Symmetric frameworks on cylinders and cones

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joint work with Bernd Schulze

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Outline

- frameworks on surfaces
    - rigidity
    - Combinatorial characterisations
        - Laman
        - Cylinders and cones
        - Ellipsoid
- Symmetric frameworks
    - Forced symmetry
        - Orbit surface matrix
        - gain graphs
        - Necessary conditions
    - Planes and spheres
    - Cylinders and cones
        - Gain graph constructions
        - Sufficiency
        - Conjectures

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  - Think of $\mathcal{M}$ as either a plane or sphere, a cylinder, a cone or torus, or an ellipsoid.

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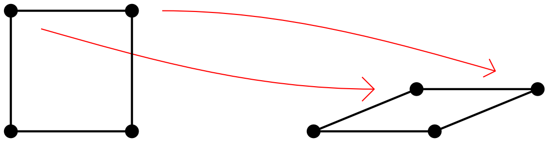

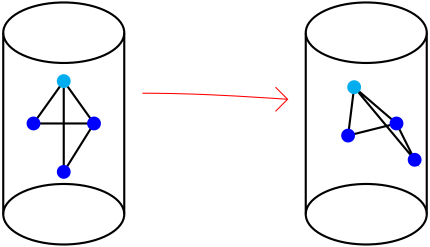

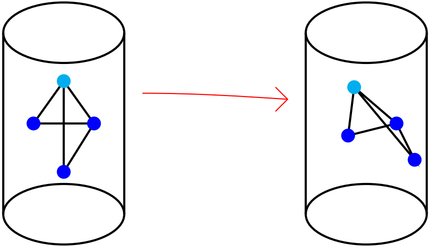

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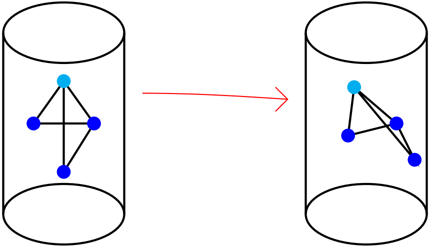

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- Let $\mathbb{Q}(p)$ denote the field of extension of $\mathbb{Q}$ formed by adjoining the coordinates of the vertices of (G, p). (G, p) is *generic* if $\text{td}[\mathbb{Q}(p) : \mathbb{Q}] = 2n$.

The rigidity matrix $R_{\mathcal{M}}(G,p)$ is an $(|E|+|V|) \times 3|V|$ matrix with columns labelled $x_1, y_1, z_1, \dots, x_n, y_n, z_n$ where the entries in the row corresponding to the edge ij are 0 except in the triple for i where the entries are $x_i - x_j, y_i - y_j, z_i - z_j$ and the triple for j where the entries are $x_j - x_i, y_j - y_i, z_j - z_i$. The entries in the row corresponding to the vertex i are 0 except in the triple for i where the entry is $N(p_i)$; the normal to $\mathcal{M}$ at the point p_i.

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- The constants above correspond to the number of isometries of the surface.
- For generic p, (G,p) is rigid if and only if it is infinitesimally rigid.

Characterising Generic Rigidity

Theorem: Laman 1970, Whiteley 1988, N. Owen and Power 2012, 2013

Let $G = (V, E)$, let $\mathcal{M}$ have k isometries for $k \in \{1, 2, 3\}$ and let (G, p) be a generic framework on $\mathcal{M}$. Then (G, p) is minimally rigid if and only if $|E| = 2|V| - k$ and for every subgraph (V', E'), with at least one edge, $|E'| \leq 2|V'| - k$.

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- The basic proof strategy:
- In the plane -
  - Maxwell direction: rigid implies counting.
  - Sufficiency falls into two steps:
  - Reduction - Henneberg-Laman recursive construction of graphs.
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- On the sphere -
  - Equivalence of geometries for rigidity.
- On other surfaces -
  - Elaboration of the scheme for the plane.

Lemma

Let $G = (V, E)$, let $\mathcal{M}$ have 0 isometries and let (G, p) be a generic minimally rigid framework on $\mathcal{M}$. Then G satisfies $|E| = 2|V|$ and for every subgraph (V', E'), with at least one edge, $|E'| \leq 2|V'|$.

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- The theorems on the previous slide become trivial for 1-dimensional frameworks:
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- For 3-dimensional frameworks it is an open problem to characterise generic minimal rigidity as a property of the graph.

Symmetric Graphs

- An *automorphism* of G is a permutation π of the vertex set $V(G)$ of G such that $\{u, v\} \in E(G)$ if and only if $\{\pi(u), \pi(v)\} \in E(G)$.
- The set of all automorphisms of G forms a group, called the automorphism group $Aut(G)$ of G.
- An *action* of a group S on G is a group homomorphism $\theta : S \rightarrow Aut(G)$.
- If $\theta(s)(v) \neq v$ for all $v \in V(G)$ and all non-trivial elements s of the group S, then the action θ is called *free*.
- If S acts on G by θ, then we say that the graph G is S*-symmetric* (with respect to θ).
- the *quotient graph* G/S is the multi-graph which has the set $V(G)/S$ of vertex orbits as its vertex set and the set $E(G)/S$ of edge orbits as its edge set.

Gain Graphs

- Given a group S and a graph $H = (V, E)$, an *S-gain graph* is a pair (H, Φ), where H is a directed multi-graph and $\Phi : E(H) \to S$ is a map which assigns an element of S to each edge of H.

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- The *covering graph* H^{ϕ} is the graph with vertex set $V \times S$ and, for $v, u \in V$ and $g, h \in S$, an edge from (v, g) to (u, h) if and only if there is an edge vu with gain m and $g * m = h$.

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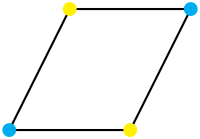

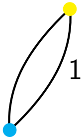

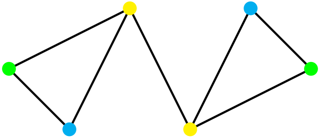

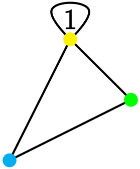
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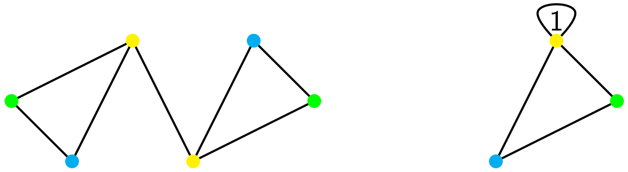

- The *net-gain* on a cycle of (H, ϕ) is the (group) product of the gains on the edges in the cycle.
- A graph is *balanced* if every cycle has net-gain 0 and is unbalanced if some cycle has non-zero net-gain.

Symmetric Frameworks

- A *symmetry operation* of the framework (G, p) on $\mathcal{M}$ is an isometry x of $\mathbb{R}^3$ which maps $\mathcal{M}$ onto itself such that for some $\alpha_x \in \mathrm{Aut}(G)$, we have $x(p_i) = p_{\alpha_x(i)} \quad \text{for all } i \in V(G)$.
- The set of all symmetry operations of a framework (G, p) on $\mathcal{M}$ forms a subgroup of the orthogonal group $O(\mathbb{R}^d)$.
- If there exists an action $\theta : S \to \mathrm{Aut}(G)$ so that $x(p(v)) = p(\theta(x)(v))$ for all $v \in V(G)$ and all $x \in S$ then (G, p) is *S-symmetric*.

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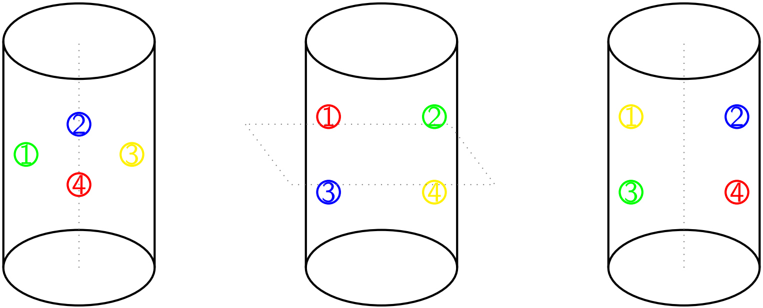
- (G, p) is S-generic (with n and hence p large enough) if $\mathrm{td}[\mathbb{Q}(p) : \mathbb{Q}(S)] = 2n/|S|$.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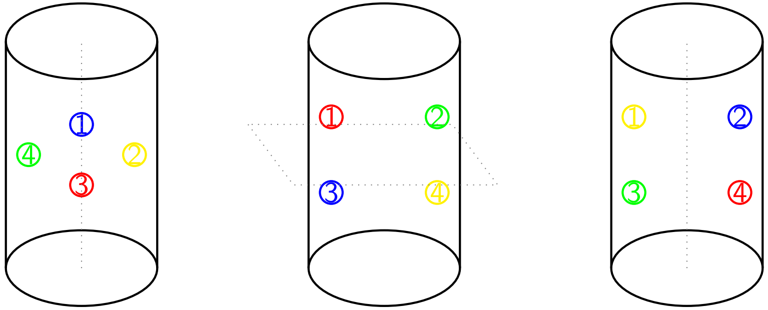

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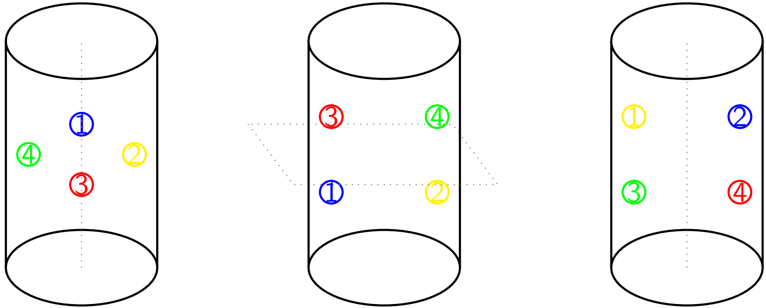

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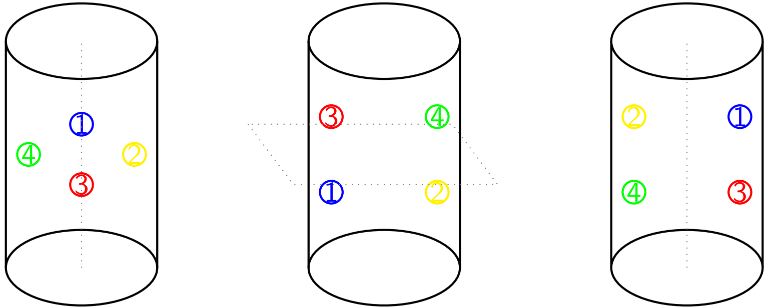
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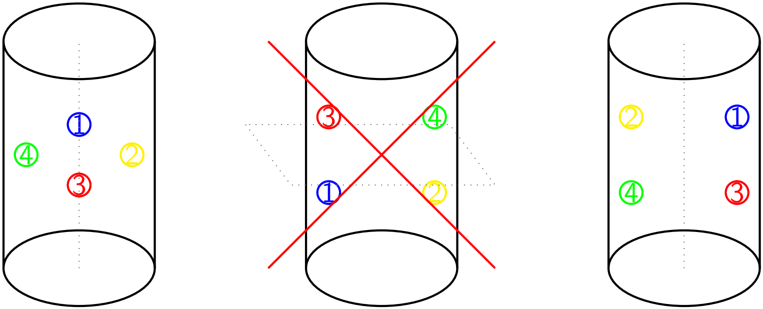

Figure : Restrict to groups about the z-axis

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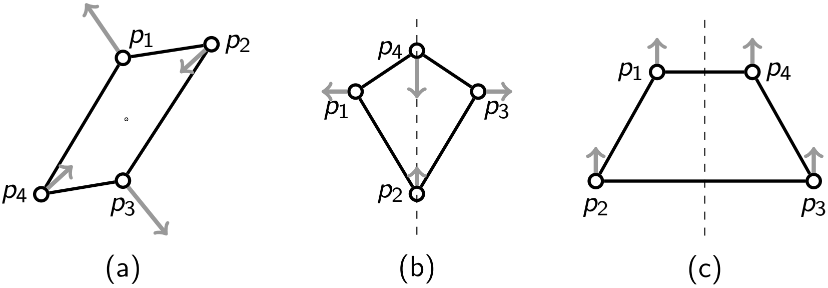
- (G, p) is *(forced) S-symmetrically continuously rigid* if every continuous edge-length preserving motion of the vertices that preserves the symmetry is an isometry of $\mathcal{M}$.
- For generic frameworks, symmetric infinitesimal rigidity and symmetric continuous rigidity coincide.



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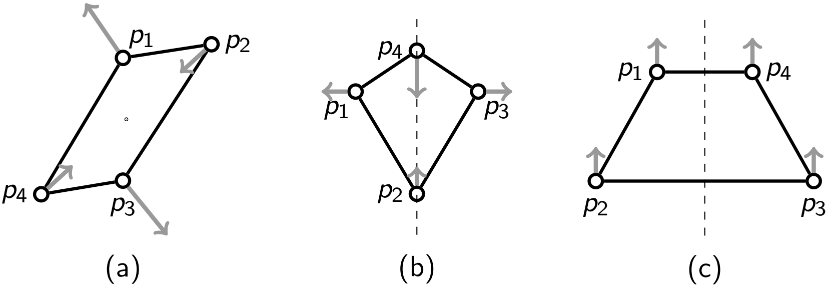

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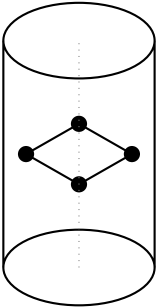

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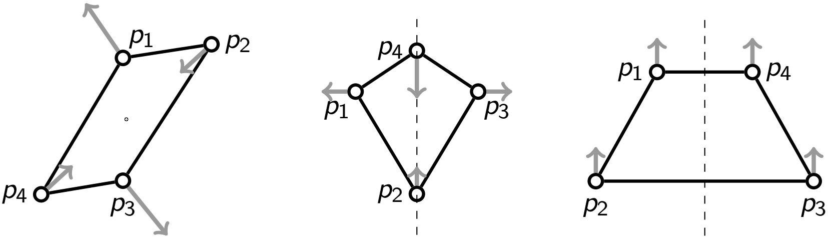

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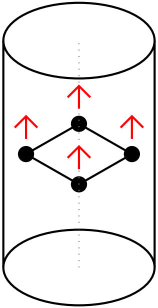

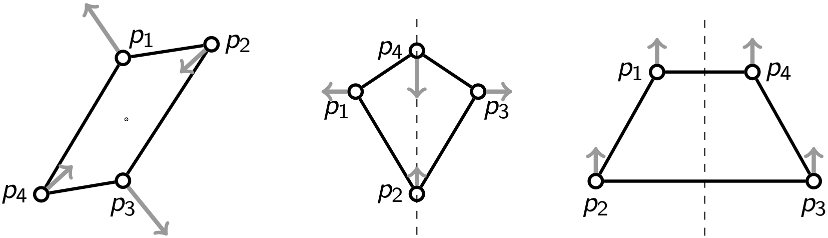

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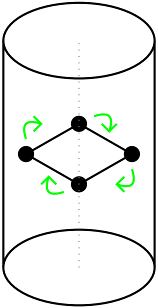

The Orbit Rigidity Matrix

For each edge orbit $Se = \{se|\, s \in S\}$ of G, the *orbit rigidity matrix* $\mathbf{O}(G, p, S)$ of (G, p) has the following corresponding ($3v_0$-dimensional) row vector:
If the two end-vertices of the edge e lie in distinct vertex orbits, then there exists an edge in Se that is of the form $\{a, sb\}$ for some $s \in S$, where $a, b \in O_{V(G)}$. The row we write in $\mathbf{O}(G, p, S)$ is:

$$\begin{pmatrix} 0 \ldots 0 & \overset{a}{\big(p_a - s(p_b)\big)} & 0 \ldots 0 & \overset{b}{\big(p_b - s^{-1}(p_a)\big)} & 0 \ldots 0 \end{pmatrix}.$$

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$$\begin{pmatrix} 0 \ldots 0 & \overset{a}{\big(2p_a - s(p_a) - s^{-1}(p_a)\big)} & 0 \ldots 0 \end{pmatrix}.$$

The Orbit Surface Matrix

Let (G, p) be a framework with quotient S-gain graph (G_0, Φ). The *orbit-surface rigidity matrix* $\mathbf{O}_{\mathcal{M}}(G, p, S)$ of (G, p) is the $(|E(G_0)| + |V(G_0)|) \times 3|V(G_0)|$ block matrix

$$\begin{bmatrix} \mathbf{O}(G, p, S) \\ \mathcal{N}_0(p_0) \end{bmatrix}$$

where $\mathbf{O}(G, p, S)$ is the standard orbit rigidity matrix for the framework and symmetry group considered in $\mathbb{R}^3$ and $\mathcal{N}_0(p_0)$ represents the surface normals to the framework joints corresponding to the vertex representatives.

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Theorem: N. and Schulze 2013+

Let (G, p) be a S-symmetric framework on $\mathcal{M}$. The solutions to $\mathbf{O}_{\mathcal{M}}(G, p, S)u = 0$ are isomorphic to the space of S-symmetric infinitesimal motions of (G, p).

Necessary Conditions

Theorem: Malestein and Theran 2012+, Jordan, Kaszanitsky and Tanigawa 2012+, N. and Schulze 2013+

Let $\mathcal{M}$ be an irreducible algebraic variety admitting ℓ isometries. Let S be a cyclic symmetry group of $\mathbb{R}^3$ acting on $\mathcal{M}$ such that under S, $\mathcal{M}$ admits ℓ_S symmetric isometries. Let (G, p) be a framework on $\mathcal{M}$ with quotient S-gain graph (G_0, Φ). Let (G, p) be a generic forced-S-symmetric isostatic framework. Then G_0 satisfies:

1. $|E(G_0)| = 2|V(G_0)| - \ell_S$
2. $|E(G_0')| \leq 2|V(G_0')| - \ell_S$ for every unbalanced subgraph G_0' and
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A similar results holds for dihedral groups and products of cyclic groups. However the subgraph conditions are more complicated.

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- Malestein and Theran 2012+ proved forced symmetry rigidity results for C_n, C_s symmetry in the plane using matroid representability techniques.
- Jordan, Kaszanitsky and Tanigawa 2012+ proved, again for frameworks in the plane:
    - forced rigidity results for C_n, C_s using inductive constructions,
    - the D_h (odd order) case and
    - they found counterexamples (to the natural class of graphs) for even order dihedral groups.

Spheres

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- So theorems for symmetry groups in the plane 'lift' to theorems for symmetries of the sphere.
- There are symmetries of the sphere that do not occur as symmetry groups in the plane. These groups can be covered using our method.

Cylinders

An S-gain graph G_0 is (k, ℓ, ℓ_S)-*gain-tight* if it satisfies

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- About the z-axis, we want
  - $(2,2,2)$-gain tight for C_n symmetry on the cylinder,
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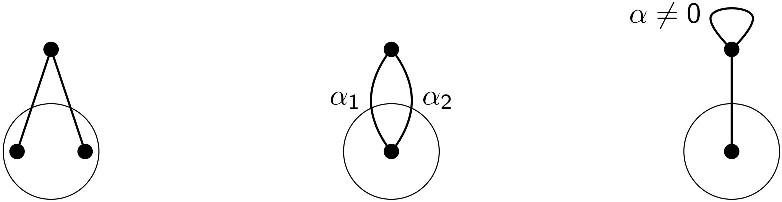
- We want to recursively characterise all such graphs using simple operations.

Henneberg operations



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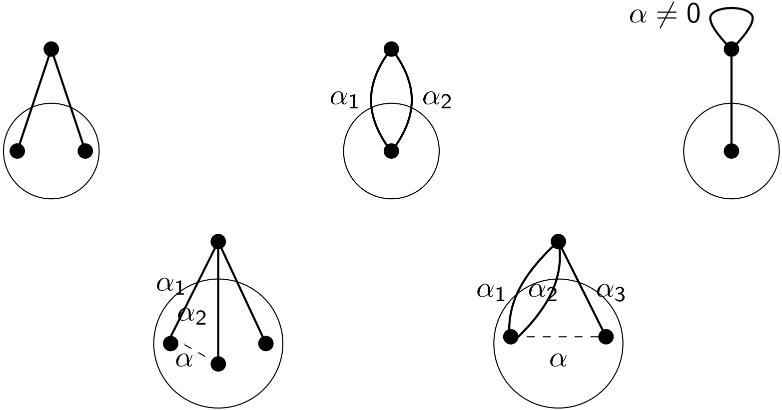


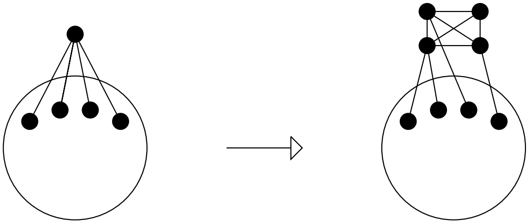
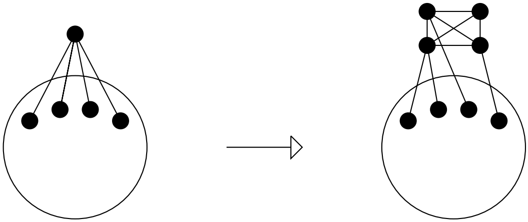


Figure : The vertex-to-K_4 move, in this case expanding a degree 4 vertex.



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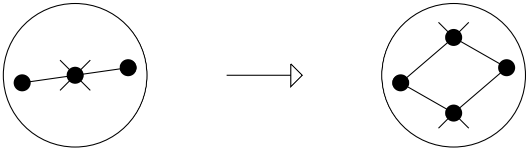

Figure : The vertex-to-4-cycle operation.

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Let S be a cyclic group, G a simple graph and G_0 the corresponding S-gain graph. Then G_0 is $(2,2,2)$-gain-tight if and only if G_0 can be constructed sequentially from K_1 by H1a, H1b, H2a, H2b, vertex-to-K_4 and vertex-to-4-cycle operations.

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I'll now sketch the proof for the $(2,2,1)$-gain-tight case. The $(2,2,2)$-gain-tight case is simpler.

Idea of the Proofs 1

To *switch* v with $s \in S$ means to change the gain function Φ on $E(H)$ as follows:

$$\Phi'(e) = \begin{cases} s \cdot \Phi(e) \cdot s^{-1} & \text{if } e \text{ is a loop incident with } v \\ s \cdot \Phi(e) & \text{if } e \text{ is a non-loop incident from } v \\ \Phi(e) \cdot s^{-1} & \text{if } e \text{ is a non-loop incident to } v \\ \Phi(e) & \text{otherwise} \end{cases}$$

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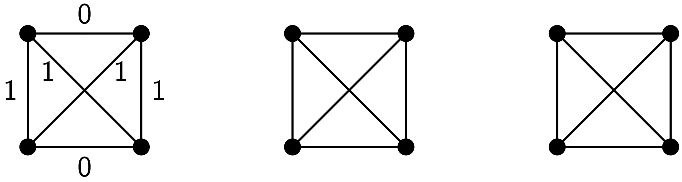

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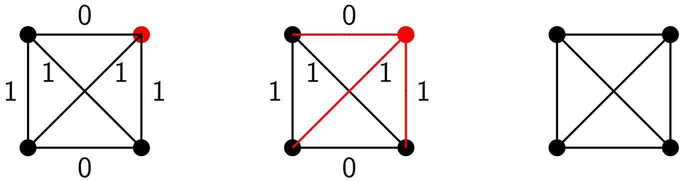

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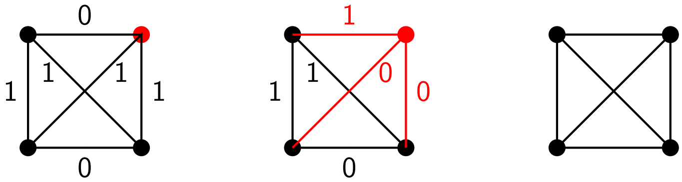

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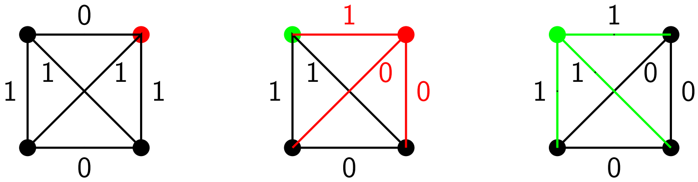

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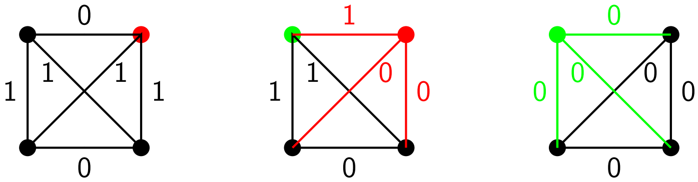

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- In each case the minimum degree is 2 or 3.
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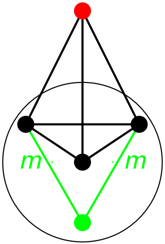
- H2a and H2b work unless all degree 3 vertices are contained in copies of K_4 with 0-gains on each edge.

Idea of the proofs 3

- Now we try to contract a copy of K_4.
- If this fails there are vertices $a, b \in K$ and a vertex $x \notin K$ such that $(ax)_m, (bx)_m \in E(H_0)$.



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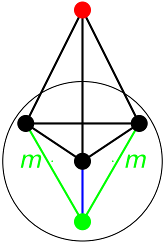

- Let c be the final vertex in K. In the $(2, 2, 2)$-gain-tight we are done. In the $(2, 2, 1)$-gain-tight it must be that $cx \in E(H_0)$.
- Repeat for each degree 3 vertex to show that H_0 contains a bridge.

Conjectures

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Let $\mathcal{M}$ be the unit cylinder defined by the polynomial $x^2 + y^2 = 1$. Let S be the cyclic group C_n representing n-fold rotation around the z-axis. Let (G, p) be a framework on $\mathcal{M}$ with quotient S-gain graph (G_0, Φ). Then (G, p) is minimally rigid if and only if G_0 is $(2, 2, 2)$-gain-tight.

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Let $\mathcal{M}$ be the unit cylinder defined by the polynomial $x^2 + y^2 = 1$. Let S be order 2 reflection group with mirror a plane containing the z-axis. Let (G, p) be a framework on $\mathcal{M}$ with quotient S-gain graph (G_0, Φ). Then (G, p) is minimally rigid if and only if G_0 is $(2, 2, 1)$-gain-tight.

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- about the z-axis, we want
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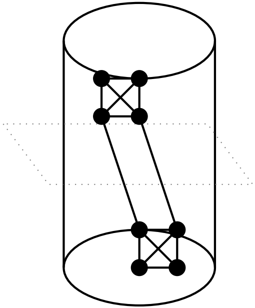
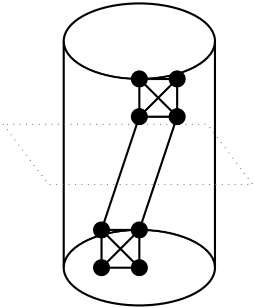
- To prove the 3 conjectures it remains to show that we can apply the inductive operations symmetrically and preserve rigidity.

Extensions

- Firstly the restriction to groups about the z-axis of the cylinder or cone was just for convenience.
- Groups with symmetry plane orthogonal to the z-axis are doable, but can have different isometry counts.
  - Reflection about a plane orthogonal to the cone is easier than reflection through the axis.
  - 2-fold rotation orthogonal to the z-axis of the cylinder.
    - The necessary count is $(2, 2, 0)$-gain-tight.
    - These are 4-regular and hence require X-replacement and variants...





- Abelian groups for the cylinder or cone with no isometries.
  - These require $(2, \ell, 0)$-gain-tight graphs...
- Dihedral groups. Are there generalisations of Jordan, Kaszanitsky and Tanigawa's counterexamples?
- Extensions from the cone to other surfaces admitting exactly 1 isometry: the torus, the elliptical cylinder, parabaloids, helicoids,...
  - For example: any minimally rigid framework (G, p) on the cone with mirror symmetry about a plane through the z-axis must have quotient graph H being $(2, 1, 0)$-gain-tight. However for the elliptical cylinder, the same group must have $(2, 1, 1)$-gain-tight quotient graphs.
- The ellipsoid and other surfaces admitting no isometries.

Thanks for listening!