Representing semigroups and groups by endomorphisms of Fraïssé limits

Part I. Semigroup embeddings

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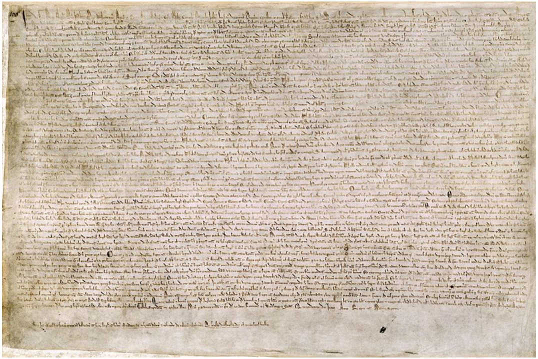
Magna Carta (June 15, 1215, Runnymede, John I)



All these customs and liberties that we have granted shall be observed in our kingdom in so far as concerns our own relations with our subjects. Let all men of our kingdom, whether clergy or laymen, observe them similarly in their relations with their own men.

Isn't this a bit like homogeneity?

One of the main motifs of this jolly get-together... ☺

Let $\mathcal{A}$ be a (countable) first order structure. $\mathcal{A}$ is said to be (ultra)homogeneous if any isomorphism

$$\iota : \mathcal{B} \to \mathcal{B}'$$

between its finitely generated substructures is a restriction of an automorphism α of $\mathcal{A}$: $\iota = \alpha|_B$.

Remark

If we restrict to relational structures, 'finitely generated' becomes simply 'finite'.

Classification programme for countable ultrahomogeneous structures

- finite graphs (Gardiner, 1976)
- posets (Schmerl, 1979)
- undirected graphs (Lachlan & Woodrow, 1980)
- tournaments (Lachlan, 1984)
- directed graphs (Cherlin, 1998 – Memoirs of AMS, 160+ pp.)
- semilattices (Droste, Kuske, Truss, 1999)
- finite groups (Cherlin & Felgner, 2000)
- permutations (Cameron, 2002)
- multipartite graphs (Jenkinson, Truss, Seidel, 2012)
- coloured multipartite graphs (Lockett, Truss, 2014)
- lattices – 'unclassifiable' (Abogatma, Truss, 2015)
- ...

Fraïssé theory

Fact

For any countably infinite ultrahomogeneous structure $\mathcal{A}$, its age $\mathrm{Age}(\mathcal{A})$ (the class of its finitely generated substructures) has the following properties:

- it has countably many isomorphism types;
- it is closed for taking (copies of) substructures;
- it has the joint embedding property (JEP);
- it has the amalgamation property (AP).

A class of finite(ly generated) structures with such properties is called a Fraïssé class.

Theorem (Fraïssé)

Let $\mathbf{C}$ *be a Fraïssé class. Then there exists a unique countably infinite ultrahomogeneous structure* $\mathcal{F}$ *such that* $\mathrm{Age}(\mathcal{F}) = \mathbf{C}$.

Fraïssé theory (continued)

The structure $\mathcal{F}$ from the previous theorem is called the Fraïssé limit of **C**.

Classical examples:

- finite chains $\longrightarrow (\mathbb{Q}, <)$
- finite undirected graphs $\longrightarrow$ the Rado (random) graph R
- finite posets $\longrightarrow$ the random poset
- finite tournaments $\longrightarrow$ the random tournament
- finite metric spaces with rational distances $\longrightarrow$ the rational Urysohn space $\mathbb{U}_{\mathbb{Q}}$
- finite permutations $\longrightarrow$ the random permutation

Fraïssé limits over finite relational languages are ω-categorical, have quantifier elimination, oligomorphic automorphism groups,...

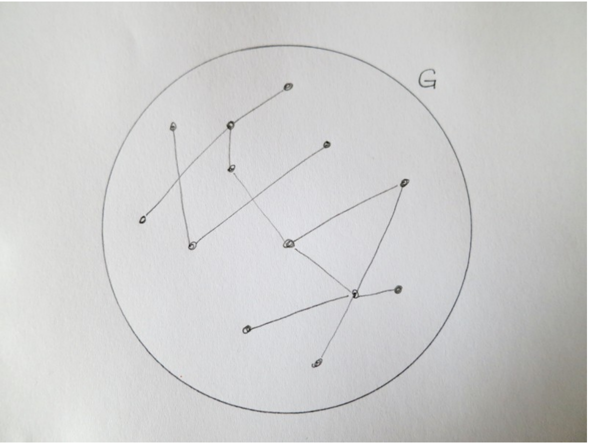
The objective of this mini-course...

...is to study the structure of $\mathrm{End}(\mathcal{F})$ for various Fraïssé limits $\mathcal{F}$ using 'only' algebraic semigroup theory (and, of course, basic model theory, combinatorics, categories, etc.), but not topology.

Our goal for today: Discuss whether the monoid/semigroup $\mathrm{End}(\mathcal{F})$ is countably universal.

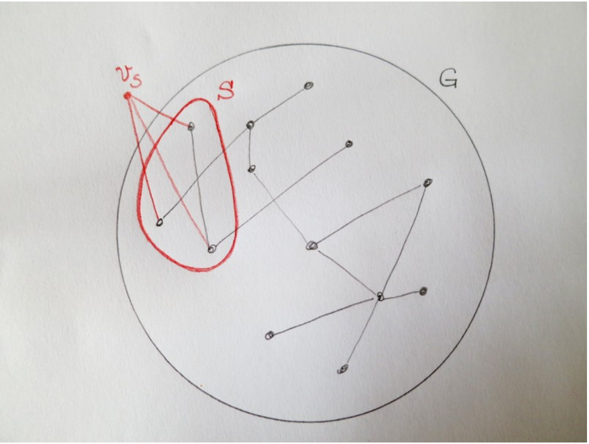
Chiefly, this is achieved by embedding $\mathcal{T}_{\aleph_0}$.

Easy example: the case of the random graph R



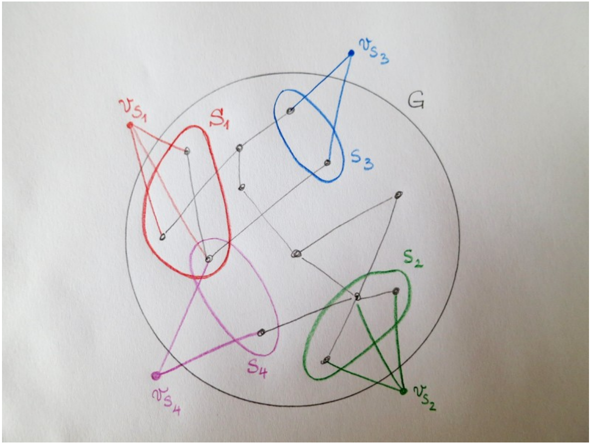
Take any countable graph G.

Easy example: the case of the random graph R



For any finite subset (and induced subgraph) S, invent a new vertex v_S that is adjacent to all vertices from S and to no other vertex from G.

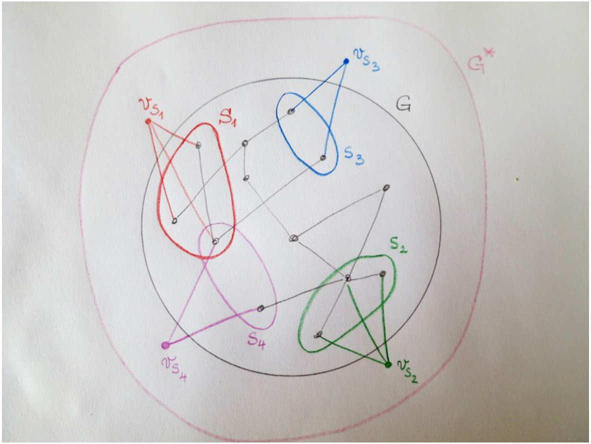
Easy example: the case of the random graph R



Do this for all finite $S \subseteq V(G)$.

($i \neq j \implies v_{S_i}$ and v_{S_j} are not adjacent.)

Easy example: the case of the random graph R



This way, we obtain G^*.

Easy example: the case of the random graph R

Now suppose we have given $\phi \in \mathrm{End}(G)$.

We can extend ϕ to G^* by sending, for each finite $S \subseteq V(G)$,

$$\phi^* : v_S \mapsto v_{S\phi}.$$

ϕ^* is easily seen to be a graph endomorphism of G^*.

Furthermore,

$$\Psi : \phi \mapsto \phi^*$$

is an (injective) monoid homomorphism $\mathrm{End}(G) \to \mathrm{End}(G^*)$.
Hence, $\mathrm{End}(G)$ embeds into $End(G^*)$.

Easy example: the case of the random graph R

Now iterate the star construction:

$$G_0 = G, \quad G_{n+1} = G_n^* \ \ (n \geq 0),$$

and let R_G be the direct limit of these graphs on vertices $\bigcup_{n\geq 0} V(G_n)$.

As is well known, R_G is one of the standard ways to build the random graph 'around' G, i.e. $R_G \cong R$. Hence, for any countable graph G, $\mathrm{End}(G)$ embeds into $\mathrm{End}(R)$.

By taking G to be the null graph on a countably infinite set of vertices, we get

Theorem (Bonato, Delić, ID, 2006)

$\mathcal{T}_{\aleph_0}$ – and thus any countable semigroup – embeds in $\mathrm{End}(R)$.

How 'bout some generalisation?

I.D.

D. Mašulović

'A universality result for endomorphism monoids of some ultrahomogeneous structures',
Proc. Edinburgh Math. Soc. 55 (2012), 635–656.

Spans and pushouts

A span is a following configuration of objects and morphisms in a category $\mathscr{C}$:

$$Y \xleftarrow{f} X \xrightarrow{g} Z$$

The pushout of this span is an object P along with two morphisms $i_1 : Y \to P$ and $i_2 : Z \to P$ with the following properties:

(1) The diagram

$$\begin{tikzcd} Y \arrow[r, "i_1"] & P \\ X \arrow[u, "f"] \arrow[r, "g"'] & Z \arrow[u, "i_2"'] \end{tikzcd}$$

commutes;

Spans and pushouts

(2) For any object Q and morphisms $j_1 : Y \to Q$ and $j_2 : Z \to Q$ for which the part of diagram below involving X, Y, Z, Q is commutative, there exists a unique morphism $u : P \to Q$ making the whole diagram

$$\begin{tikzcd}
 & & Q \\
Y \arrow[rru, bend left, "j_1"] \arrow[r, "i_1"] & P \arrow[ru, "u"] & \\
X \arrow[u, "f"] \arrow[r, "g"'] & Z \arrow[u, "i_2"'] \arrow[ruu, bend right, "j_2"'] &
\end{tikzcd}$$

commutative.

Abstract nonsense $\Longrightarrow$ pushout (if it exists) is unique.

Amalgams and AP

A span in a concrete category $\mathscr{C}$ (of structures and homomorphisms) where f, g are embeddings is called an amalgam.

AP: For any amalgam (A, B, C, f, g) in $\mathscr{C}$, $\exists D \in \mathscr{C}$ & embeddings $i_1 : B \hookrightarrow D$ and $i_2 : C \hookrightarrow D$ such that

$$\begin{tikzcd} B \arrow[r, hook, "i_1"] & D \\ A \arrow[u, hook, "f"] \arrow[r, hook, "g"] & C \arrow[u, hook, "i_2"'] \end{tikzcd}$$

commutes.

Strict AP

For a Fraïssé class $\mathbf{C}$ let $\overline{\mathbf{C}}$ denote the class (category) of all countable structures A with $\mathrm{Age}(A) \subseteq \mathbf{C}$.

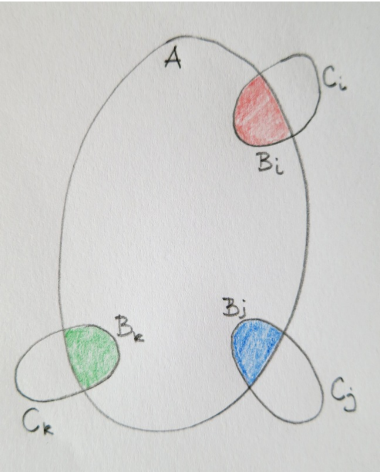
Loosely speaking, the strict AP for $\mathbf{C}$ asserts amalgamation by pushouts, i.e. the existence of the pushout in $\overline{\mathbf{C}}$ of any amalgam in $\mathbf{C}$. So, for any amalgam (A, B, C, f, g) in $\mathbf{C}$ there exists a structure $P \in \overline{\mathbf{C}}$ and embeddings $i_1 : B \hookrightarrow P$ and $i_2 : C \hookrightarrow P$ such that

$$\begin{tikzcd} B \arrow[r, hook, "i_1"] & P \\ A \arrow[u, hook, "f"] \arrow[r, hook, "g"] & C \arrow[u, hook, "i_2"'] \end{tikzcd}$$

is a pushout square.

The strict AP can be shown to extend to the case when $B, C \in \overline{\mathbf{C}}$.

Rooted multi-amalgams over $\mathbf{C}$



- $A \in \overline{\mathbf{C}}$ – the *root*,
- $B_i, C_i \in \mathbf{C}$,
- $A \cap C_i = B_i$,
- $i \neq j \;\Rightarrow$ $(C_i \setminus B_i) \cap (C_j \setminus B_j) = \varnothing$.

Notation: $(A, (B_i, C_i)_{i \in I})$

Rooted multi-amalgams over **C**

Free **C**-sum:

$$(A, (B_i, C_i)_{i \in I}) \ \rightsquigarrow \ D = \coprod^{*}(A, (B_i, C_i)_{i \in I}) \in \overline{\mathbf{C}}$$

(a) There are embeddings $f : A \to D$ and $g_i : C_i \to D$, $i \in I$, such that $f|_{B_i} = g_i|_{B_i}$ for any $i \in I$;

(b) for any structure $D' \in \overline{\mathbf{C}}$ and any homomorphisms $\varphi : A \to D'$, $\psi_i : C_i \to D'$, $i \in I$, such that for any $i \in I$ we have $\varphi|_{B_i} = \psi_i|_{B_i}$, there exists a unique homomorphism $\delta : D \to D'$ extending all the given homomorphisms, that is, such that we have $f\delta = \varphi$ and $g_i\delta = \psi_i$ for all $i \in I$.

Lemma

If **C** *is a Fraïssé class with strict AP, then every rooted multi-amalgam over* **C** *has the free* **C***-sum in* $\overline{\mathbf{C}}$*.*

Strict AP examples

- Finite (simple) graphs: the free sum is just the amalgam itself;
- Finite posets: the free sum of $(A, (B_i, C_i)_{i \in I})$ = take the union of order relations on A and C_i's (a reflexive and antisymmetric relation) and construct its transitive closure;
- Algebraic structures: If $\mathbf{V}$ is a variety of algebras, then strict AP = ordinary AP for $\mathbf{V}_{f.g.}$, and the free sum of a rooted multi-amalgam is just the free algebra freely generated by the partial algebra $(A, (B_i, C_i)_{i \in I})$ (Grätzer) $\Longrightarrow$ finite semilattices / distributive lattices / Boolean algebras, ...

The star construction

Let $A \in \overline{\mathbf{C}}$ be an arbitrary countable structure.

Let $\{(B_i, C_i) : \ i \in I\}$ be an enumeration of all pairs consisting of a finitely generated substructure B_i of A and a one-point extension $C_i \in \mathbf{C}$ of B_i such that if $B_i = B_j = B$ then C_i and C_j are not B-isomorphic. By renaming elements if necessary, $(A, (B_i, C_i)_{i \in I})$ becomes a rooted multi-amalgam over $\mathbf{C}$.

So, let $A^* = \coprod^* (A, (B_i, C_i)_{i \in I}) \in \overline{\mathbf{C}}$.

The star construction

Let $A^{(0)} = A$ and $A^{(n+1)} = (A^{(n)})^*$ for all $n \geq 0$.

We can identify $A^{(n)}$ with its appropriate copy within $A^{(n+1)}$, so in that sense we can form the structure

$$F(A) = \bigcup_{n \geq 0} A^{(n)}.$$

Proposition

Let $\mathbf{C}$ be a Fraïssé class with the strict AP and let $A \in \overline{\mathbf{C}}$ be arbitrary. Then $F(A)$ is isomorphic to the Fraïssé limit of $\mathbf{C}$.

Homomorphism extensions

Our motivation: Given $\varphi \in \mathrm{End}(A)$, extend it to $\hat{\varphi} \in \mathrm{End}(A^*)$ in a 'neat way'.

By the defining properties of free **C**-sums, for this it suffices to define homomorphisms $\psi_i : C_i \to A^*$ that agree with φ on B_i. This emphasises the importance of spans of the following type:

$$C \xleftarrow{\mathbf{1}_B} B \xrightarrow{f} B'$$

where C is a one-point extension of B.

One-point homomorphism extension property (1PHEP)

The class $\mathbf{C}$ enjoys the 1PHEP if for any $B, B', C \in \mathbf{C}$ such that C is a one-point extension of B, and any surjective homomorphism $f : B \to B'$ there exists an extension C' of B' and a surjective homomorphism $f^* : C \to C'$ such that $f^*|_B = f$; in other words, the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, two heads, "f^*"] & C' \\ B \arrow[u, hook, "\mathbf{1}_B"] \arrow[r, two heads, "f"] & B' \arrow[u, hook, "\mathbf{1}_{B'}"'] \end{tikzcd}$$

(Here C' is either a one-point extension of B', or $C' = B'$.)

Remark

1PHEP is in Fraïssé classes equivalent to homo-amalgamation property (HAP), intimately related to homomorphism-homogeneity.

Strict 1PHEP

We require that any span of the form

$$C \xleftarrow{\;i\;} B \xrightarrow{\;f\;} B'$$

where $B, B', C \in \mathbf{C}$ and C is a one-point extension of B, has a pushout $P \in \mathbf{C}$ with respect to $\overline{\mathbf{C}}$ as a concrete category, and if

$$\begin{tikzcd} C \arrow[r, "f'"] & P \\ B \arrow[u, hook, "i"] \arrow[r, two heads, "f"] & B' \arrow[u, "i'"'] \end{tikzcd}$$

is a pushout square in $\overline{\mathbf{C}}$, then i' is an embedding and the homomorphism f' is surjective.

The main result

Theorem (ID+DM, 2012)

Let $\mathbf{C}$ *be a Fraïssé class satisfying the following three properties:*

(i) $\mathbf{C}$ *enjoys the strict AP.*

(ii) $\mathbf{C}$ *enjoys the strict 1PHEP.*

(iii) *For any* $B, C \in \mathbf{C}$ *such that* C *is a one-point extension of* B*, the pointwise stabilizer* $\mathrm{Aut}_B(C)$ *(of* B *in* $\mathrm{Aut}(C)$*) is trivial.*

Then for any $A \in \overline{\mathbf{C}}$ *there is an embedding of* $\mathrm{End}(A)$ *into* $\mathrm{End}(A^*)$*. Consequently, if* F *is the Fraïssé limit of* $\mathbf{C}$ *then* $\mathrm{End}(A)$ *embeds into* $\mathrm{End}(F)$*.*

The main result

A one-point extension C of B is uniquely generated if $x, x' \in C \setminus B$ and $\langle B, x\rangle = \langle B, x'\rangle = C$ implies $x = x'$. Notice that this automatically holds in relational structures.

Corollary

Let $\mathbf{C}$ *be a Fraïssé class satisfying the condition of uniquely generated one-point extensions. If* $\mathbf{C}$ *satisfies the strict AP and the strict 1PHEP, then for any* $A \in \overline{\mathbf{C}}$*,* $\mathrm{End}(A)$ *embeds into* $\mathrm{End}(A^*)$ *and so into the endomorphism monoid of the Fraïssé limit of* $\mathbf{C}$*.*

The random graph (revisited)

Lemma

The class of finite simple graphs satisfies the strict 1PHEP.

Take A to be the null graph on $\aleph_0$ vertices.

Corollary (Bonato, Delić, ID, 2006)

$\mathrm{End}(R)$ *embeds* $\mathcal{T}_{\aleph_0}$ *and thus any countable semigroup.*

The random poset

Lemma

The class of finite posets satisfies the strict 1PHEP.

Take A to be the antichain of size $\aleph_0$.

Corollary (ID, 2007)

$\mathrm{End}(\mathbb{P})$ *embeds* $\mathcal{T}_{\aleph_0}$ *and thus any countable semigroup.*

The (rational) Urysohn space

Let Σ be an additive submonoid of $\mathbb{R}_0^+$, and let $\mathbf{M}_\Sigma$ be the class of all finite metric spaces with distances in Σ. Here are two charming exercises in metric geometry:

Lemma

$\mathbf{M}_\Sigma$ *enjoys the strict 1PHEP.*

Lemma

$\mathbf{M}_\Sigma$ *enjoys the strict AP (even though the category of metric spaces has no coproducts!).*

Take A to be the unit $\aleph_0$-simplex.

Corollary

$\mathrm{End}(\mathbb{U}_\mathbb{Q})$ *embeds* $\mathcal{T}_{\aleph_0}$ *and thus any countable semigroup.*

Lemma

$\mathrm{End}(\mathbb{U}_\mathbb{Q})$ *embeds into* $\mathrm{End}(\mathbb{U})$.

The significance of CEP

An algebra A is said to have the congruence extension property (CEP) if any congruence θ on any subalgebra B of A is a restriction of a congruence of A.

Classical examples:

- semilattices
- distributive lattices
- Boolean algebras
- Abelian groups
- ... (a huge subject in universal algebra)

Lemma

Let $\mathbf{C}$ *be a class of finitely generated algebras with the CEP and closed under taking homomorphic images. Then* $\mathbf{C}$ *has the strict 1PHEP.*

The countable generic semilattice

Lemma

Any one-point extension of a semilattice is uniquely generated.

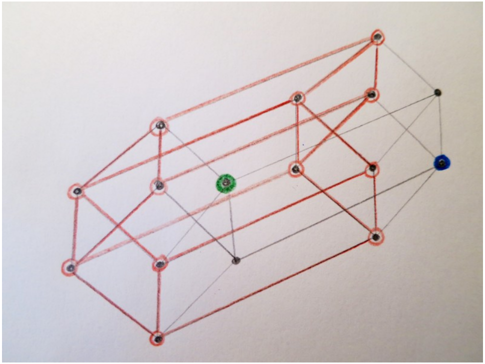
Take A to be the free semilattice of rank $\aleph_0$.

Corollary (ID, 2007)

$\mathrm{End}(\Omega)$ *embeds* $\mathcal{T}_{\aleph_0}$ *and thus any countable semigroup.*

The countable generic distributive lattice

ALAS!!! Not every one-point extension of a (finite) distributive lattice is uniquely generated.



This lattice is a one-point extension of its ‘red’ sublattice, but it is generated both by the ‘green’ and the ‘blue’ element (for example).

The countable generic distributive lattice

However...

Lemma (Noticed by ID on the night of June 18/19, 2015)

Finite one-point extensions of distributive lattices satisfy the condition (iii) of the Main Theorem!

Corollary (Solution of Problem 4.16 of ID+DM)

The monoid $\mathrm{End}(\mathbb{D})$ *embeds* $\mathcal{T}_{\aleph_0}$ *and thus any countable semigroup.*

The countable generic distributive lattice

Lemma

Let L be a finite distributive lattice that is a one-point extension of its sublattice K, and let ϕ be an automorphism of L fixing K pointwise. Then ϕ is the identity mapping.

Proof.

Let $L = \langle K, x \rangle$.

Since then any element of L is obtained as $p(x)$, where p is a unary distributive lattice polynomial with coefficients in K, the lemma follows if $\phi(x) = x$. So, assume that $\phi(x) \neq x$.

Then, as already noted,

$$\phi(x) = (x \wedge a) \vee b$$

for some $a, b \in K$ (we used the distributive laws for this).

The countable generic distributive lattice

Hence,

$$\phi(x \vee b) = \phi(x) \vee \phi(b) = \phi(x) \vee b = (x \wedge a) \vee b \vee b = \phi(x).$$

Since ϕ is an automorphism of L, we must have $x \vee b = x$.

Therefore,

$$\phi(x) = (x \wedge a) \vee b = (x \vee b) \wedge (a \vee b) = x \wedge (a \vee b) \leq x,$$

so $\phi(x) < x$.

But then

$$x > \phi(x) > \phi^2(x) > \dots$$

contradicting the finiteness of L. $\square$

THANK YOU!

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