Arrangements of pseudocircles and circles

Ross J. Kang*

Utrecht University

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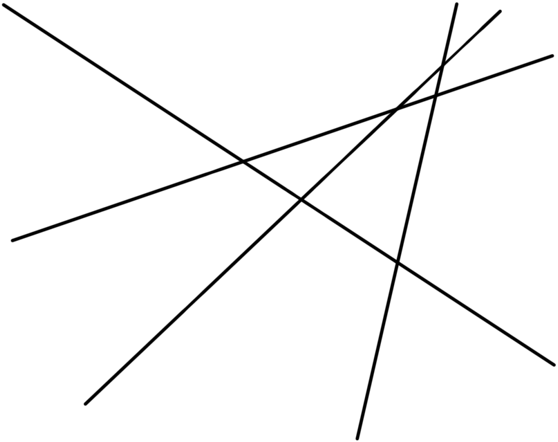
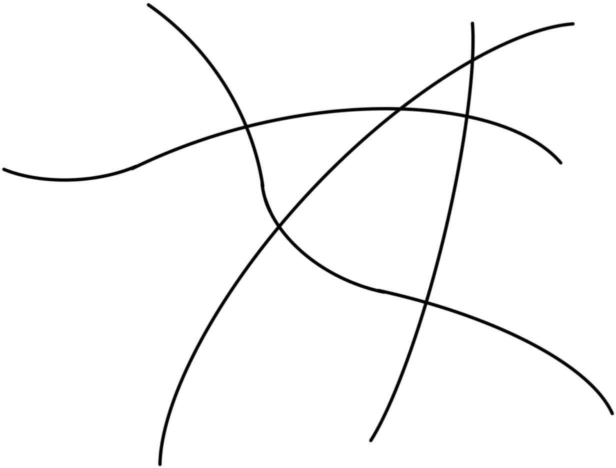
*Joint work with Tobias Müller (Utrecht). Support from NWO.

A line arrangement



A pseudoline arrangement



Pseudoline arrangements

A *pseudoline* is the image of a line under a homeomorphism of $\mathbb{R}^2$.
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2. When two pseudolines meet, they must cross.

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The study of such objects apparently goes back at least to 1826 (J. Steiner, vol. 1 of Crelle's).

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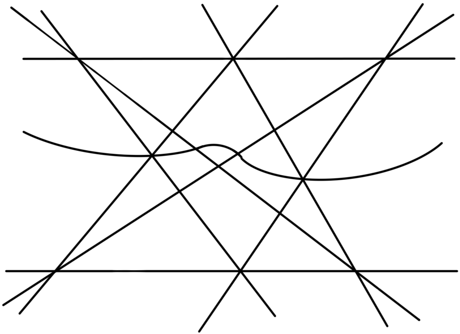

F.W. Levi (1926). Hint: consult Pappus of Alexandria (c. 340 AD).

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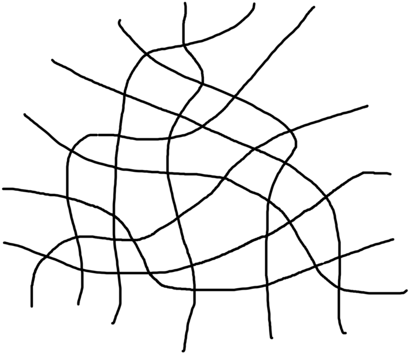

Ringel (1956).

Small pseudoline arrangements are stretchable

Goodman and Pollack (1980), after a conjecture of Grünbaum, showed every arrangement of eight pseudolines is equivalent to a line arrangement. In other words, every arrangement of eight pseudolines is *stretchable*.

Richter-Gebert (1989) showed Ringel's example is the unique non-stretchable simple arrangement of nine pseudolines.

STRETCHABILITY

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Theorem (Mnëv, 1988, cf. Shor, 1991)

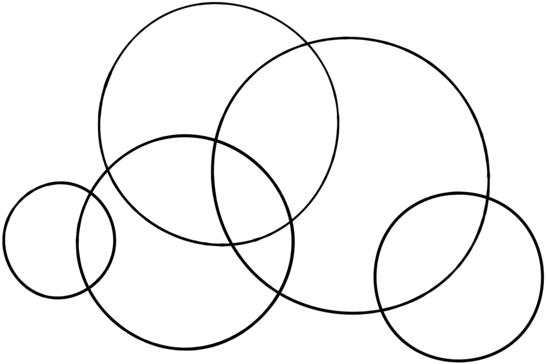
SIMPLE STRETCHABILITY is NP-hard.

Note 1: The theorem may be viewed as a corollary to a deep topological theorem of Mnëv, but Shor gave a simpler and direct proof.

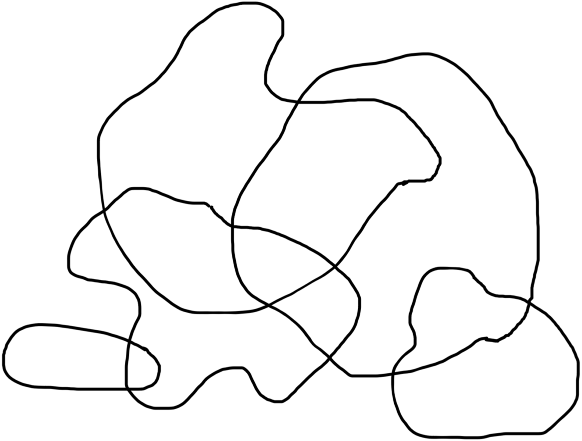
Note 2: In fact, SIMPLE STRETCHABILITY is complete for the computational class "existential theory of the reals"[†].

[†] Given an existential first-order sentence over the real numbers, is it true?

A circle arrangement



A pseudocircle arrangement



Pseudocircle arrangements

A *pseudocircle* is a Jordan curve.
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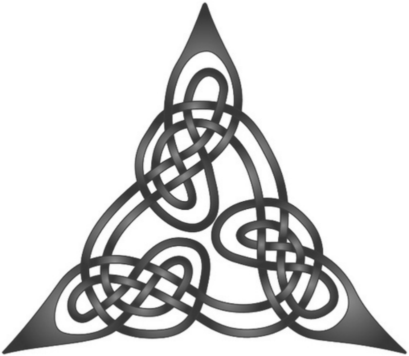
Note for context: one may interpret simple line arrangements as arrangements of great circles on a sphere.

Not a pseudocircle arrangement



St. John's knot.

Pseudocircle arrangements

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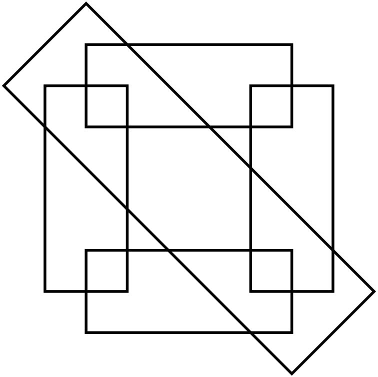


Linhart and Ortner (2005) conjectured this to be the smallest non-circleable arrangement.

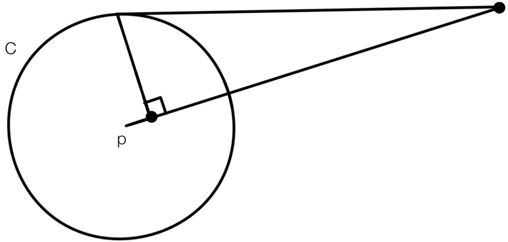
Small pseudocircle arrangements are circleable

Theorem (K and Müller)

Every pseudocircle arrangement on four pseudocircles is circleable.

The (lengthy) case analysis to prove this is much more involved than Goodman and Pollack's (compact) argument for pseudoline arrangements, but fortunately it is shortened by the use of circle inversions.

Circle inversions



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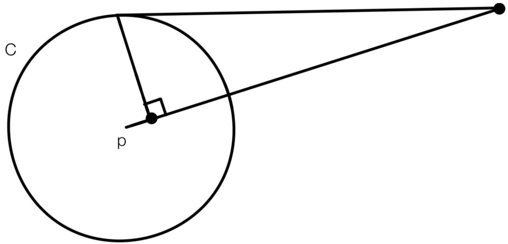

Fun facts about an inversion in a circle C with centre p: It

- exchanges the interior and exterior of C;
- maps circles through p to lines, circles not through p to circles, lines through p to lines, lines not through p to circles;
- is conformal.

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Notes on CIRCLEABILITY and CONVEXIBILITY

Note 1: Circle inversions come in handy again, as do wiring diagrams of pseudoline arrangements.

Note 2: The polynomial-time reductions are from SIMPLE STRETCHABILITY, so these computational recognition problems are also complete for "existential theory of the reals".

Note 3: Indirectly, the last theorem shows there to be a non-convexible pseudocircle arrangement, answering another question of Linhart and Ortner (2008). The theorem also directly answers a stronger, earlier question of Bultena, Grünbaum and Ruskey (1998).

One more pseudocircle result

We learned of the following conjecture, of Russian folklore, from Artem Pyatkin.

> *Given an arrangement of convex pseudocircles, the corresponding intersection graph can be realised as the intersection graph of a collection of circles.*

(The intersection graph has vertex set the collection of pseudocircles, two vertices adjacent if the corresponding pseudocircles intersect.)

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Theorem (K and Müller)

The following depicts a counter-example to the above conjecture.

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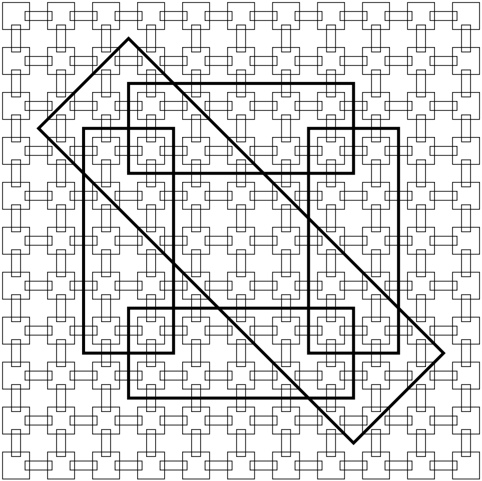

Open problems

1. What are all the smallest non-circleable pseudocircle arrangements?
2. What is the size of a smallest non-circleable arrangement of (simple) convex pseudocircles?
3. What is the size of a smallest non-convexible arrangement of (simple) pseudocircles?
4. What is the size of a smallest convex pseudocircle arrangement, the corresponding intersection graph of which cannot be realised as the intersection graph of a collection of circles?
5. What is the computation complexity of recognising if a convex pseudocircle arrangement is realisable as the intersection graph of a collection of circles?

Thank you!