a metric

$$\|f\|_{\mathrm{WP}}^2 = \sigma^2(f).$$

Properties of the metric

How does the metric compare with the Weil-Petersson metric on Teichmüller space?

Properties of the metric: completeness

Theorem

There exist graphs G for which $\| \cdot \|_{\mathrm{WP}}$ is incomplete.

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In fact, the metric is incomplete for the graph with one vertex and two edges.

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In fact, this occurs for the "dumbbell" graph.

However, for the "belt buckle" graph, the curvature is negative.