Truemper configurations

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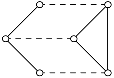
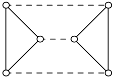
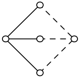
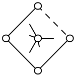
Graph Theory and Interactions — Durham Symposium
July 2013

Outline

Truemper configurations

The following graphs are called **Truemper configurations**

- Pyramid: 
- Prism: 
- Theta: 
- Wheel: 

For more about them: see the survey of Vušković.

Original motivation

Theorem (Truemper, 1982)

Let β be a $\{0,1\}$ vector whose entries are in one-to-one correspondence with the chordless cycles of a graph G. Then there exists a subset F of the edge set of G such that $|F \cap C| \equiv \beta_C \pmod 2$ for all chordless cycles C of G, if and only if every induced subgraph G' of G that is a Truemper configuration or K_4 there exists a subset F' of the edge set of G' such that $|F' \cap C| \equiv \beta_C \pmod 2$, for all chordless cycles C of G'.

Our motivation

We are interested in Truemper configurations as **induced subgraphs** of graphs that we study.

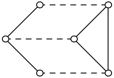
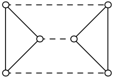
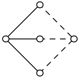
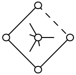
Something classical:

- consider a class of graphs where some Truemper configurations are excluded
- to study a generic graph G from the class, suppose that it contains a Truemper configuration H that is authorized
- prove that $G \setminus H$ must attach to H in a very specifc way, so that if H is present, we "understand" the graph.
- continue the study for graphs where H is excluded.

Five classical classes

- Even-hole-free graphs
  (Conforti, Cornuéjols, Kapoor and Vušković 2002)
- Perfect graphs
  (Chudnovsky, Robertson, Seymour and Thomas 2002)
- Claw-free graphs
  (Chudnovsky and Seymour 2005)
- ISK4-free graphs
  (Lévêque, Maffray and NT 2012)
- Bull-free graphs
  (Chudnovsky 2012)

Detecting Truemper configurations

- Pyramid  Polynomial, $O(n^9)$, Chudnovsky and Seymour, 2002
- Prism  NP-complete, Maffray, NT, 2003 Follows from a construction of Bienstock
- Theta  Polynomial, $O(n^{11})$, Chudnovsky and Seymour, 2006
- Wheel  NP-complete, Diot, Tavenas and Trotignon, 2013

Outline

Our project

Studying the $2^4 = 16$ classes of graphs obtained by excluding Truemper configurations.

Example : graphs with no prism and no theta, ...

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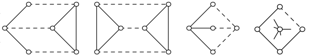
Example : graphs with no prism and no theta, ...

Good news: here 16=15.

Universally signable graphs (1)

A graph is **universally signable** if it contains no Truemper configuration ().

Examples of such graphs:

- cliques
- chordless cycles
- any graph obtained by gluing two previoulsy built graphs along a clique

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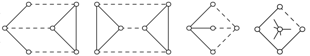
A graph is **universally signable** if it contains no Truemper configuration ().

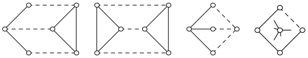
Examples of such graphs:

- cliques
- chordless cycles
- any graph obtained by gluing two previoulsy built graphs along a clique

Theorem (Conforti, Cornuéjols, Kapoor and Vušković 1999)

If G is universally signable then G is a clique or G is a chordless cycle, or G has a clique cutset.

Universally signable graphs (2)

Universally signable graphs: no 

Theorem (Conforti, Cornuéjols, Kapoor and Vušković 1999)

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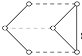
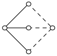
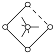
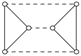
Consequences and open questions:

- Many algorithms (recognition in time $O(nm)$, colouring, max stable set, . . .)
- A nice property: every universally signable graph has a **simplicial extreme** (= vertex of degree 2 or whose neighbourhood is a clique).
- Question: recognition in linear time ?

“Only-prism” graphs

A graph is **only prism** if it contains no pyramid , no theta  and no wheel .

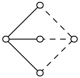
So, the prism  is the only allowed Truemper configuration.

Theorem (Diot, Radovanović, NT and Vušković 2013)

If a graph G is only-prism, then G is the line graph of a triangle-free chodless graph, or G has a clique cutset.

A **chodless** graph is a graph such that every cycle is chordless. The theorem is reversible: any graph obtained by repeatedly gluing line graphs of a triangle-free chodless graphs along cliques is in the class.

Theta-free graphs (1)

Theta: 

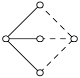
No structural description of theta-free graphs is known so far. But:

Theorem (Chudnovsky and Seymour 2005)

There exits an $O(n^{11})$-time algorithm that decides whether a graph is theta-free.

- Can one be faster?
- Is there a polytime algorithm for computing a max stable set in theta-free graphs?

Theta-free graphs (2)

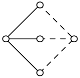
Theta: 

Theorem (Kühn and Osthus 2004)

There exists a function f such that every theta-free graph G satisfies $\chi(G) \leq f(\omega(G))$.

- the existence of f in the theorem above is non-trivial, for many classes of graphs there is no such f.
- could the function f in the theorem above be a polynomial? A quadratic function?

Theta-free graphs (2)

Theta: 

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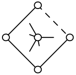
Theorem (Radovanović and Vušković 2010)

If f^* be the smallest possible function in the theorem above, then $f^*(2) = 3$. Rephrased: every {theta, triangle}-free graph is 3-colourable.

Wheel-free graphs

Little is known about wheel-free graphs. Wheel: 

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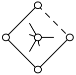
Little is known about wheel-free graphs. Wheel: 

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Theorem (Chudnovsky 2012)

If G is a wheel-free graph, then G contains a multisimplicial vertex (= a vertex whose neighborhood is a disjoint union of cliques).

Outline

LexBFS

- LexBFS is a variant on BFS introduced by Rose, Tarjan and Lueker in 1976.
- LexBFS is a linear time algorithm whose input is any graph and whose output is a linear ordering of the vertices.

Theorem (Brandstädt, Dragan and Nicolai 1997)

An ordering $\prec$ of the vertices of a graph G is a LexBFS ordering if and only if it satisfies the following property: for all $a, b, c \in V$ such that $c \prec b \prec a$, $ca \in E$ and $cb \notin E$, there exists a vertex d in G such that $d \prec c$, $db \in E$ and $da \notin E$.

A property of LexBFS

Notation: $N[x] = N(x) \cup \{x\}$.

Theorem (Berry and Bordat 2000)

If a graph G is not a clique and z is the last vertex of a LexBFS ordering of G, then there exists a connected component C of $G \setminus N[z]$ such that for every neighbor x of z, either $N[x] = N[z]$, or $N(x) \cap C \neq \emptyset$.

Definition

Let $\mathcal{F}$ be a set of graphs. A graph G is locally $\mathcal{F}$-decomposable if for every vertex v of G, every $F \in \mathcal{F}$ contained in $N(v)$ and every connected component C of $G \setminus N[v]$, there exists $y \in V(F)$ such that y has non-neighbors in both F and C.

Our main result

Theorem (Aboulker, Charbit, NT and Vušković 2012)

Suppose that $\mathcal{F}$ is a set of non-clique graphs, G is a locally $\mathcal{F}$-decomposable graph, and v is the last vertex in a LexBFS ordering of G. Then $N(v)$ is $\mathcal{F}$-free.

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Example: if $\mathcal{F} = \{$ $\}$, then **chordal** (=hole-free) graphs are precisely the locally $\mathcal{F}$-decomposable graphs. It follows:

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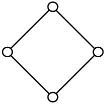
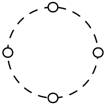
Example: if $\mathcal{F} = \{$ $\}$, then **chordal** (=hole-free) graphs are precisely the locally $\mathcal{F}$-decomposable graphs. It follows:

Theorem (Rose, Tarjan and Lueker 1976)

If G is chordal graph, then the last vertex in a LexBFS-order of G is simplicial (=its neighborhood is a clique).

Application 1: square-free perfect graphs

Partenoff, Roussel, Rusu, 1999

- Graph G : a perfect graph with no 
- $\mathcal{F}$: 
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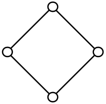
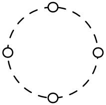
- Graph G : a perfect graph with no 
- $\mathcal{F}$: 
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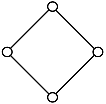
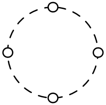
- Graph G : a perfect graph with no 
- $\mathcal{F}$: 
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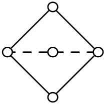
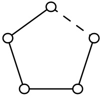
Question: what about perfect graphs in general ?
Fails because of replication.

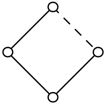
Application 2: a more general class of perfect graphs

Maffray, NT, Vušković, 2008

- Graph G : a perfect graph with no 
- $\mathcal{F}$: 
- An elimination ordering, vertices whose neighbourhood is long-hole-free
- Consequence: maximum clique in time $O(n^7)$

Application 3: even-hole-free graphs

da Silva, Vušković, 2007

- Graph G: even-hole-free
- $\mathcal{F}$: 
- An elimination ordering: a vertex whose neighborhood is chordal
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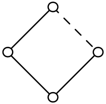
Application 3: even-hole-free graphs

da Silva, Vušković, 2007

- Graph G: even-hole-free
- $\mathcal{F}$: 
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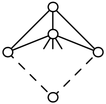
Question: Adario-Berry, Chudnovsky, Havet, Reed, Seymour's theorem: every even-hole-free graph admits a bisimplicial vertex
Proof with some graph searching method?

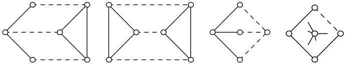
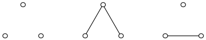
Application 4: wheel-free graphs

Aboulker, Charbit, NT, Vušković, 2012

- G: a graph with no 
- $\mathcal{F}$: 
- An elimination ordering: a vertex whose neighbouhood is multisimplicial
- Consequence: maximum clique in time $O(nm)$

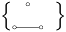
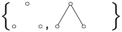
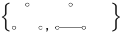
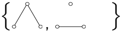
Application 5: universally signable graphs

Aboulker, Charbit, Chudnovsky, NT, Vušković, 2012

- Graphs with no 
- $\mathcal{F}$: 
- An elimination ordering: a vertex of degree 2 or simplicial
- Consequence: colouring in linear time.

Application 6: when $\mathcal{F}$ is finite

- Provided that $\mathcal{F}$ is finite, it seems that the description of locally $\mathcal{F}$-decomposable graphs is quite automatic.
- In the next slide, all possible sets $\mathcal{F}$ of non-clique graphs on three vertices are studied.
- Interestingly, for every set $\mathcal{F}$, the class of locally $\mathcal{F}$-decomposable graphs is defined by exluding some Truemper configurations.
- If $\mathcal{F}$ contains graphs on at least 4 vertices, such a behavior disappears.

(1-wheel, theta, pyramid)-free		no stable set of size 3
3-wheel-free		disjoint union of cliques
(2-wheel, prism, pyramid)-free		complete multipartite
(1-wheel, 3-wheel, theta, pyramid)-free		disjoint union of at most two cliques
(1-wheel, 2-wheel, prism, theta, pyramid)-free		stable sets of size at most 2 with all possible edges between them
(2-wheel, 3-wheel, prism, pyramid)-free		clique or stable set
(wheel, prism, theta, pyramid)-free		clique or stable set of size 2

Thanks

Thanks for your attention.