Ramsey Classes by Partite Construction II

Honza Hubička

Mathematics and Statistics
University of Calgary
Calgary

Institute of Computer Science
Charles University
Prague

Joint work with Jaroslav Nešetřil

Permutation Groups and Transformation Semigroups 2015

Ramsey classes

We consider relational structures in language L without function symbols.

Definition

A class $\mathcal{C}$ (of finite relational structures) is Ramsey iff

$$\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Ramsey classes

We consider relational structures in language L without function symbols.

Definition

A class $\mathcal{C}$ (of finite relational structures) is Ramsey iff

$$\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

$\binom{\mathbf{B}}{\mathbf{A}}$ is set of all substructures of $\mathbf{B}$ isomorphic to $\mathbf{A}$.

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$: For every 2-coloring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ is monochromatic.

Ramsey classes

We consider relational structures in language L without function symbols.

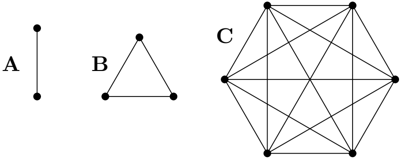
Definition

A class $\mathcal{C}$ (of finite relational structures) is Ramsey iff

$$\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

$\binom{\mathbf{B}}{\mathbf{A}}$ is set of all substructures of **B** isomorphic to **A**.

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$: For every 2-coloring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ is monochromatic.



Ramsey classes

We consider relational structures in language L without function symbols.

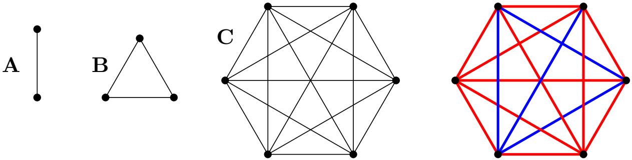
Definition

A class $\mathcal{C}$ (of finite relational structures) is Ramsey iff

$$\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

$\binom{\mathbf{B}}{\mathbf{A}}$ is set of all substructures of **B** isomorphic to **A**.

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$: For every 2-coloring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ is monochromatic.



Nešetřil-Rödl Theorem

A structure **A** is called complete (or irreducible) if every pair of distinct vertices belong to a relation of **A**.

$\mathrm{Forb}_E(\mathcal{E})$ is a class of all finite structures **A** such that there is no embedding from **E** $\in \mathcal{E}$ to **A**.

Theorem (Nešetřil-Rödl Theorem, 1977)

- *Let L be a finite relational language.*
- *Let $\mathcal{E}$ be a set of complete ordered L-structures.*
- *The then class $\mathrm{Forb}_E(\mathcal{E})$ is a Ramsey class.*

Nešetřil-Rödl Theorem

A structure **A** is called complete (or irreducible) if every pair of distinct vertices belong to a relation of **A**.

$\mathrm{Forb}_E(\mathcal{E})$ is a class of all finite structures **A** such that there is no embedding from $\mathbf{E} \in \mathcal{E}$ to **A**.

Theorem (Nešetřil-Rödl Theorem, 1977)

- *Let L be a finite relational language.*
- *Let $\mathcal{E}$ be a set of complete ordered L-structures.*
- *The then class $\mathrm{Forb}_E(\mathcal{E})$ is a Ramsey class.*

Proof by partite construction.

Unary closures = relations with out-degree 1

Unary closure description $\mathcal{C}$ is a set of pairs (R^U, R^B) where R^U is unary relation and R^B is binary relation.

We say that structure **A** is $\mathcal{C}$-closed if for every pair (R^U, R^B) the B-outdegree of every vertex of **A** that is in U is 1.

Theorem (H., Nešetřil, 2015)

Let $\mathcal{E}$ be a family of complete ordered structures and $\mathcal{U}$ an unary closure description. Then the class of all $\mathcal{C}$-closed structures in $\mathrm{Forb}_E(\mathcal{E})$ has Ramsey lift.

Unary closures = relations with out-degree 1

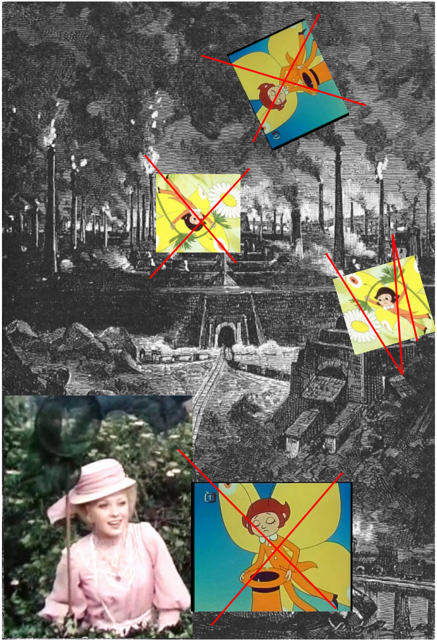
Unary closure description $\mathcal{C}$ is a set of pairs (R^U, R^B) where R^U is unary relation and R^B is binary relation.

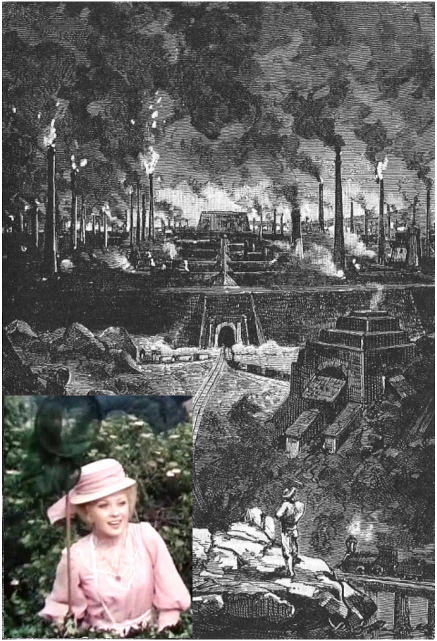
We say that structure **A** is $\mathcal{C}$-closed if for every pair (R^U, R^B) the B-outdegree of every vertex of **A** that is in U is 1.

Theorem (H., Nešetřil, 2015)

Let $\mathcal{E}$ be a family of complete ordered structures and $\mathcal{U}$ an unary closure description. Then the class of all $\mathcal{C}$-closed structures in $\mathrm{Forb}_E(\mathcal{E})$ has Ramsey lift.

All Cherlin Shelah Shi classes with unary closure can be described this way!





Map of Ramsey Classes

boolean algebras

interval graphs

permutations

cyclic orders

unions of complete

linear orders

Unary CSS classes

metric spaces

acyclic graphs

partial orders

K_n-free graphs

graphs

restricted ⟵⟶ free

Further applications

- Known Cherlin-Shelah-Shi classes:
  - bouquets
  - bowties extended by path
  - known examples without unary closure

Further applications

- Known Cherlin-Shelah-Shi classes:
  - bouquets
  - bowties extended by path
  - known examples without unary closure
- *n*-ary functions
  (structures with a function symbol (A, f) where $f : A^n \to A$)
  - Consider (A, f) as a relational structure with $(n+1)$-ary relation where every n-tuple has a closure vertex
  - Because the algebraic closure is not locally finite Fraïssé limit is not ω-categorical

Further applications

- Known Cherlin-Shelah-Shi classes:
  - bouquets
  - bowties extended by path
  - known examples without unary closure
- *n*-ary functions
  (structures with a function symbol (A, f) where $f : A^n \to A$)
  - Consider (A, f) as a relational structure with $(n+1)$-ary relation where every n-tuple has a closure vertex
  - Because the algebraic closure is not locally finite Fraïssé limit is not ω-categorical
- In some cases algebraic closure is introduced as a scaffolding and does not appear in the final Ramsey class:
  - Structures with infinitely many equivalence classes

Further applications

- Known Cherlin-Shelah-Shi classes:
  - bouquets
  - bowties extended by path
  - known examples without unary closure
- *n*-ary functions
  (structures with a function symbol (A, f) where $f : A^n \to A$)
  - Consider (A, f) as a relational structure with $(n+1)$-ary relation where every n-tuple has a closure vertex
  - Because the algebraic closure is not locally finite Fraïssé limit is not ω-categorical
- In some cases algebraic closure is introduced as a scaffolding and does not appear in the final Ramsey class:
  - Structures with infinitely many equivalence classes
  - $\mathbb{QQ}$

Structures with forbidden homomorphisms

Let $\mathcal{F}$ be a family of relational structures. We denote by $\mathrm{Forb}_H(\mathcal{F})$ the class of all finite structures $\mathbf{A}$ such that there is no $\mathbf{F} \in \mathcal{F}$ having a homomorphism $\mathbf{F} \to \mathbf{A}$.

Structures with forbidden homomorphisms

Let $\mathcal{F}$ be a family of relational structures. We denote by $\mathrm{Forb}_H(\mathcal{F})$ the class of all finite structures $\mathbf{A}$ such that there is no $\mathbf{F} \in \mathcal{F}$ having a homomorphism $\mathbf{F} \to \mathbf{A}$.

Theorem (Cherlin,Shelah,Shi 1998)

For every finite family $\mathcal{F}$ of finite connected relational structures there is an ω-categorical structure that is universal for $\mathrm{Forb}_H(\mathcal{F})$.

Structures with forbidden homomorphisms

Let $\mathcal{F}$ be a family of relational structures. We denote by $\mathrm{Forb}_H(\mathcal{F})$ the class of all finite structures **A** such that there is no **F** $\in \mathcal{F}$ having a homomorphism **F** $\to$ **A**.

Theorem (Cherlin,Shelah,Shi 1998)

For every finite family $\mathcal{F}$ of finite connected relational structures there is an ω-categorical structure that is universal for $\mathrm{Forb}_H(\mathcal{F})$.

- Every ω-categorical structure can be lifted to homogeneous.
- Explicit homogenization is given by H. and Nešetřil (2009).

Structures with forbidden homomorphisms

Let $\mathcal{F}$ be a family of relational structures. We denote by $\mathrm{Forb}_H(\mathcal{F})$ the class of all finite structures **A** such that there is no **F** $\in \mathcal{F}$ having a homomorphism **F** $\to$ **A**.

Theorem (Cherlin,Shelah,Shi 1998)

For every finite family $\mathcal{F}$ of finite connected relational structures there is an ω-categorical structure that is universal for $\mathrm{Forb}_H(\mathcal{F})$.

- Every ω-categorical structure can be lifted to homogeneous.
- Explicit homogenization is given by H. and Nešetřil (2009).

Theorem (Nešetřil, 2010)

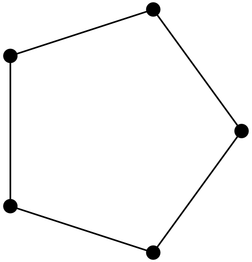
For every finite family $\mathcal{F}$ of finite connected relational structures there is a Ramsey lift of $\mathrm{Forb}_H(\mathcal{F})$.

Explicit homogenization of $\mathrm{Forb}_H(\mathbf{C}_5)$

Basic concept:

- Amalgamation of two structures in $\mathrm{Forb}_H(\mathcal{F})$ fails iff the free amalgam contains a homomorphic copy of structure $\mathbf{F} \in \mathcal{F}$.
- Use extra relations to prevent such amalgams

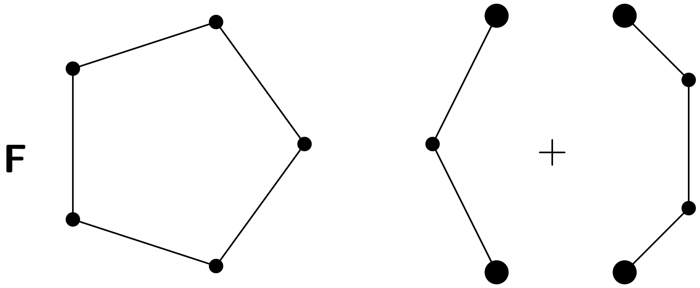
F



Explicit homogenization of $\mathrm{Forb}_H(\mathbf{C}_5)$

Basic concept:

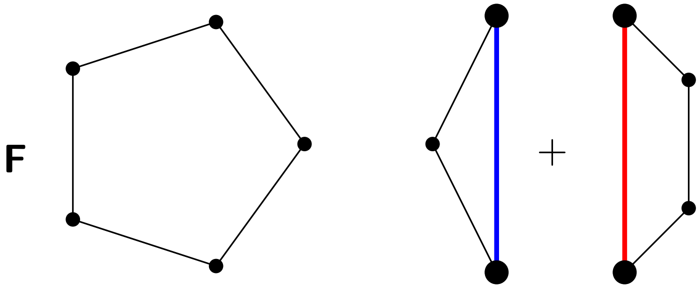
- Amalgamation of two structures in $\mathrm{Forb}_H(\mathcal{F})$ fails iff the free amalgam contains a homomorphic copy of structure $\mathbf{F} \in \mathcal{F}$.
- Use extra relations to prevent such amalgams



Explicit homogenization of $\mathrm{Forb}_H(\mathbf{C}_5)$

Basic concept:

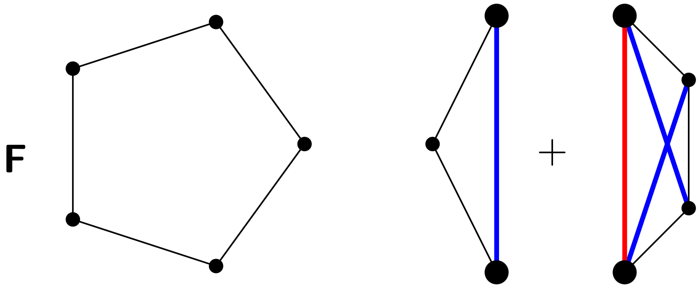
- Amalgamation of two structures in $\mathrm{Forb}_H(\mathcal{F})$ fails iff the free amalgam contains a homomorphic copy of structure $\mathbf{F} \in \mathcal{F}$.
- Use extra relations to prevent such amalgams



Explicit homogenization of $\mathrm{Forb}_H(\mathbf{C}_5)$

Basic concept:

- Amalgamation of two structures in $\mathrm{Forb}_H(\mathcal{F})$ fails iff the free amalgam contains a homomorphic copy of structure $\mathbf{F} \in \mathcal{F}$.
- Use extra relations to prevent such amalgams

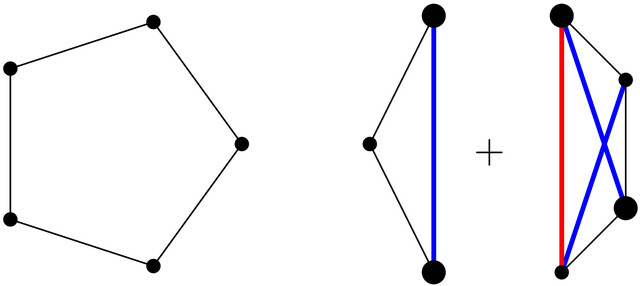


Explicit homogenization of $\mathrm{Forb}_H(\mathbf{C}_5)$

Basic concept:

- Amalgamation of two structures in $\mathrm{Forb}_H(\mathcal{F})$ fails iff the free amalgam contains a homomorphic copy of structure $\mathbf{F} \in \mathcal{F}$.
- Use extra relations to prevent such amalgams

F

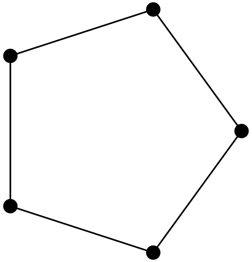


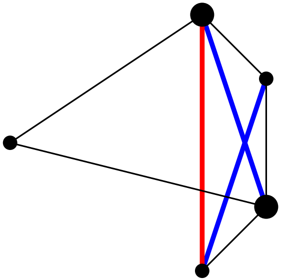
Explicit homogenization of $\mathrm{Forb}_H(\mathbf{C}_5)$

Basic concept:

- Amalgamation of two structures in $\mathrm{Forb}_H(\mathcal{F})$ fails iff the free amalgam contains a homomorphic copy of structure $\mathbf{F} \in \mathcal{F}$.
- Use extra relations to prevent such amalgams

F



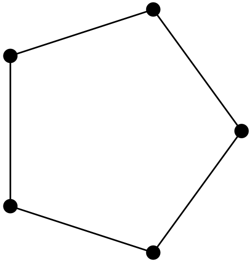


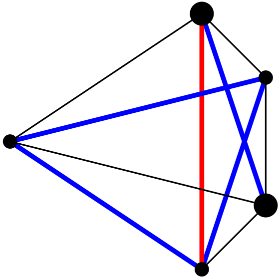
Explicit homogenization of $\mathrm{Forb}_H(\mathbf{C}_5)$

Basic concept:

- Amalgamation of two structures in $\mathrm{Forb}_H(\mathcal{F})$ fails iff the free amalgam contains a homomorphic copy of structure $\mathbf{F} \in \mathcal{F}$.
- Use extra relations to prevent such amalgams

F





Explicit homogenization of $\mathrm{Forb}_H(\mathcal{F})$

Definition

Let C be a vertex cut in structure $\mathbf{A}$. Let $\mathbf{A}_1 \neq \mathbf{A}_2$ be two components of $\mathbf{A}$ produced by cut C. We call C minimal separating cut for $\mathbf{A}_1$ and $\mathbf{A}_2$ in $\mathbf{A}$ if $C = N_{\mathbf{A}}(A_1) = N_{\mathbf{A}}(A_2)$.

Explicit homogenization of $\mathrm{Forb}_H(\mathcal{F})$

Definition

Let C be a vertex cut in structure $\mathbf{A}$. Let $\mathbf{A}_1 \neq \mathbf{A}_2$ be two components of $\mathbf{A}$ produced by cut C. We call C minimal separating cut for $\mathbf{A}_1$ and $\mathbf{A}_2$ in $\mathbf{A}$ if $C = N_{\mathbf{A}}(A_1) = N_{\mathbf{A}}(A_2)$.

A rooted structure $\mathcal{P}$ is a pair $(\mathbf{P}, \overrightarrow{R})$ where $\mathbf{P}$ is a relational structure and $\overrightarrow{R}$ is a tuple consisting of distinct vertices of $\mathbf{P}$. $\overrightarrow{R}$ is called the root of $\mathcal{P}$.

Explicit homogenization of $\mathrm{Forb}_H(\mathcal{F})$

Definition

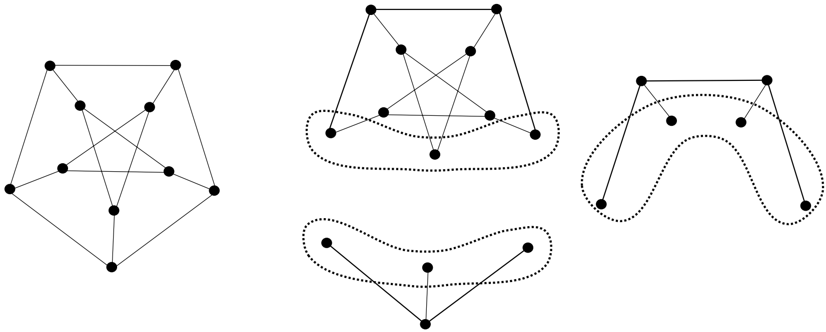
Let C be a vertex cut in structure $\mathbf{A}$. Let $\mathbf{A}_1 \neq \mathbf{A}_2$ be two components of $\mathbf{A}$ produced by cut C. We call C minimal separating cut for $\mathbf{A}_1$ and $\mathbf{A}_2$ in $\mathbf{A}$ if $C = N_{\mathbf{A}}(A_1) = N_{\mathbf{A}}(A_2)$.

A rooted structure $\mathcal{P}$ is a pair $(\mathbf{P}, \overrightarrow{R})$ where $\mathbf{P}$ is a relational structure and $\overrightarrow{R}$ is a tuple consisting of distinct vertices of $\mathbf{P}$. $\overrightarrow{R}$ is called the root of $\mathcal{P}$.

Definition

Let $\mathbf{A}$ be a connected relational structure and R a minimal separating cut for component $\mathbf{C}$ in $\mathbf{A}$. A piece of a relational structure $\mathbf{A}$ is then a rooted structure $\mathcal{P} = (\mathbf{P}, \overrightarrow{R})$, where the tuple $\overrightarrow{R}$ consists of the vertices of the cut R in a (fixed) linear order and $\mathbf{P}$ is a structure induced by $\mathbf{A}$ on $C \cup R$.

Pieces of Petersen graph



Explicit homogenization of $\mathrm{Forb}_H(\mathcal{F})$

- Enumerate by $\mathcal{P}_1, \ldots \mathcal{P}_N$ all isomorphism types of pieces structures in $\mathcal{F}$.
- Add lifted relations E^1, E^2,... E^N where arities correspond to sizes of roots of pieces $\mathcal{P}_1, \ldots \mathcal{P}_N$.

Explicit homogenization of $\mathrm{Forb}_H(\mathcal{F})$

- Enumerate by $\mathcal{P}_1, \ldots \mathcal{P}_N$ all isomorphism types of pieces structures in $\mathcal{F}$.
- Add lifted relations E^1, E^2,... E^N where arities correspond to sizes of roots of pieces $\mathcal{P}_1, \ldots \mathcal{P}_N$.
- Canonical lift of structure **A**, denoted by $\overline{\mathbf{A}}$, adds $\vec{t} \in E^i_{\mathbf{A}}$ if and only if there is a rooted homomorphism from $\mathcal{P}$ to **A** mapping root of the piece to $\vec{t}$.

Explicit homogenization of $\mathrm{Forb}_H(\mathcal{F})$

- Enumerate by $\mathcal{P}_1, \ldots \mathcal{P}_N$ all isomorphism types of pieces structures in $\mathcal{F}$.
- Add lifted relations E^1, E^2,... E^N where arities correspond to sizes of roots of pieces $\mathcal{P}_1, \ldots \mathcal{P}_N$.
- Canonical lift of structure **A**, denoted by $\overline{\mathbf{A}}$, adds $\vec{t} \in E^i_{\mathbf{A}}$ if and only if there is a rooted homomorphism from $\mathcal{P}$ to **A** mapping root of the piece to $\vec{t}$.
- A sublift **X** of $\overline{\mathbf{A}}$ is maximal if there is no extend **A** to $\mathbf{B} \in \mathrm{Forb}_H(\mathcal{F})$ such that $\overline{\mathbf{B}}$ induces more lifted relations on X. In this case also **A** is call a witness of **X**.

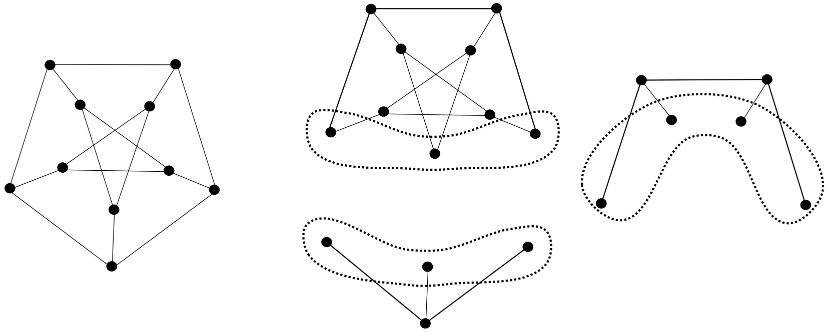
Explicit homogenization of $\mathrm{Forb}_H(\mathcal{F})$

- Enumerate by $\mathcal{P}_1, \ldots \mathcal{P}_N$ all isomorphism types of pieces structures in $\mathcal{F}$.
- Add lifted relations E^1, E^2,... E^N where arities correspond to sizes of roots of pieces $\mathcal{P}_1, \ldots \mathcal{P}_N$.
- Canonical lift of structure **A**, denoted by $\overline{\mathbf{A}}$, adds $\vec{t} \in E^i_{\mathbf{A}}$ if and only if there is a rooted homomorphism from $\mathcal{P}$ to **A** mapping root of the piece to $\vec{t}$.
- A sublift **X** of $\overline{\mathbf{A}}$ is maximal if there is no extend **A** to $\mathbf{B} \in \mathrm{Forb}_H(\mathcal{F})$ such that $\overline{\mathbf{B}}$ induces more lifted relations on X. In this case also **A** is call a witness of **X**.

Lemma

The class of all maximal sublifts of canonical lifts of structures in $\mathrm{Forb}_H(\mathcal{F})$ is an strong amalgamation class.

Explicit homogenization of Petersen-free graph



Homogenization will consist of two ternary relations and one quaternary relation denoting the rooted homomorphisms from the pieces above.

Infinitely many pieces

Example

- Let $\mathcal{C}_o$ be the class of odd cycles.
- The pieces are even and odd paths rooted by the end

Infinitely many pieces

Example

- Let $\mathcal{C}_o$ be the class of odd cycles.
- The pieces are even and odd paths rooted by the end

Only two lifted relations needed: all even paths can be tracked by E^1 and all odd by E^2.

Example

- Let $\mathcal{C}_o$ be the class of odd cycles.
- The pieces are even and odd paths rooted by the end

Only two lifted relations needed: all even paths can be tracked by E^1 and all odd by E^2.

- A piece $\mathcal{P} = (\mathbf{P}, \overrightarrow{R})$ is incompatible with a rooted structure $\mathcal{A}$ if there is a free amalgam of $\mathcal{P}$ and $\mathcal{A}$ unifying the roots it is isomorphic to some $\mathbf{F} \in \mathcal{F}$.

Infinitely many pieces

Example

- Let $\mathcal{C}_o$ be the class of odd cycles.
- The pieces are even and odd paths rooted by the end

Only two lifted relations needed: all even paths can be tracked by E^1 and all odd by E^2.

- A piece $\mathcal{P} = (\mathbf{P}, \overrightarrow{R})$ is incompatible with a rooted structure $\mathcal{A}$ if there is a free amalgam of $\mathcal{P}$ and $\mathcal{A}$ unifying the roots it is isomorphic to some $\mathbf{F} \in \mathcal{F}$.
- $\mathcal{I}_{\mathcal{P}}$ is set of all rooted structures incompatible with $\mathcal{P}$.

Infinitely many pieces

Example

- Let $\mathcal{C}_o$ be the class of odd cycles.
- The pieces are even and odd paths rooted by the end

Only two lifted relations needed: all even paths can be tracked by E^1 and all odd by E^2.

- A piece $\mathcal{P} = (\mathbf{P}, \overrightarrow{R})$ is incompatible with a rooted structure $\mathcal{A}$ if there is a free amalgam of $\mathcal{P}$ and $\mathcal{A}$ unifying the roots it is isomorphic to some $\mathbf{F} \in \mathcal{F}$.
- $\mathcal{I}_{\mathcal{P}}$ is set of all rooted structures incompatible with $\mathcal{P}$.

For two pieces $\mathcal{P}_1$ and $\mathcal{P}_2$ put $\mathcal{P}_1 \sim \mathcal{P}_2$ if and only if $\mathcal{I}_{\mathcal{P}_1} = \mathcal{I}_{\mathcal{P}_2}$ and put $\mathcal{P}_1 \preceq \mathcal{P}_2$ if and only if $\mathcal{I}_{\mathcal{P}_2} \subseteq \mathcal{I}_{\mathcal{P}_1}$.

Definition

A family of finite structures $\mathcal{F}$ is called regular if $\sim$ is locally finite.

The existence of ω-categorical universal graph

Theorem (H., Nešetřil, 2015)

Let $\mathcal{F}$ be class of connected structures that is closed for homomorphic images. Then there is an ω-categorical universal structure in $\mathrm{Forb}_H(\mathcal{F})$ if and only if $\mathcal{F}$ is regular.

The existence of ω-categorical universal graph

Theorem (H., Nešetřil, 2015)

Let $\mathcal{F}$ be class of connected structures that is closed for homomorphic images. Then there is an ω-categorical universal structure in $\mathrm{Forb}_H(\mathcal{F})$ if and only if $\mathcal{F}$ is regular.

- The finite case:

  Cherlin, Shelah, Shi, 1999: Universal graphs with forbidden subgraphs and algebraic closure

  Covington, 1990: Homogenizable relational structures

The existence of ω-categorical universal graph

Theorem (H., Nešetřil, 2015)

Let $\mathcal{F}$ be class of connected structures that is closed for homomorphic images. Then there is an ω-categorical universal structure in $\mathrm{Forb}_H(\mathcal{F})$ if and only if $\mathcal{F}$ is regular.

- The finite case:

  Cherlin, Shelah, Shi, 1999: Universal graphs with forbidden subgraphs and algebraic closure

  Covington, 1990: Homogenizable relational structures

- The finite case of relational trees:

  Nešetřil, Tardif, 2000: Duality theorems for finite structures (characterising gaps and good characterisations)

The existence of ω-categorical universal graph

Theorem (H., Nešetřil, 2015)

Let $\mathcal{F}$ be class of connected structures that is closed for homomorphic images. Then there is an ω-categorical universal structure in $\mathrm{Forb}_H(\mathcal{F})$ if and only if $\mathcal{F}$ is regular.

- The finite case:

  Cherlin, Shelah, Shi, 1999: Universal graphs with forbidden subgraphs and algebraic closure

  Covington, 1990: Homogenizable relational structures

- The finite case of relational trees:

  Nešetřil, Tardif, 2000: Duality theorems for finite structures (characterising gaps and good characterisations)

The existence of ω-categorical universal graph

Theorem (H., Nešetřil, 2015)

Let $\mathcal{F}$ be class of connected structures that is closed for homomorphic images. Then there is an ω-categorical universal structure in $\mathrm{Forb}_H(\mathcal{F})$ if and only if $\mathcal{F}$ is regular.

- The finite case:

  Cherlin, Shelah, Shi, 1999: Universal graphs with forbidden subgraphs and algebraic closure

  Covington, 1990: Homogenizable relational structures

- The finite case of relational trees:

  Nešetřil, Tardif, 2000: Duality theorems for finite structures (characterising gaps and good characterisations)

- The infinite case of relational trees:

  P. L. Erdös, Pálvölgyi, Tardif, Tardos, 2012: On infinite-finite tree-duality pairs of relational structures

The Ramsey Property

Consider special case of $\mathrm{Forb}_H(\mathbf{C}_5)$.

- The homogenization is a metric space with distances 1,2,3.

The Ramsey Property

Consider special case of $\mathrm{Forb}_H(\mathbf{C}_5)$.

- The homogenization is a metric space with distances 1,2,3.
- Describe the metric space by forbidden triangles implying image of 5-cycle

$$1 - 1 - 1, \qquad 1 - 2 - 2, \qquad 3 - 1 - 1$$

and non-metric triangles:

$$3 - 1 - 1$$

The Ramsey Property

Consider special case of $\mathrm{Forb}_H(\mathbf{C}_5)$.

- The homogenization is a metric space with distances 1,2,3.
- Describe the metric space by forbidden triangles implying image of 5-cycle

$$1-1-1, \qquad 1-2-2, \qquad 3-1-1$$

  and non-metric triangles:

$$3-1-1$$

- Let **A** and **B** be such metric spaces. Applying Nešetřil-Rödl theorem obtain $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

The Ramsey Property

Consider special case of $\mathrm{Forb}_H(\mathbf{C}_5)$.

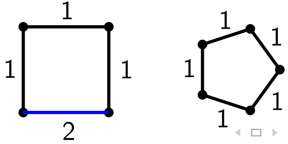
- The homogenization is a metric space with distances 1,2,3.
- Describe the metric space by forbidden triangles implying image of 5-cycle

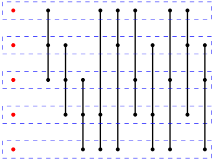
$$1-1-1, \qquad 1-2-2, \qquad 3-1-1$$

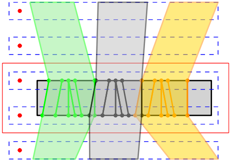
and non-metric triangles:

$$3-1-1$$

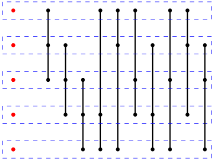
- Let **A** and **B** be such metric spaces. Applying Nešetřil-Rödl theorem obtain $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.
- Little trouble: **C** is not a complete structure and may not be complete to a metric space at all!

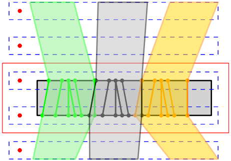






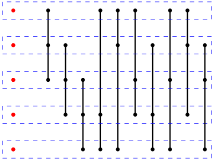
- Nešetřil-Rödl Theorem:
  $\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.
  By mean of forbidden irreducible substructures force $\mathbf{C}_0$ to be 3-colored graph without triangles
  $111, 122, 311$

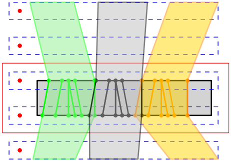




- Nešetřil-Rödl Theorem:
  $\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.
  By mean of forbidden irreducible substructures force $\mathbf{C}_0$ to be 3-colored graph without triangles
  $111, 122, 311$
- Construct C_0-partite $\mathbf{P}_0$

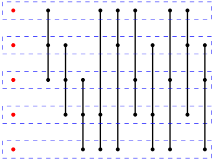
The Induced Partite Construction

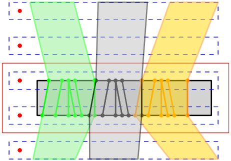




- Nešetřil-Rödl Theorem:
  $\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.
  By mean of forbidden irreducible substructures force $\mathbf{C}_0$ to be 3-colored graph without triangles
  $111, 122, 311$
- Construct C_0-partite $\mathbf{P}_0$
- Enumerate by $\mathbf{A}_1, \ldots \mathbf{A}_N$ all possible projections of copies of $\mathbf{A}$ in $\mathbf{P}_0$

The Induced Partite Construction

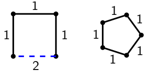




- Nešetřil-Rödl Theorem:
  $\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.
  By mean of forbidden irreducible substructures force $\mathbf{C}_0$ to be 3-colored graph without triangles
  $111, 1{\color{blue}22}, {\color{red}3}11$
- Construct C_0-partite $\mathbf{P}_0$
- Enumerate by $\mathbf{A}_1, \ldots \mathbf{A}_N$ all possible projections of copies of $\mathbf{A}$ in $\mathbf{P}_0$
- Construct C_0-partite $\mathbf{P}_1 \ldots \mathbf{P}_N$:
  - $\mathbf{B}_i$: partite system induced on $\mathbf{P}_{i-1}$ by all copies of all with projection to $\mathbf{A}_i$
  - Partite lemma: $\mathbf{C}_i \longrightarrow (\mathbf{B}_i)^{\mathbf{A}_i}_2$
  - $\mathbf{P}_i$ is built by repeating the free amalgamation of $\mathbf{P}_i$ over all copies of $\mathbf{B}_i$ in $\mathbf{C}_i$

The Induced Partite Construction

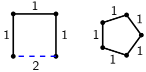
The problematic forbidden subgraphs:



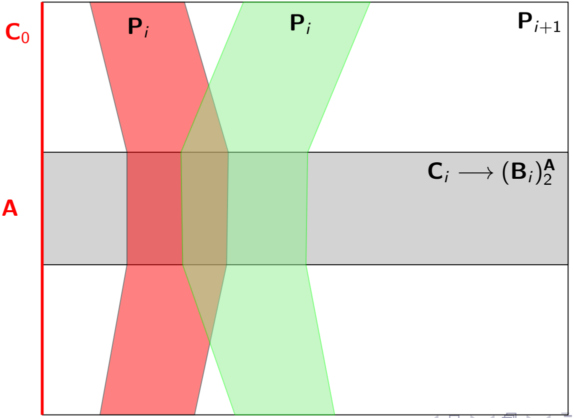
Can we prove by induction that $\mathbf{C}_0$-partite pictures $1, \ldots, N$ will omit these given that $\mathbf{C}_0$ do not contain triangles $111, 122, 311$?

The Induced Partite Construction

The problematic forbidden subgraphs:

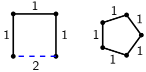


Can we prove by induction that $\mathbf{C}_0$-partite pictures 1, . . . , N will omit these given that $\mathbf{C}_0$ do not contain triangles 111, 122, 311?

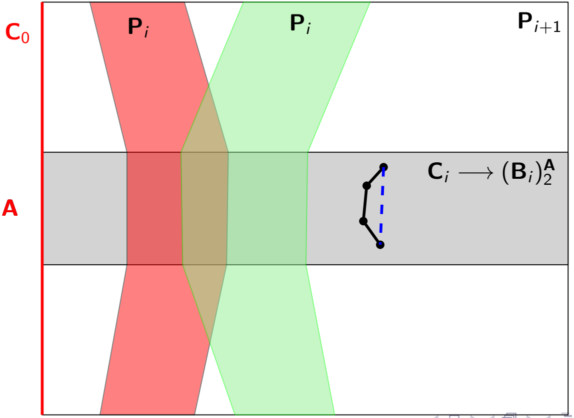


The Induced Partite Construction

The problematic forbidden subgraphs:

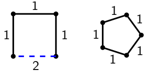


Can we prove by induction that $\mathbf{C}_0$-partite pictures 1,. . . , N will omit these given that $\mathbf{C}_0$ do not contain triangles 111, 122, 311?

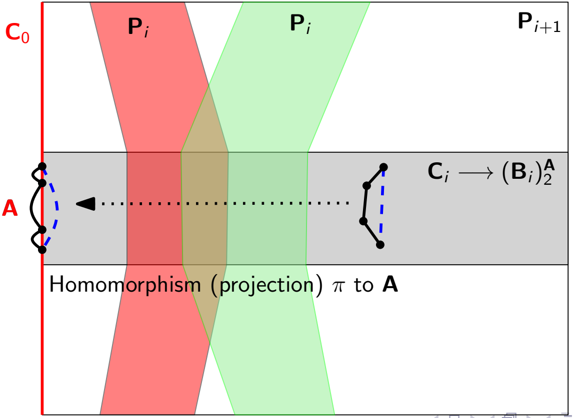


The Induced Partite Construction

The problematic forbidden subgraphs:

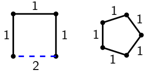


Can we prove by induction that $\mathbf{C}_0$-partite pictures 1, ..., N will omit these given that $\mathbf{C}_0$ do not contain triangles 111, 122, 311?

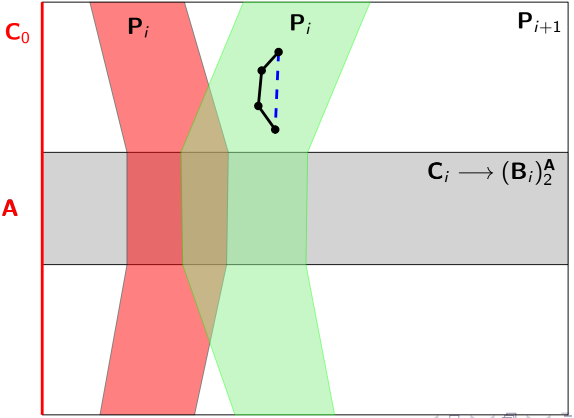


The Induced Partite Construction

The problematic forbidden subgraphs:

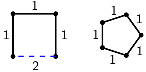


Can we prove by induction that $\mathbf{C}_0$-partite pictures 1,. . . , N will omit these given that $\mathbf{C}_0$ do not contain triangles 111, 122, 311?

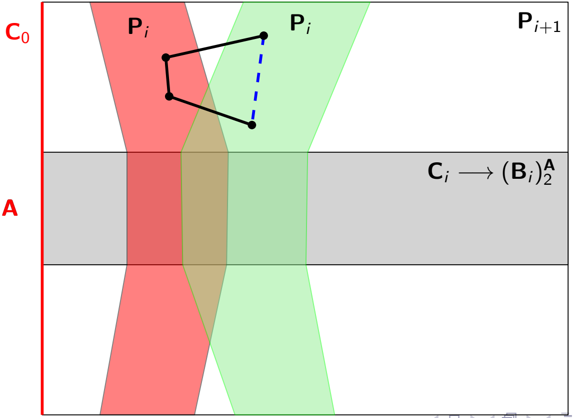


The Induced Partite Construction

The problematic forbidden subgraphs:

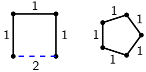


Can we prove by induction that $\mathbf{C}_0$-partite pictures 1, . . . , N will omit these given that $\mathbf{C}_0$ do not contain triangles 111, 122, 311?

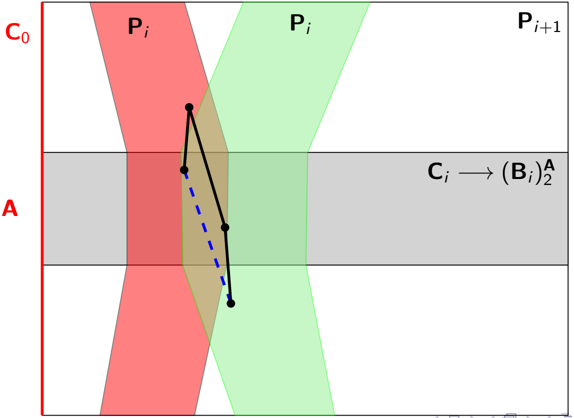


The Induced Partite Construction

The problematic forbidden subgraphs:

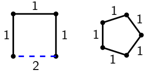


Can we prove by induction that $\mathbf{C}_0$-partite pictures $1, \ldots, N$ will omit these given that $\mathbf{C}_0$ do not contain triangles $111, 122, 311$?

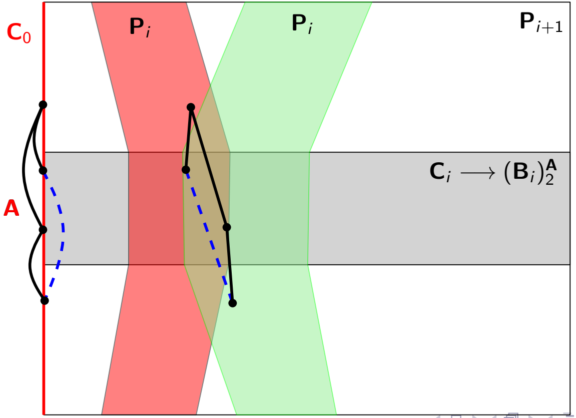


The Induced Partite Construction

The problematic forbidden subgraphs:

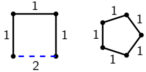


Can we prove by induction that $\mathbf{C}_0$-partite pictures $1, \ldots, N$ will omit these given that $\mathbf{C}_0$ do not contain triangles 111, 122, 311?

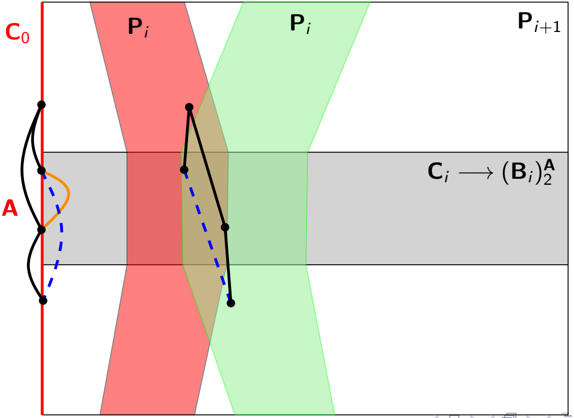


The Induced Partite Construction

The problematic forbidden subgraphs:

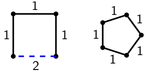


Can we prove by induction that $\mathbf{C}_0$-partite pictures $1, \ldots, N$ will omit these given that $\mathbf{C}_0$ do not contain triangles 111, $1{\color{blue}22}$, ${\color{red}3}11$?

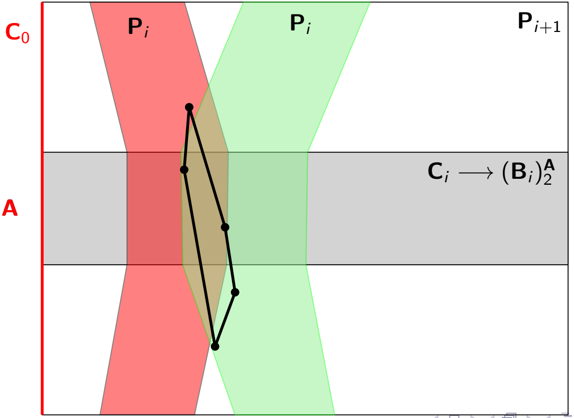


The Induced Partite Construction

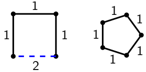
The problematic forbidden subgraphs:



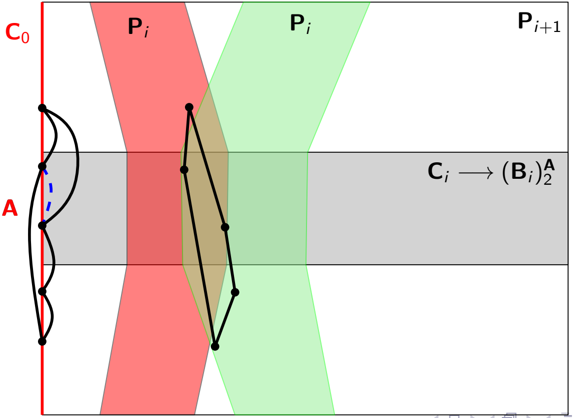
Can we prove by induction that $\mathbf{C}_0$-partite pictures 1,. . . , N will omit these given that $\mathbf{C}_0$ do not contain triangles 111, 122, 311?



The problematic forbidden subgraphs:



Can we prove by induction that $\mathbf{C}_0$-partite pictures 1, . . . , N will omit these given that $\mathbf{C}_0$ do not contain triangles 111, 122, 311?



The Iterated Induced Partite Construction

- Fix $\mathcal{F}$.
- Produce ordered homogenizing lift for $\mathrm{Forb}_H(\mathcal{F})$. Denote by $\mathcal{L}$ the class of all maximal sublifts of canonical lifts of structures in $\mathrm{Forb}_H(\mathcal{F})$.
- Fix **A** and **B** in the lifted language.

The Iterated Induced Partite Construction

- Fix $\mathcal{F}$.
- Produce ordered homogenizing lift for $\mathrm{Forb}_H(\mathcal{F})$. Denote by $\mathcal{L}$ the class of all maximal sublifts of canonical lifts of structures in $\mathrm{Forb}_H(\mathcal{F})$.
- Fix **A** and **B** in the lifted language.
- Produce forbidden configurations ("cuttings" of structures $\mathbf{F} \in \mathcal{F}$) and sort them as $\mathbf{F}_1, \ldots, \mathbf{F}_M$ in a way so number of minimal separating cuts increase.

The Iterated Induced Partite Construction

- Fix $\mathcal{F}$.
- Produce ordered homogenizing lift for $\mathrm{Forb}_H(\mathcal{F})$. Denote by $\mathcal{L}$ the class of all maximal sublifts of canonical lifts of structures in $\mathrm{Forb}_H(\mathcal{F})$.
- Fix **A** and **B** in the lifted language.
- Produce forbidden configurations ("cuttings" of structures $\mathbf{F} \in \mathcal{F}$) and sort them as $\mathbf{F}_1, \ldots, \mathbf{F}_M$ in a way so number of minimal separating cuts increase.
- Use Nešetřil-Rödl theorem to find $\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ that does not contain any irreducible structures not allowed in $\mathcal{L}$.

The Iterated Induced Partite Construction

- Fix $\mathcal{F}$.
- Produce ordered homogenizing lift for $\mathrm{Forb}_H(\mathcal{F})$. Denote by $\mathcal{L}$ the class of all maximal sublifts of canonical lifts of structures in $\mathrm{Forb}_H(\mathcal{F})$.
- Fix **A** and **B** in the lifted language.
- Produce forbidden configurations ("cuttings" of structures $\mathbf{F} \in \mathcal{F}$) and sort them as $\mathbf{F}_1, \dots, \mathbf{F}_M$ in a way so number of minimal separating cuts increase.
- Use Nešetřil-Rödl theorem to find $\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ that does not contain any irreducible structures not allowed in $\mathcal{L}$.
- Repeat partite construction to obtain $\mathbf{C}_1, \dots \mathbf{C}_M$ such that $\mathbf{C}_i \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. $\mathbf{C}_i$ is built from $\mathbf{C}_{i-1}$.

The Iterated Induced Partite Construction

- Fix $\mathcal{F}$.
- Produce ordered homogenizing lift for $\mathrm{Forb}_H(\mathcal{F})$. Denote by $\mathcal{L}$ the class of all maximal sublifts of canonical lifts of structures in $\mathrm{Forb}_H(\mathcal{F})$.
- Fix $\mathbf{A}$ and $\mathbf{B}$ in the lifted language.
- Produce forbidden configurations ("cuttings" of structures $\mathbf{F} \in \mathcal{F}$) and sort them as $\mathbf{F}_1, \ldots, \mathbf{F}_M$ in a way so number of minimal separating cuts increase.
- Use Nešetřil-Rödl theorem to find $\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ that does not contain any irreducible structures not allowed in $\mathcal{L}$.
- Repeat partite construction to obtain $\mathbf{C}_1, \ldots \mathbf{C}_M$ such that $\mathbf{C}_i \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. $\mathbf{C}_i$ is built from $\mathbf{C}_{i-1}$.
- By whack-a-mole argument show that $\mathbf{C}_i$ does not contain $\mathbf{F}_i$.

The Iterated Induced Partite Construction

- Fix $\mathcal{F}$.
- Produce ordered homogenizing lift for $\mathrm{Forb}_H(\mathcal{F})$. Denote by $\mathcal{L}$ the class of all maximal sublifts of canonical lifts of structures in $\mathrm{Forb}_H(\mathcal{F})$.
- Fix **A** and **B** in the lifted language.
- Produce forbidden configurations ("cuttings" of structures $\mathbf{F} \in \mathcal{F}$) and sort them as $\mathbf{F}_1, \ldots, \mathbf{F}_M$ in a way so number of minimal separating cuts increase.
- Use Nešetřil-Rödl theorem to find $\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ that does not contain any irreducible structures not allowed in $\mathcal{L}$.
- Repeat partite construction to obtain $\mathbf{C}_1, \ldots \mathbf{C}_M$ such that $\mathbf{C}_i \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. $\mathbf{C}_i$ is built from $\mathbf{C}_{i-1}$.
- By whack-a-mole argument show that $\mathbf{C}_i$ does not contain $\mathbf{F}_i$.
- Because $\mathbf{C}_i$ has homomorphism to $\mathbf{C}_{i-1}$ we know that $\mathbf{C}_i$ does not contain $\mathbf{F}_1, \ldots, \mathbf{F}_i$.

The Iterated Induced Partite Construction

- Fix $\mathcal{F}$.
- Produce ordered homogenizing lift for $\mathrm{Forb}_H(\mathcal{F})$. Denote by $\mathcal{L}$ the class of all maximal sublifts of canonical lifts of structures in $\mathrm{Forb}_H(\mathcal{F})$.
- Fix **A** and **B** in the lifted language.
- Produce forbidden configurations ("cuttings" of structures $\mathbf{F} \in \mathcal{F}$) and sort them as $\mathbf{F}_1, \ldots, \mathbf{F}_M$ in a way so number of minimal separating cuts increase.
- Use Nešetřil-Rödl theorem to find $\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ that does not contain any irreducible structures not allowed in $\mathcal{L}$.
- Repeat partite construction to obtain $\mathbf{C}_1, \ldots \mathbf{C}_M$ such that $\mathbf{C}_i \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. $\mathbf{C}_i$ is built from $\mathbf{C}_{i-1}$.
- By whack-a-mole argument show that $\mathbf{C}_i$ does not contain $\mathbf{F}_i$.
- Because $\mathbf{C}_i$ has homomorphism to $\mathbf{C}_{i-1}$ we know that $\mathbf{C}_i$ does not contain $\mathbf{F}_1, \ldots, \mathbf{F}_i$.
- Turn $\mathbf{C}_i$ into an maximal lift $\mathbf{C} \in \mathcal{L}$.

Infinite families of forbidden substructures

Definition

Class $\mathcal{F}$ (of relational structures) is locally finite in class $\mathcal{C}$ if for every $\mathbf{A} \in \mathcal{C}$ there is only finitely many structures $\mathbf{F} \in \mathcal{F}$, $\mathbf{F} \to \mathbf{A}$.

Infinite families of forbidden substructures

Definition

Class $\mathcal{F}$ (of relational structures) is locally finite in class $\mathcal{C}$ if for every $\mathbf{A} \in \mathcal{C}$ there is only finitely many structures $\mathbf{F} \in \mathcal{F}$, $\mathbf{F} \to \mathbf{A}$.

Theorem (H., Nešetřil, 2015)

Let $\mathcal{E}$ be a family of complete ordered structures, $\mathcal{F}$ be a regular family of connected structures. Assume that $\mathcal{F}$ is locally finite in $\mathrm{Forb}_E(\mathcal{E})$. Then class $\mathrm{Forb}_E(\mathcal{E}) \cap \mathrm{Forb}_H(\mathcal{F})$ has Ramsey lift.

Example

Theorem (Nešetřil Rödl, 1984)

Partial orders have Ramsey lift.

$\mathbf{P} = (V, \leq, \prec, \perp)$

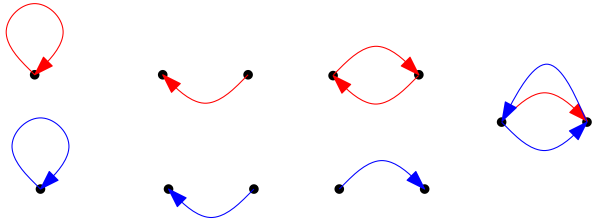
Example

Theorem (Nešetřil Rödl, 1984)

Partial orders have Ramsey lift.

$\mathbf{P} = (V, \leq, \prec, \perp)$

Forbidden complete substructures $\mathcal{E}$:



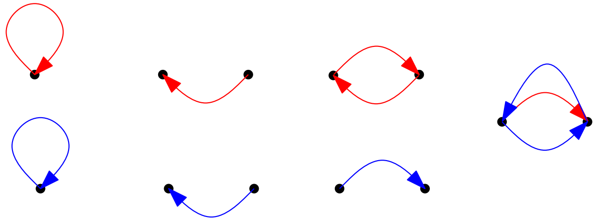
Example

Theorem (Nešetřil Rödl, 1984)

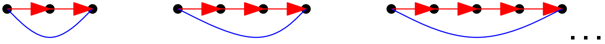
Partial orders have Ramsey lift.

$\mathbf{P} = (V, \leq, \prec, \perp)$

Forbidden complete substructures $\mathcal{E}$:



Forbidden homomorphic images $\mathcal{F}$:



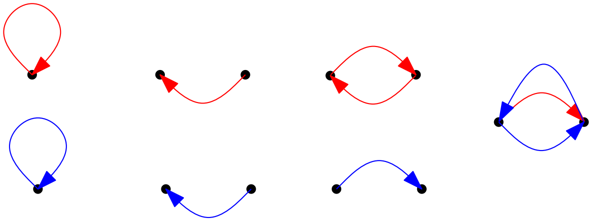
Example

Theorem (Nešetřil Rödl, 1984)

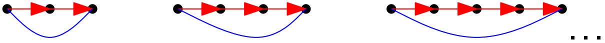
Partial orders have Ramsey lift.

$\mathbf{P} = (V, \leq, \prec, \perp)$

Forbidden complete substructures $\mathcal{E}$:



Forbidden homomorphic images $\mathcal{F}$:



Structures in $\mathrm{Forb}_E(\mathcal{E}) \cup \mathrm{Forb}_H(\mathcal{F})$ can be completed into partial orders without affecting existing $\prec$ and $\perp$ relations.

General statement

Definition

Let $\mathcal{R}$ be a Ramsey class, $\mathcal{H}$ be a family of finite ordered connected structures, and, $\mathcal{C}$ an closure description.
$\mathcal{K}$ is $(\mathcal{R}, \mathcal{F}, \mathcal{C})$-multiamalgamation class if:

1. $\mathcal{K}$ is a subclass of the class of all $\mathcal{C}$-closed structures in $\mathcal{R} \cap \mathrm{Forb}_H(\mathcal{F})$.

General statement

Definition

Let $\mathcal{R}$ be a Ramsey class, $\mathcal{H}$ be a family of finite ordered connected structures, and, $\mathcal{C}$ an closure description.
$\mathcal{K}$ is $(\mathcal{R}, \mathcal{F}, \mathcal{C})$-multiamalgamation class if:

1. $\mathcal{K}$ is a subclass of the class of all $\mathcal{C}$-closed structures in $\mathcal{R} \cap \mathrm{Forb}_H(\mathcal{F})$.
2. $\mathcal{F}$ is regular and locally finite in $\mathcal{R}$

General statement

Definition

Let $\mathcal{R}$ be a Ramsey class, $\mathcal{H}$ be a family of finite ordered connected structures, and, $\mathcal{C}$ an closure description.
$\mathcal{K}$ is $(\mathcal{R}, \mathcal{F}, \mathcal{C})$-multiamalgamation class if:

1. $\mathcal{K}$ is a subclass of the class of all $\mathcal{C}$-closed structures in $\mathcal{R} \cap \mathrm{Forb}_H(\mathcal{F})$.
2. $\mathcal{F}$ is regular and locally finite in $\mathcal{R}$
3. **Completetion property:** Let **B** be structure from $\mathcal{K}$, **C** be $\mathcal{C}$-semi-closed structure with homomorphism to some structure in $\mathcal{R} \cap \mathrm{Forb}_H(\mathcal{F})$ such that every vertex of **C** as well as every tuple in every relation of **C** is contained in a copy of **B**. Then there exists $\overline{\mathbf{C}} \in \mathcal{K}$ and a homomorphism $h : \mathbf{C} \to \overline{\mathbf{C}}$ such that h is an embedding on every copy of **B**.

General statement

Definition

Let $\mathcal{R}$ be a Ramsey class, $\mathcal{H}$ be a family of finite ordered connected structures, and, $\mathcal{C}$ an closure description.
$\mathcal{K}$ is $(\mathcal{R}, \mathcal{F}, \mathcal{C})$-multiamalgamation class if:

1. $\mathcal{K}$ is a subclass of the class of all $\mathcal{C}$-closed structures in $\mathcal{R} \cap \mathrm{Forb}_H(\mathcal{F})$.
2. $\mathcal{F}$ is regular and locally finite in $\mathcal{R}$
3. **Completetion property:** Let **B** be structure from $\mathcal{K}$, **C** be $\mathcal{C}$-semi-closed structure with homomorphism to some structure in $\mathcal{R} \cap \mathrm{Forb}_H(\mathcal{F})$ such that every vertex of **C** as well as every tuple in every relation of **C** is contained in a copy of **B**. Then there exists $\overline{\mathbf{C}} \in \mathcal{K}$ and a homomorphism $h : \mathbf{C} \to \overline{\mathbf{C}}$ such that h is an embedding on every copy of **B**.

Theorem (H. Nešetřil, 2015)

Every $(\mathcal{R}, \mathcal{F}, \mathcal{C})$-multiamalgamation class $\mathcal{K}$ has a Ramsey lift.

Example

Example

Consider relational structure with two relations R^1 and R^2 where both relations forms an acyclic graph. Further forbid all cycles consisting of one segment in R^1 and other in R^2.

Example

Example

Consider relational structure with two relations R^1 and R^2 where both relations forms an acyclic graph. Further forbid all cycles consisting of one segment in R^1 and other in R^2.

- Show that acyclic graphs in R^1 with linear extension forms an Ramsey class
- Show that acyclic graphs in R^2 with linear extension forms an Ramsey class

Example

Consider relational structure with two relations R^1 and R^2 where both relations forms an acyclic graph. Further forbid all cycles consisting of one segment in R^1 and other in R^2.

- Show that acyclic graphs in R^1 with linear extension forms an Ramsey class
- Show that acyclic graphs in R^2 with linear extension forms an Ramsey class
- Use the fact that strong amalgamation Ramsey classes can be interposed freely to build Ramsey class $\mathcal{R}$. $\mathcal{R}$ now has two independent linear orders.

Example

Example

Consider relational structure with two relations R^1 and R^2 where both relations forms an acyclic graph. Further forbid all cycles consisting of one segment in R^1 and other in R^2.

- Show that acyclic graphs in R^1 with linear extension forms an Ramsey class
- Show that acyclic graphs in R^2 with linear extension forms an Ramsey class
- Use the fact that strong amalgamation Ramsey classes can be interposed freely to build Ramsey class $\mathcal{R}$. $\mathcal{R}$ now has two independent linear orders.
- Show that the family of all bi-colored oriented cycles $\mathcal{B}$ is regular
- Show that the class in question is $(\mathcal{R}, \mathcal{B}, \emptyset)$-multiamalgamation class

How complex can be Ramsey lift?

…it contains at least an homogenizing lift

- **free linear order**:
  graphs, digraphs, $\mathrm{Forb}_H(\mathcal{F})$ classes, metric spaces, …
- **convex linear order**:
  classes with unary relations
- **unary predicate and convex linear order**:
  n-partite graphs, dense cyclic order
- **linear extension**:
  acyclic graphs, partial orders and variants
- **multiple linear extensions**:
  two freely overlapped acyclic graphs possibly with additional constraints
- **ordered digraph**:
  a structure with ternary relations where neighborhood of every vertex forms a bipartite graph

Map of Ramsey Classes

boolean algebras

Unary CSS classes

interval graphs

metric spaces

permutations

acyclic graphs

cyclic orders

partial orders

unions of complete

K_n-free graphs

linear orders

graphs

restricted

free

Map of Ramsey Classes

Forb homo and degrees

boolean algebras

Unary CSS classes

interval graphs

metric spaces

permutations

acyclic graphs

cyclic orders

partial orders

unions of complete graphs

K_n-free graphs

linear orders

graphs

restricted

free

Open problems

Can we use techniques above to find Ramsey lift of the following?

- all (non-unary) Cherlin-Shelah-Shi classes,
- classes produced by Hrusovski construction,
- $\mathbf{C}_4$-free graphs where very pair of vertices has closure denoting the only vertex connected to both,
- semilattices, lattices and boolean algebras

Open problems

Can we use techniques above to find Ramsey lift of the following?

- all (non-unary) Cherlin-Shelah-Shi classes,
- classes produced by Hrusovski construction,
- $\mathbf{C}_4$-free graphs where very pair of vertices has closure denoting the only vertex connected to both,
- semilattices, lattices and boolean algebras

Can the notion of multiamalgamation be extended to handle more algebraic structures, such as groups and semigroups?

Open problems

Can we use techniques above to find Ramsey lift of the following?

- all (non-unary) Cherlin-Shelah-Shi classes,
- classes produced by Hrusovski construction,
- $\mathbf{C}_4$-free graphs where very pair of vertices has closure denoting the only vertex connected to both,
- semilattices, lattices and boolean algebras

Can the notion of multiamalgamation be extended to handle more algebraic structures, such as groups and semigroups?

. . . Can we find examples of Ramsey classes without Ramsey lift or does all homogeneous classes with finite closures permit Ramsey lifts?

Thank you!