A conjecture of Thomassen on Hamilton cycles in highly connected tournaments

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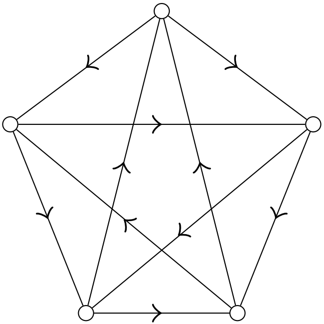
Joint work with Daniela Kühn, John Lapinskas, and Deryk Osthus

Basics



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- A Hamilton cycle (HC) in T is a consistently oriented cycle through every vertex of T.
- Interested in edge-disjoint Hamilton cycles

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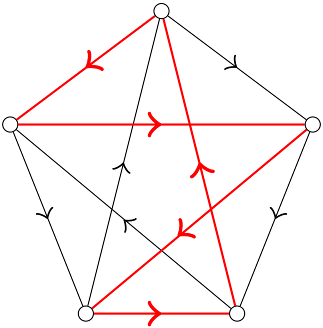

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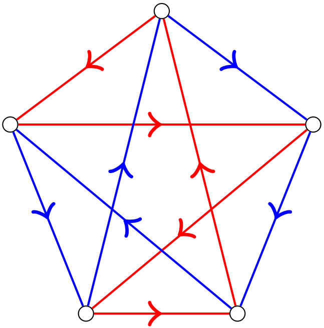

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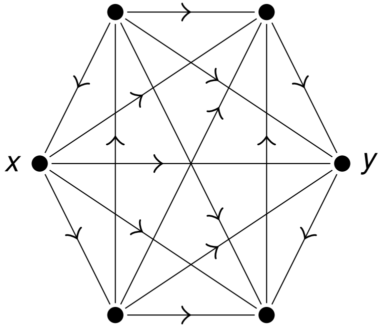
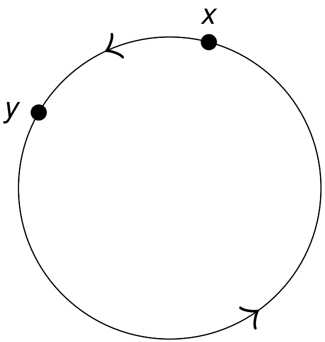
- What conditions guarantee a Hamilton cycle in a tournament?
- What prevents a Hamilton cycle in a tournament?





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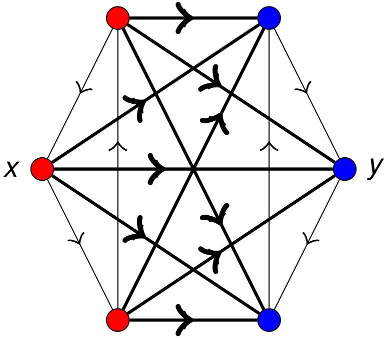

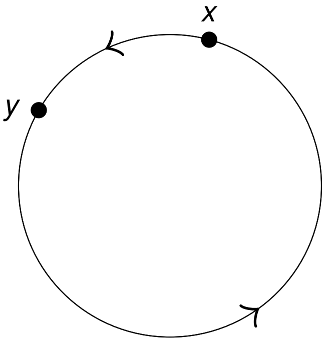

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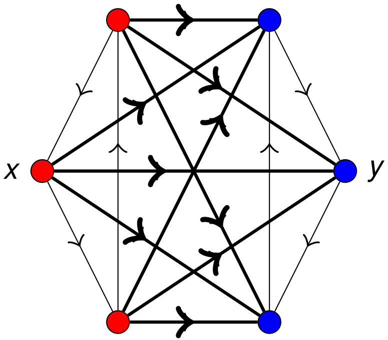

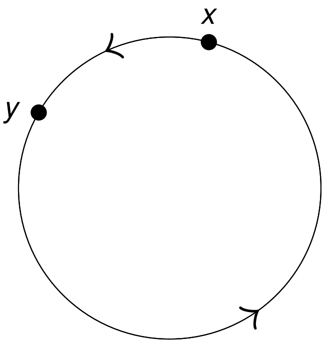

- No HC since no path from y to x.
- i.e. the tournament is not strongly connected.

Connectivity

A tournament T is strongly connected if for all $x, y \in V(T)$, $\exists$ a path from x to y and $\exists$ a path from y to x.

- Have seen T contains a HC $\implies$ T strongly connected.

Theorem (Camion, 1959)

If T is strongly connected then T contains a HC.

- The proof is about half a page long.

Can we get more (edge-disjoint) HCs?

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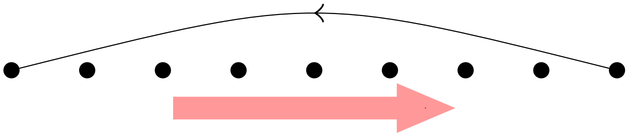

More edge-disjoint HCs

We need a stronger condition to force more edge-disjoint HCs.

A tournament T is strongly r-connected if deleting any $r-1$ vertices keeps T strongly connected.

Conjecture (Thomassen, 1982)

$\forall k, \exists f(k)$ s.t. if T is a strongly $f(k)$-connected tournament, then T contains k edge-disjoint HCs.

Know $f(1)=1$. Not known whether $f(2)$ exists.

Theorem (Kühn, Lapinskas, Osthus, P., 2013+)

For every k, $f(k)$ exists, and we can take $f(k) = O(k^2 \log^2 k)$.

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- Every strongly $(10^{12}k^2 \log^2 k)$-connected tournament contains k edge-disjoint HCs.
- $\forall k$, $f(k) > k^2/4$:
  there exists a tournament that is
  - strongly $k^2/4$-connected,
  - does not contain k edge-disjoint HCs.

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Related results

Thomassen's Conjecture with edge-connectivity?

- $\forall k$, $\exists g(k)$ s.t. every strongly $g(k)$-edge-connected tournament contains k edge-disjoint HCs.
- False. (Thomassen, 1982)

Kelly's Conjecture

- How many edge-disjoint HCs are we guaranteed in a highly connected tournament?
- Regular tournaments are highly connected.

Conjecture (Kelly, 1968)

Every regular tournament on n (odd) vertices has a Hamilton decomposition, i.e. $(n-1)/2$ edge-disjoint HCs.

- Proved by Kühn, Osthus, 2013.
- Thomassen's Conjecture motivated by Kelly's Conjecture.

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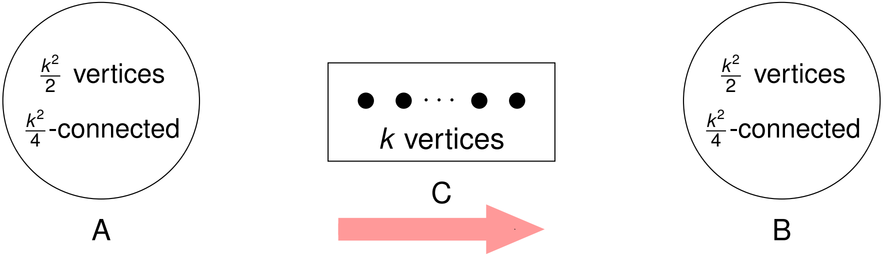

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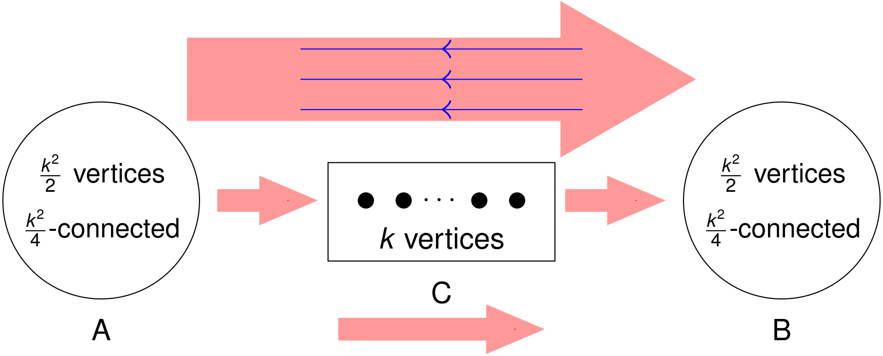

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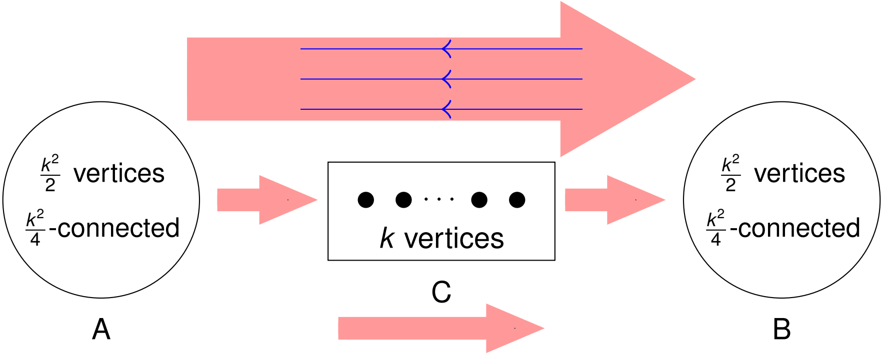

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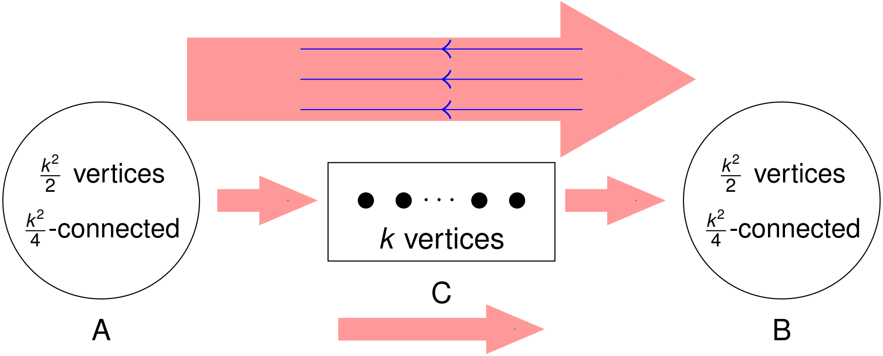

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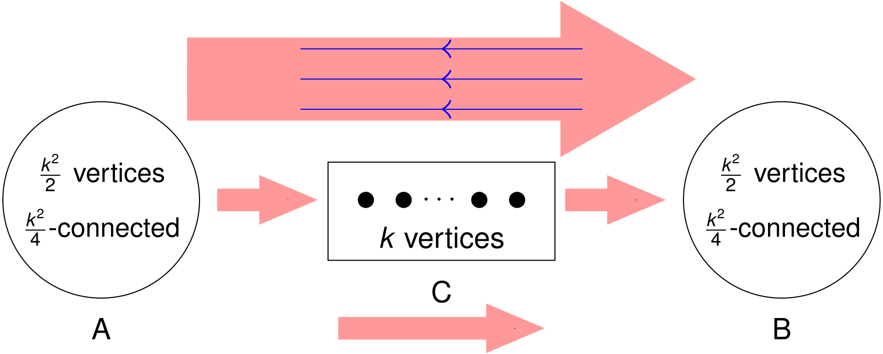

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- T_k is strongly $k^2/4$-connected
- T_k does not contain k edge-disjoint HCs: k edge-disjoint HCs require $> k^2/2$ backwards edges (because of C)

Sketch proof

- Will give a sketch of why $f(2) \leq 10^{13}$
- i.e. will sketch why every strongly 10^{13}-connected tournament contains 2 edge-disjoint HCs.
- Use essentially the same ideas for the full result.

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Are there any useful properties of all tournaments?

Theorem (Redei, 1934)

Every tournament has a Hamilton path, i.e. a consistently oriented path through every vertex.

Proposition

Let T be any tournament and let $D \subseteq T$ s.t. $\Delta(D) \leq t$. Then in $T - D$ there exist paths $Q_1, \ldots, Q_{t+1}$ s.t.

- *$Q_1, \ldots, Q_{t+1}$ are vertex-disjoint,*
- *$V(Q_1) \cup \cdots \cup V(Q_{t+1}) = V(T)$.*

The case $t = 0$ is Redei's Theorem.

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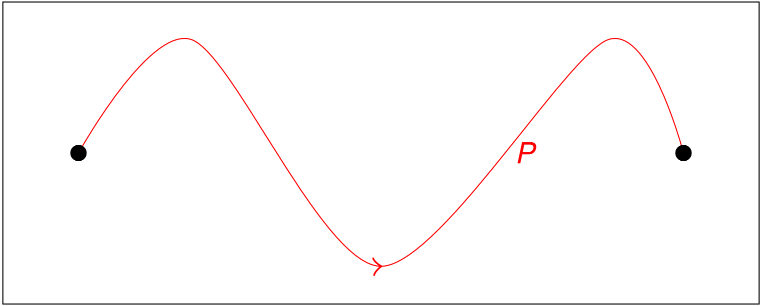
Given tournament T

- take P = Hamilton path, so $\Delta(P) = 2$



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- Proposition $\implies \exists\ Q_1, Q_2, Q_3$ s.t.
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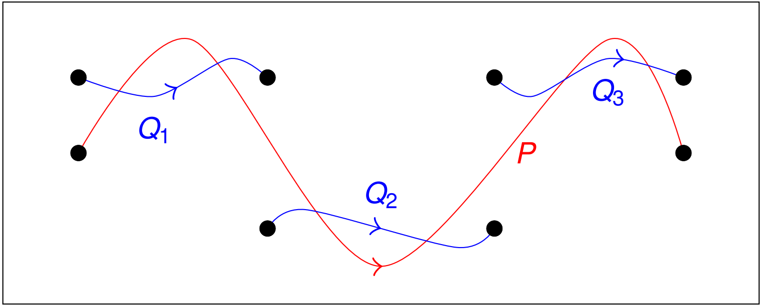
  - Q_1, Q_2, Q_3 edge-disjoint from P



Key idea of the proof - linking structure

Let T be a strongly 10^{13}-connected tournament

We can find a linking structure $L \subseteq T$ s.t.

- $|V(L)| \leq |V(T)|/100$
- $\Delta(L) \leq 4$

and where L has the following key property:

Given any 5 vertex-disjoint paths $P_1, \ldots, P_5$ outside of $V(L)$

a. each path can be extended into $V(L)$ with a single (suitable) edge
b. these extended paths can be connected into a cycle C using edges of L
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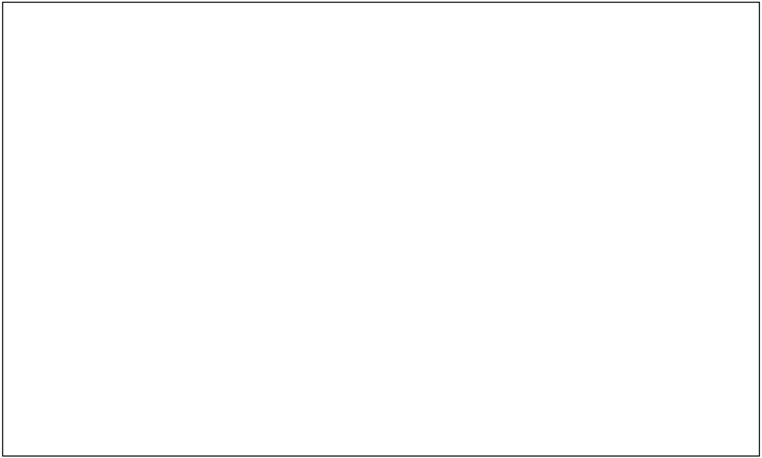

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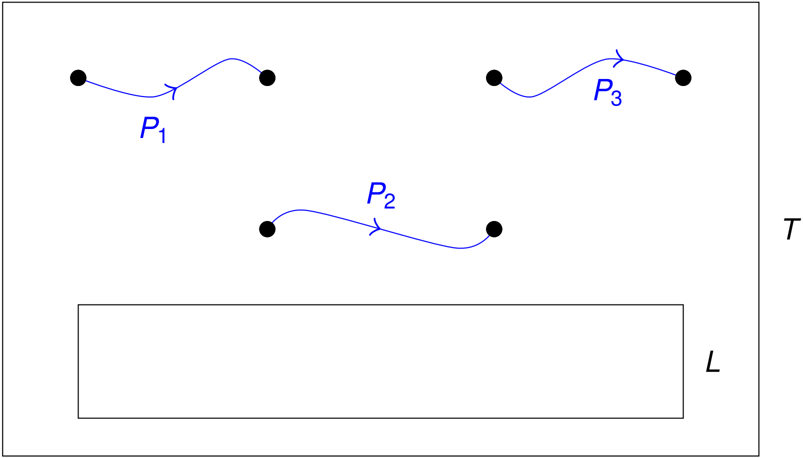

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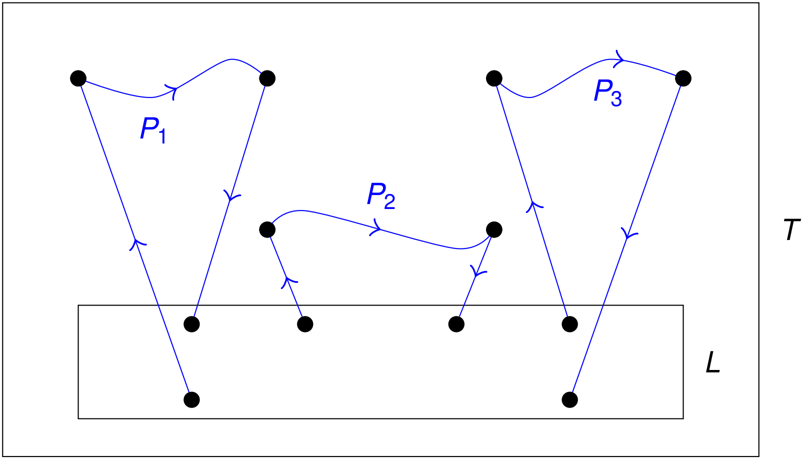

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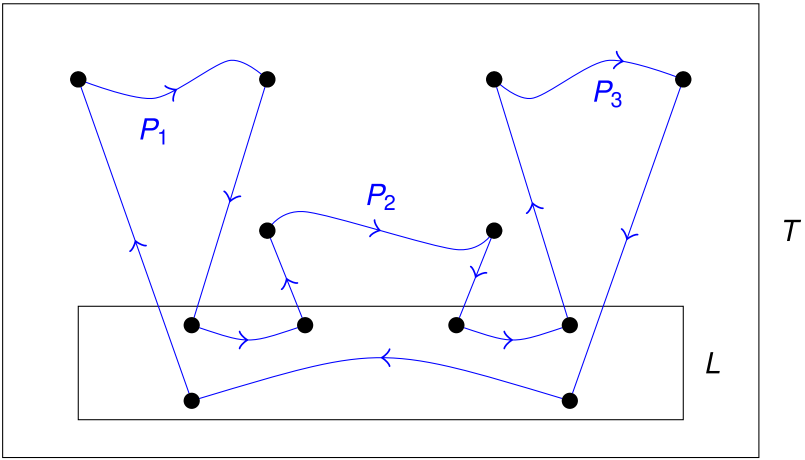

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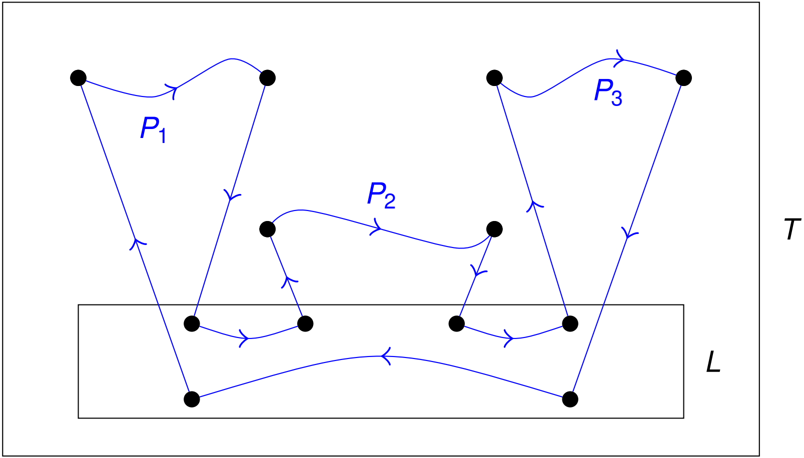
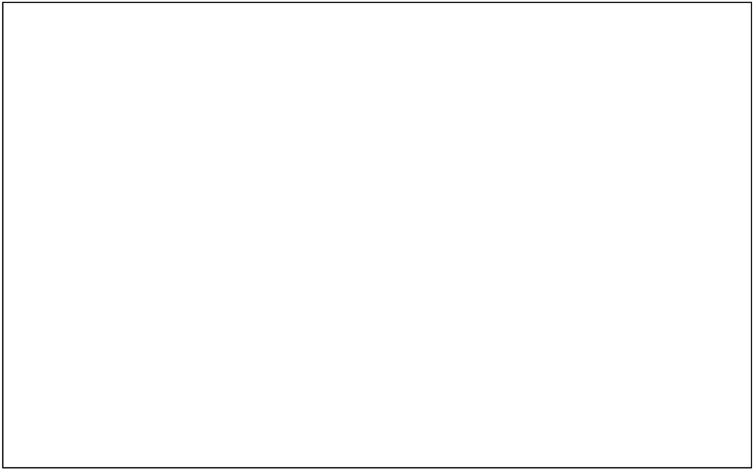


If P_1, P_2, P_3 cover all of $V(T) \setminus V(L)$ then we obtain a HC.

Obtaining two edge-disjoint HC

How do we find edge-disjoint HC_{red} and HC_{blue}?

- Find two vertex-disjoint linking structures L_{red} and L_{blue} in T.



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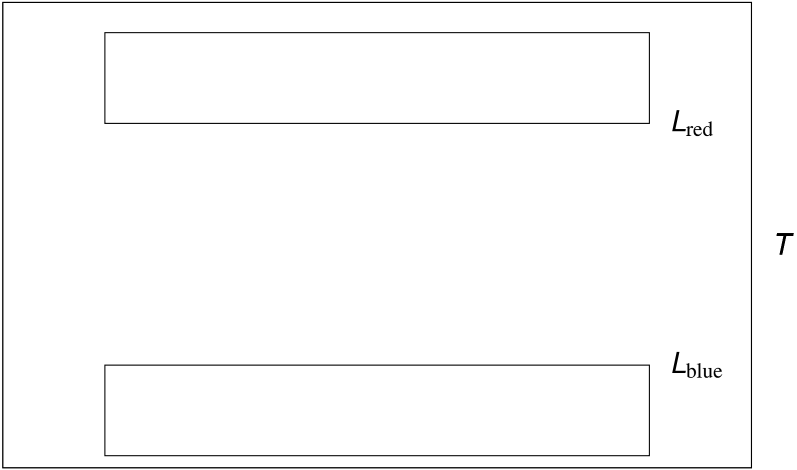
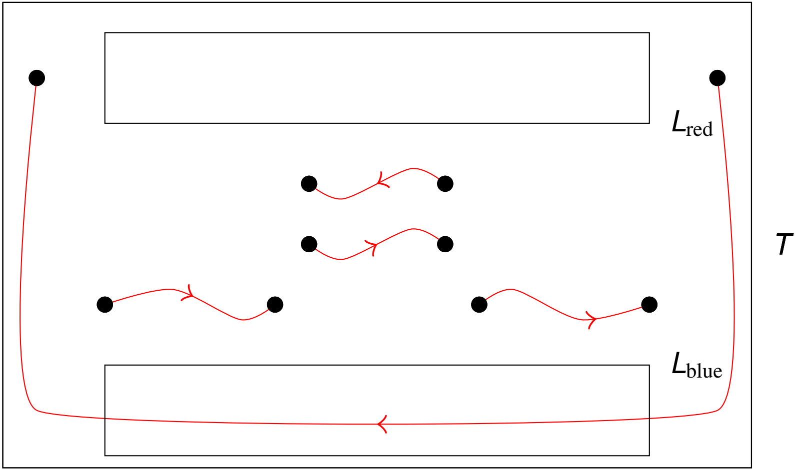


Obtaining two edge-disjoint HC

For HC_{red}: can find 5 vertex-disjoint red paths s.t.

- they cover $V(T) \setminus V(L_{\text{red}})$
- they are edge-disjoint from L_{blue} (recall $\Delta(L_{\text{blue}}) \leq 4$)



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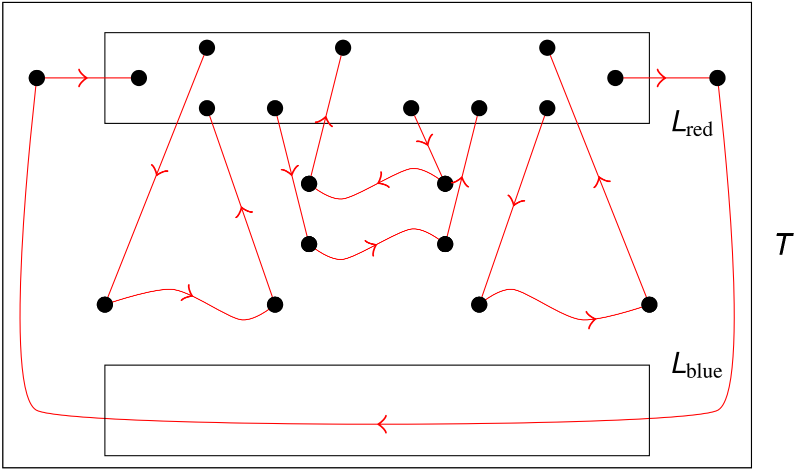

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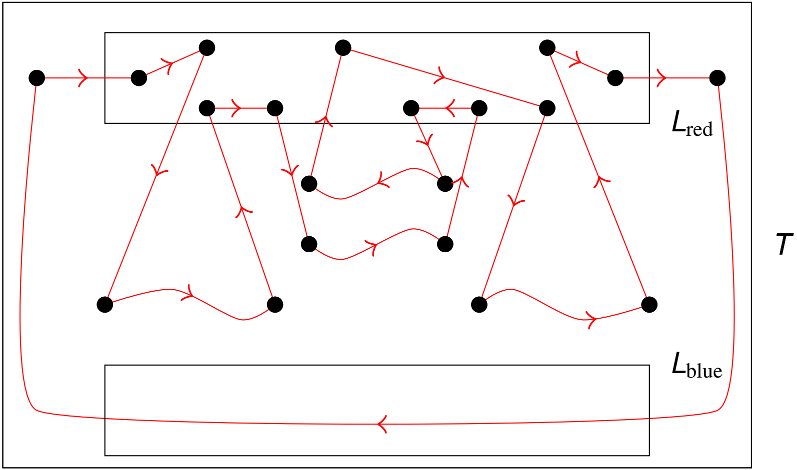

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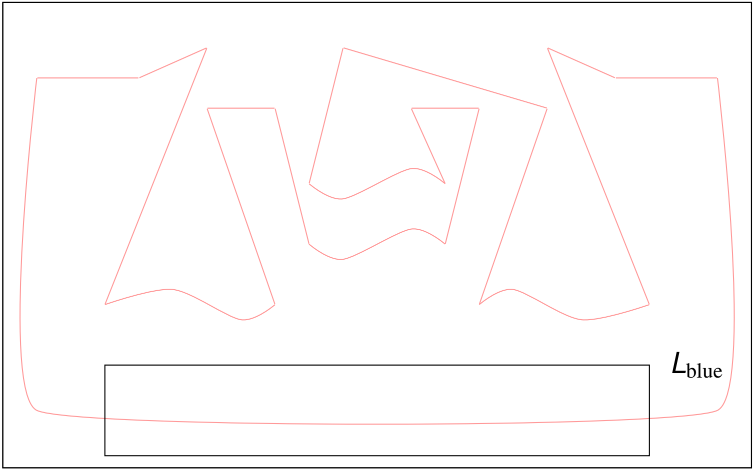

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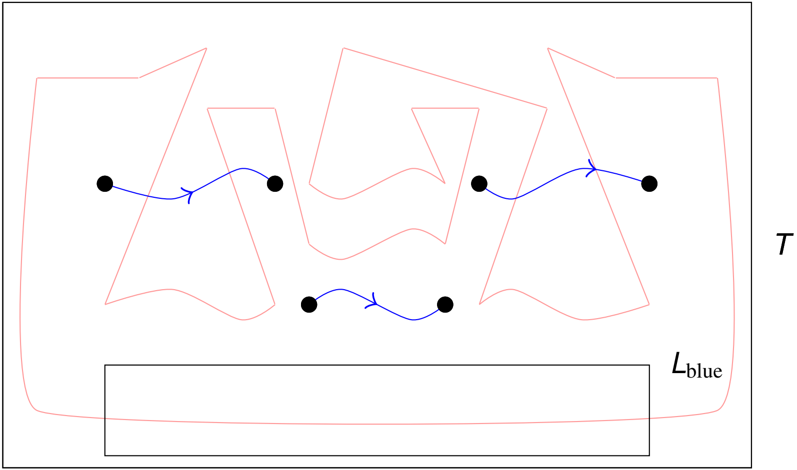

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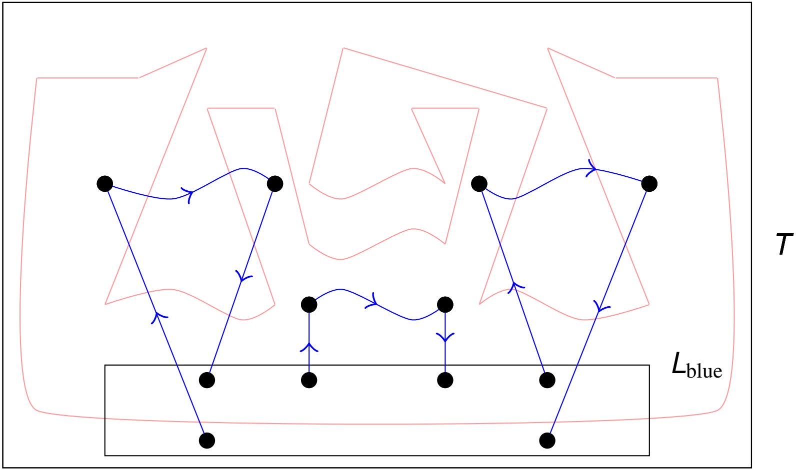

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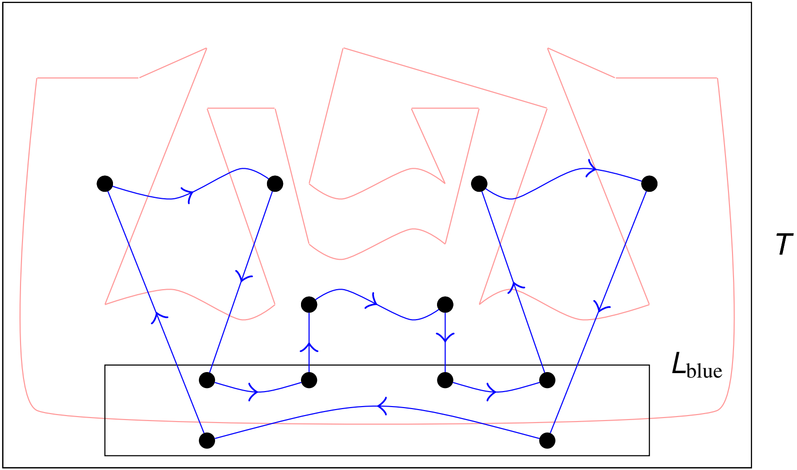
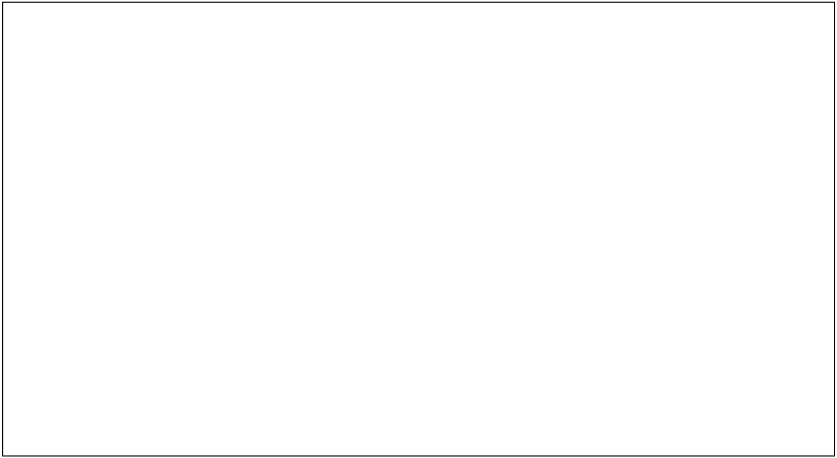


Key idea of the proof - linking structure

The linking structure L consists of



T

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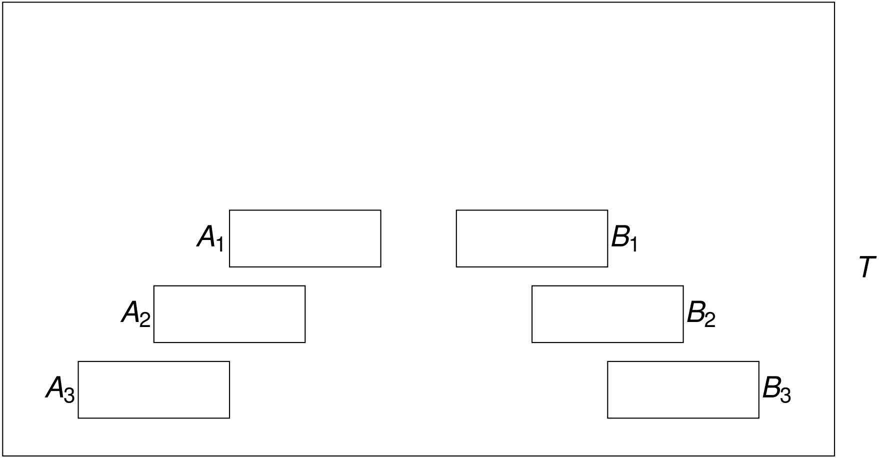
The linking structure L consists of

a 5 in-dominating sets $A_1, \ldots, A_5$ and 5 out-dominating sets $B_1, \ldots, B_5$



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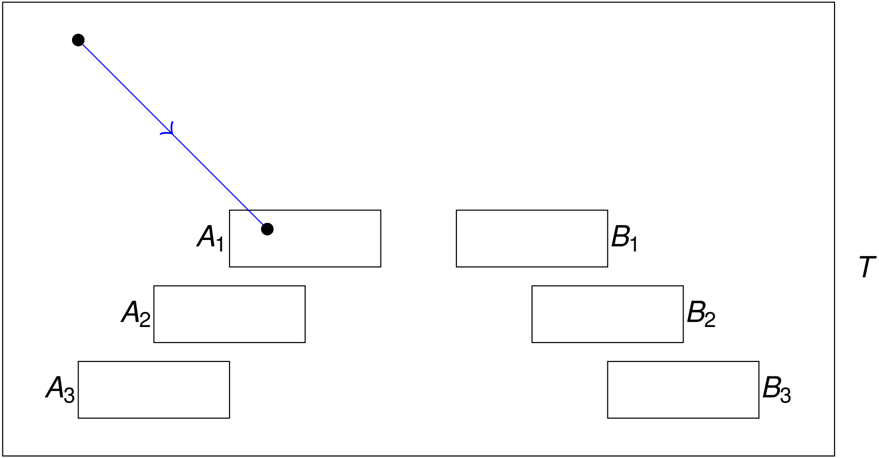

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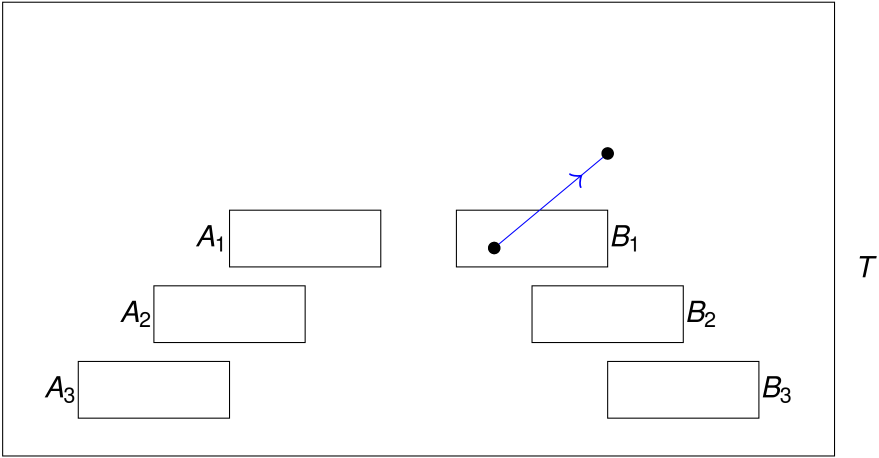

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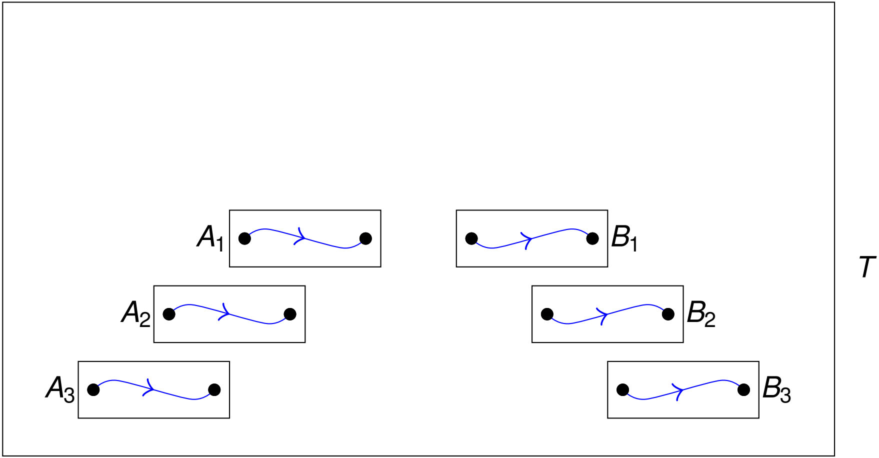

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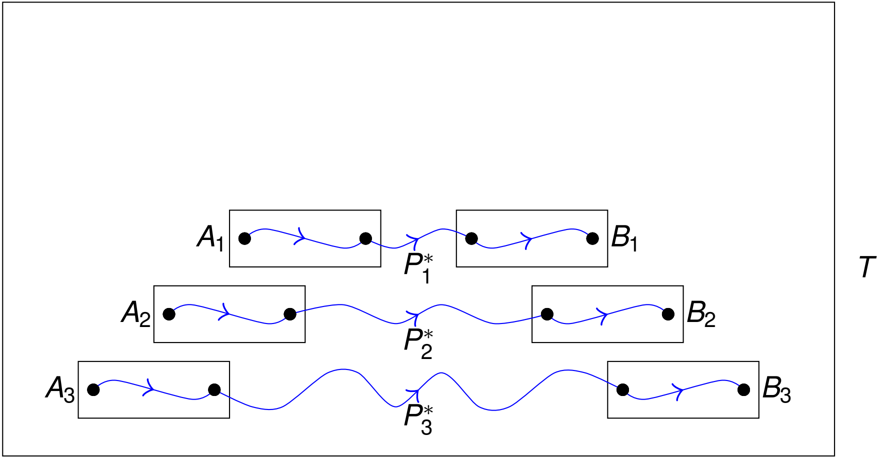

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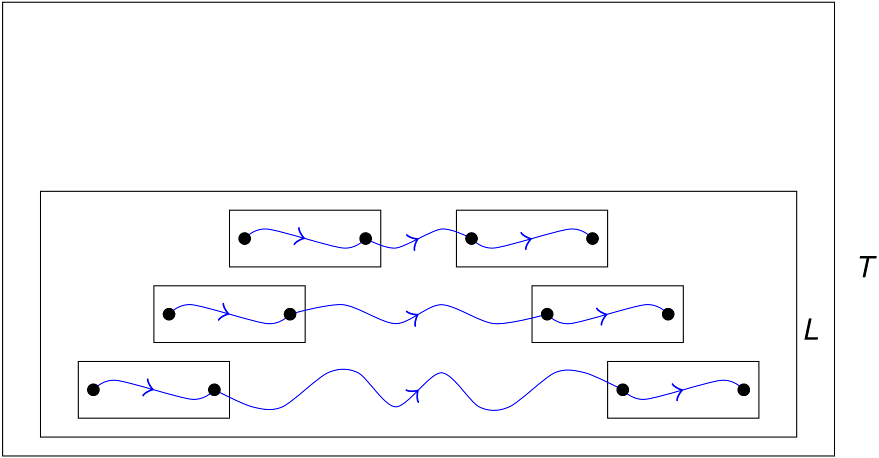

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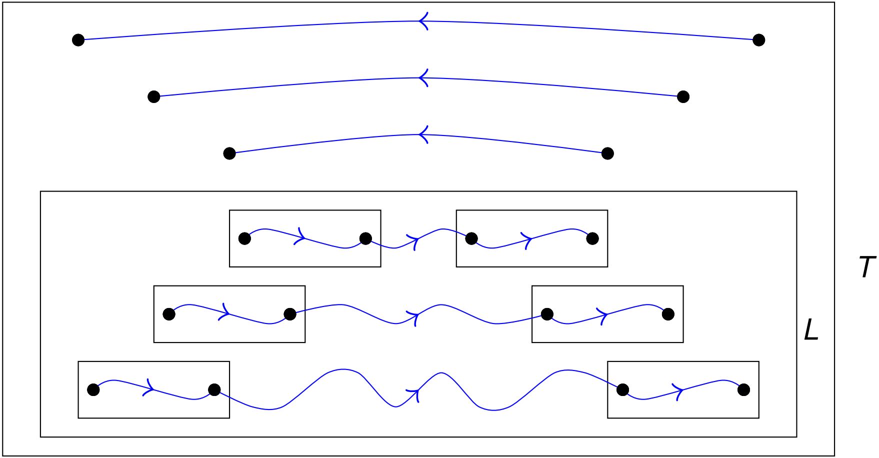

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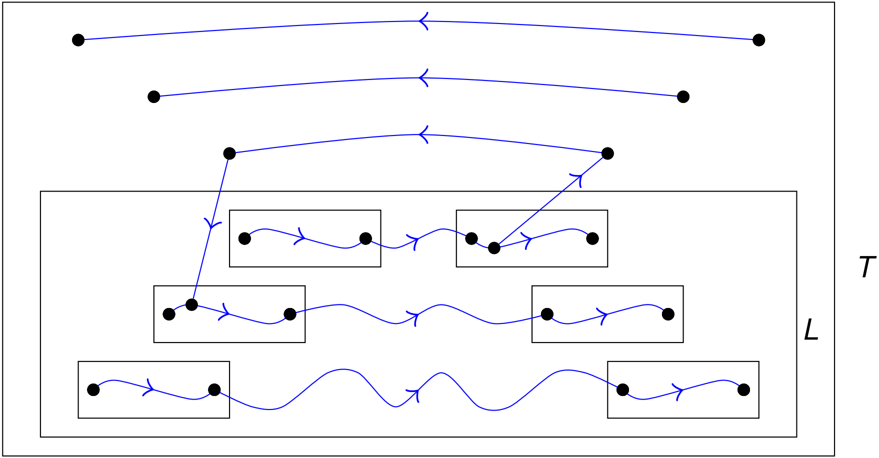

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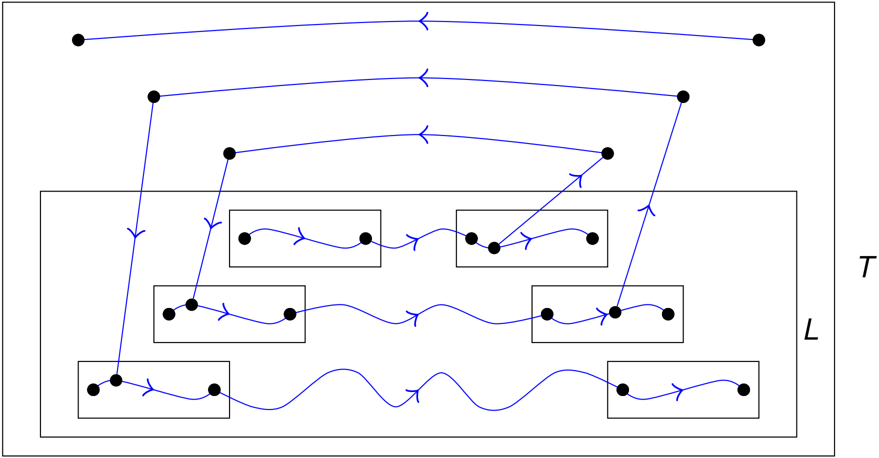

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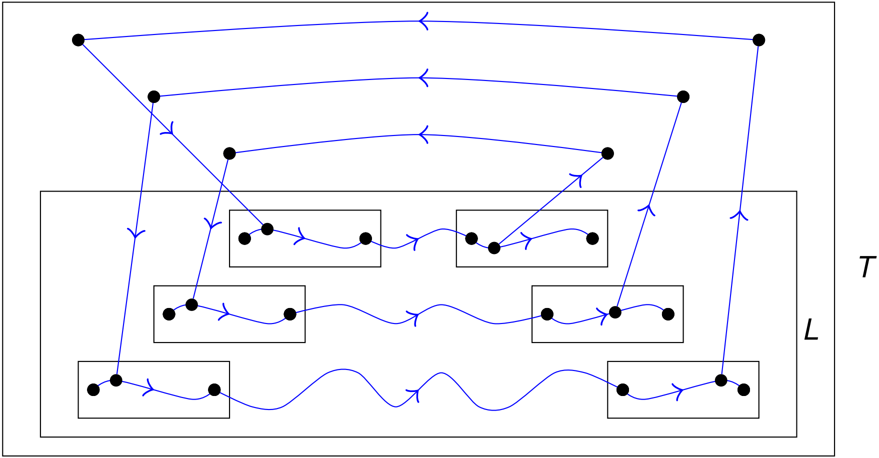
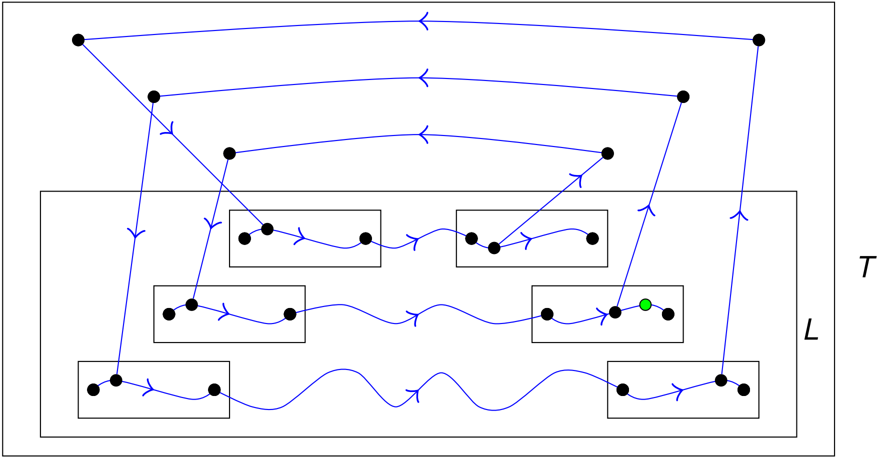


Key idea of the proof - linking structure

Given 3 paths outside *L* ...

- can extend paths into *L* to form cycle
- but the cycle does not contain all vertices of *L*.



Covering Edges

Method for absorbing vertices into cycles

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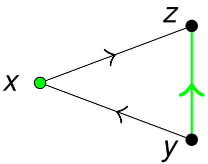
- Let $x \in V(T)$ and $yz \in E(T)$.
- yz is a covering edge for x if $yx, xz \in E(T)$.



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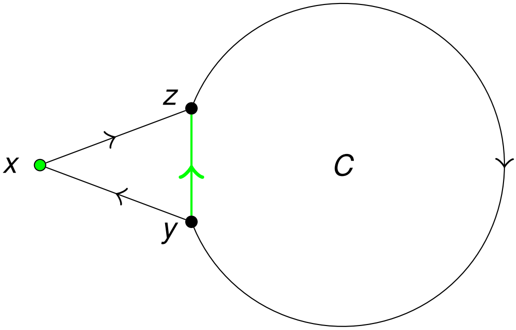

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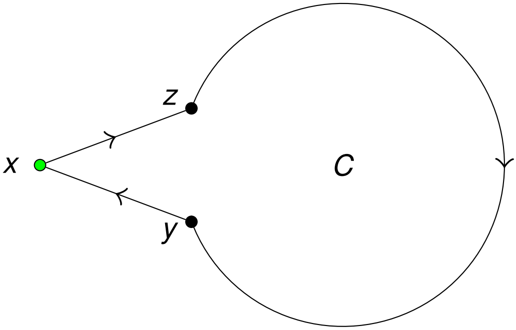

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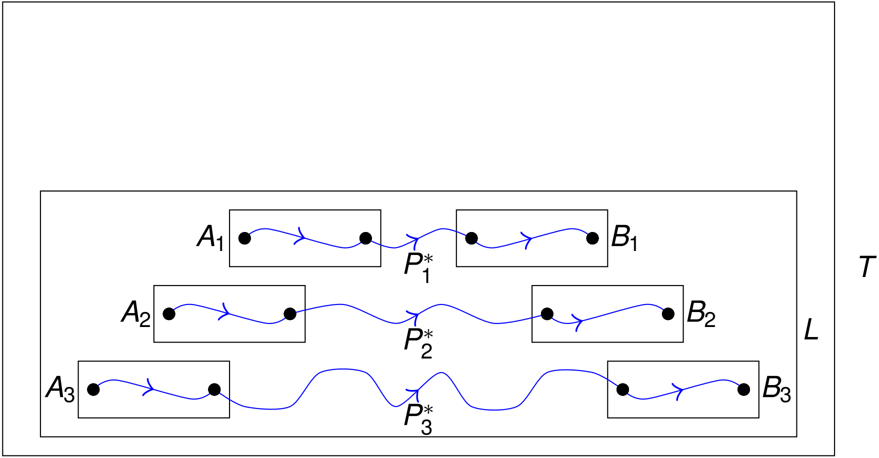
L consists of dominating sets linked by paths and ...



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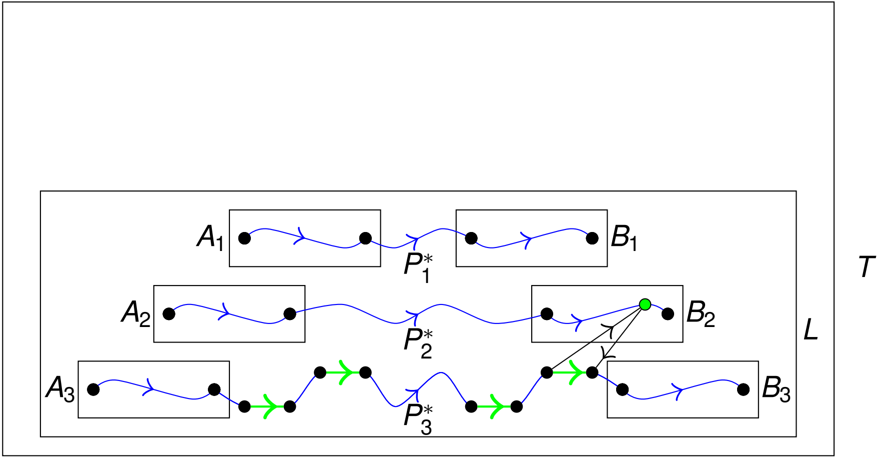
L consists of dominating sets linked by paths and ...

- c a distinct covering edge on P_3^* for each vertex in our dominating sets



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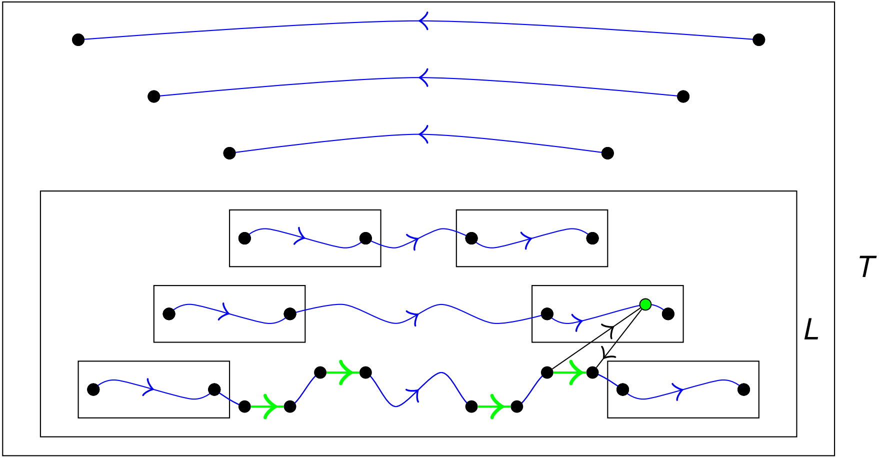

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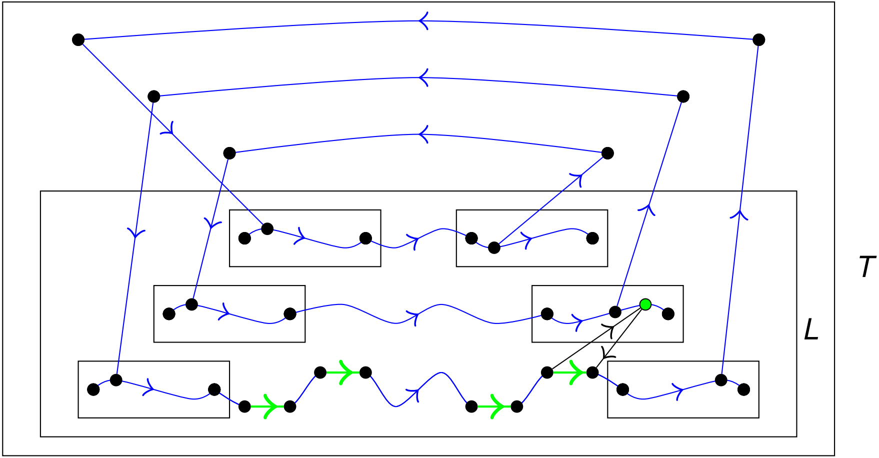
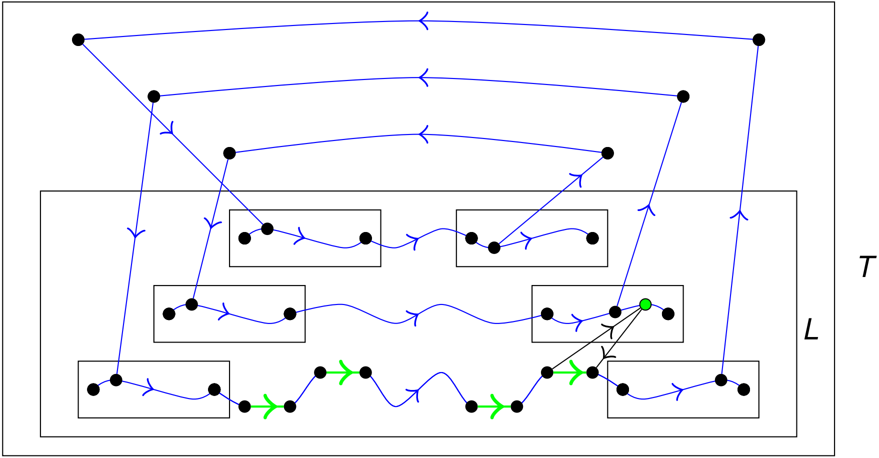


Key idea of the proof - linking structure

Given 3 paths outside L ...

- can extend paths into L to form cycle C
- and use covering edges to absorb any missing vertices into C.



Linking structure and connectivity

This completes the proof sketch except ... where do we use connectivity?

- Use strong connectivity to construct P_i^*.
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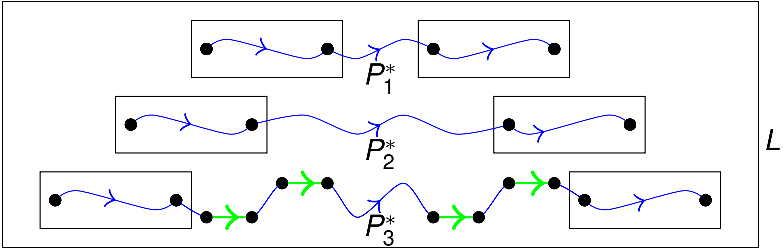

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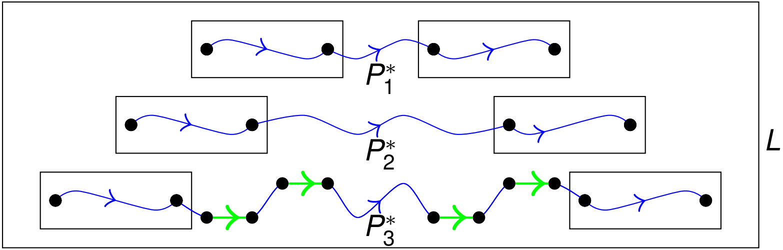

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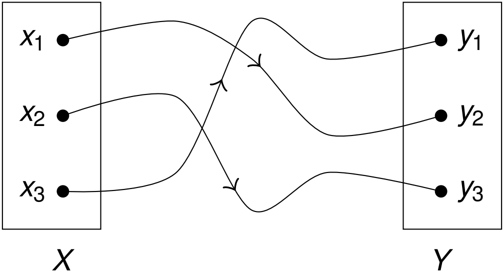
Connectivity and linkedness

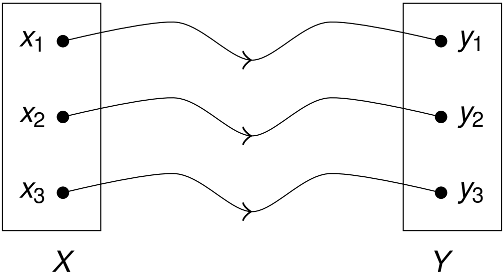
Menger's Theorem (for tournaments)



If T is strongly r-connected with $X, Y \subseteq V(T)$ of size r, then can find r vertex-disjoint X-Y-paths.

Connectivity and linkedness

Linkedness (for tournaments)



T is r-linked if given any r pairs $(x_1, y_1), \ldots, (x_r, y_r)$, there exist vertex-disjoint paths connecting x_i to y_i $\forall i$.

Connectivity and Linkedness

Theorem (Thomassen, 1984)

If a tournament is strongly $ck!$-connected, then it is k-linked.

Theorem (Kühn, Lapinskas, Osthus, P., 2013+)

If a tournament is strongly $ck \log k$-connected, then it is k-linked.

Short proof based on the idea of a sorting network.

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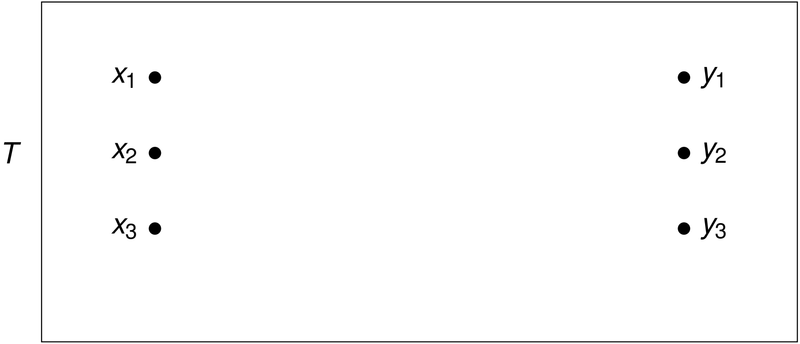
Connectivity, linkedness, and sorting networks

Given T strongly $ck \log k$-connected, find vertex-disjoint paths from x_i to y_i (example: $k = 3$)



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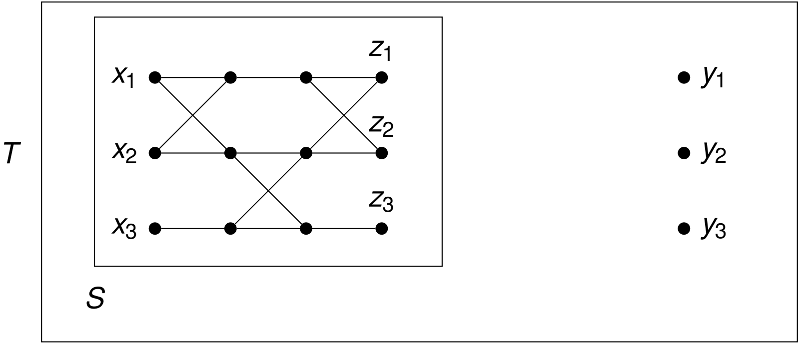

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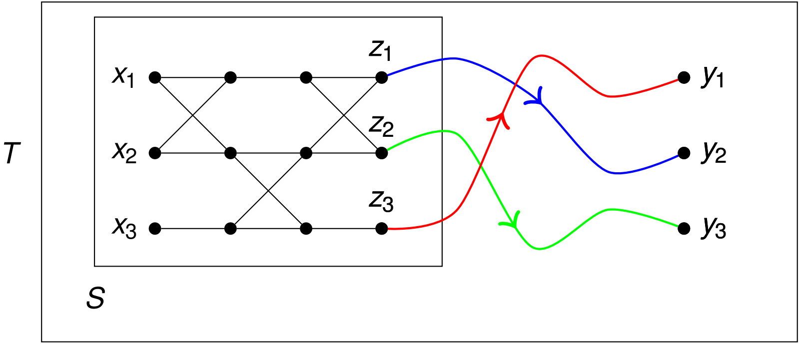

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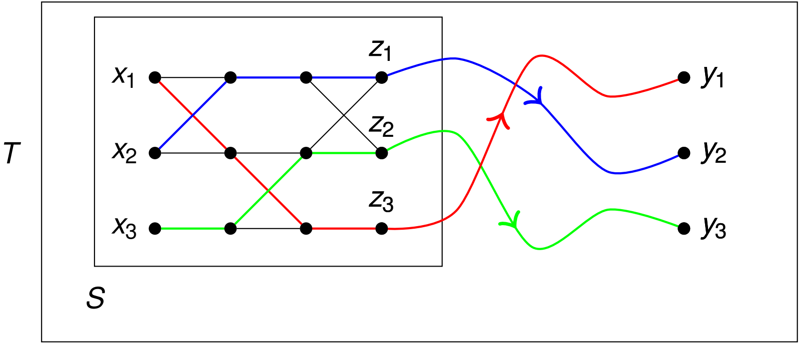

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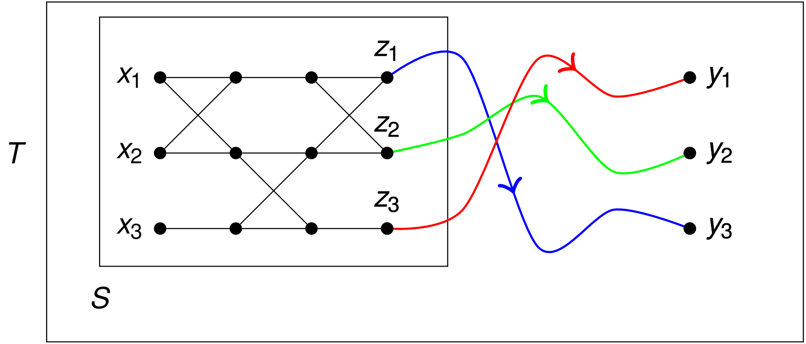

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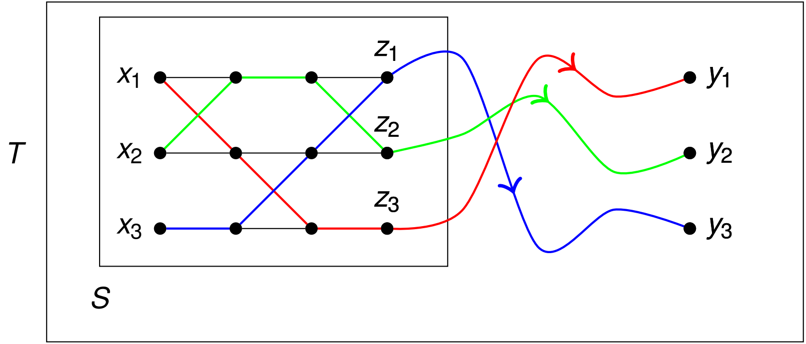

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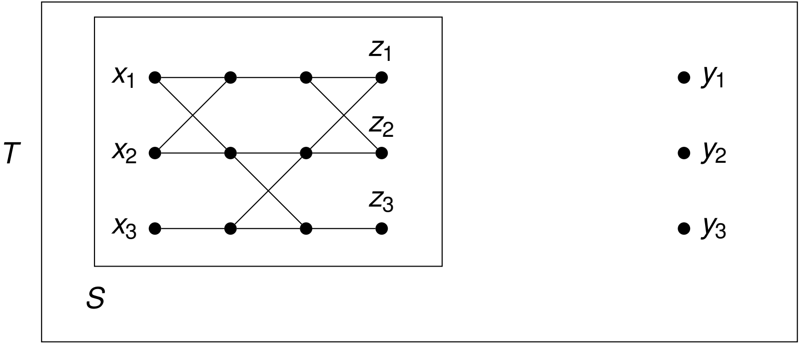

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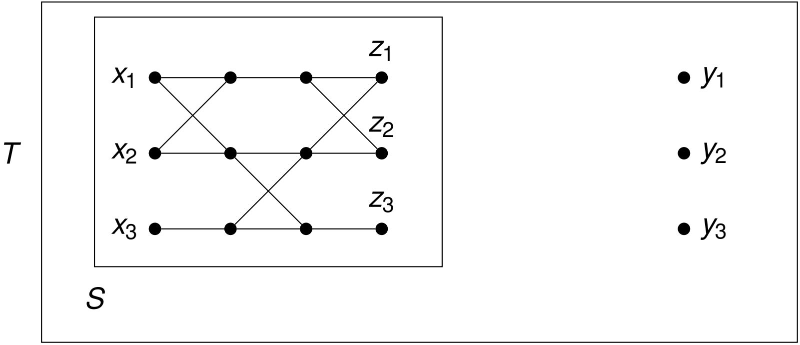

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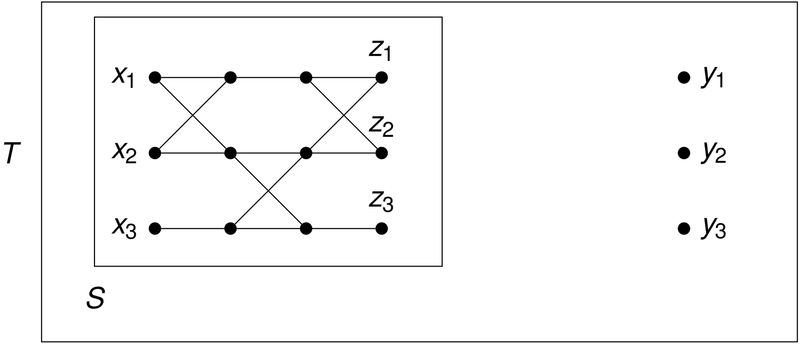

- The structure S 'simulates' a sorting network
- Crossing edges correspond to comparators
- $\exists$ sorting network with $ck \log k$ comparators (Ajtai, Komlós, Szemerédi, 1983) $\implies$ can find small S

Connectivity and Linkedness

Conjecture (Kühn, Lapinskas, Osthus, P.)

If a tournament is strongly ck-connected, then it is k-linked.

Evidence for:

- $22k$-connected graphs are k linked (Bollobás, Thomason, 1996).
- $10k$-connected graphs are k linked (Thomas, Wollan, 2005)

Evidence against:

- $\forall k\ \exists$ strongly k-connected digraph that is not 2-linked (Thomassen, 1991).

If the conjecture holds then we can take $f(k) = O(k^2 \log k)$ rather than $f(k) = O(k^2 \log^2 k)$.

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