Free homogeneous structures are generalised measurable

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Thanks and acknowledgements

Thank you for giving me this chance to speak at this wonderful conference.

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This work is distinct from, but closely related to, two ongoing collaborations: one with **Dugald Macpherson**, **Charlie Steinhorn**, and **Daniel Wolf**; and the other with **Charlotte Kestner**. These collaborations have certainly informed this work and I'm greatly indebted to those colleagues **and others**. Also thanks to EPSRC.

Afterwards, I'll post these slides on `anscombe.sdf.org/research.html`.

Apologies for using beamer, but I will give several cumbersome definitions.

Two themes and one theorem

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– definable sets can be meaningfully assigned (generalised) measure and dimension, which takes only finitely many values on any definable family

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(A) Free homogeneous structures,
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– definable sets can be meaningfully assigned (generalised) measure and dimension, which takes only finitely many values on any definable family

Main Theorem

Free homogeneous structures in finite relational languages are generalised measurable.

Coming soon to a whiteboard near you...

1. **Theme B**: Generalised measurable structures
   - Finite fields
   - Measurable structures
   - Pseudofinite fields, the random graph
   - Generalised measurable structures
2. **Theme A**: Free homogeneous structures
   - Definition
   - From now on...
3. **Proof** of Main Theorem
   - Definition of Γ_r
   - The measuring semiring T_r
   - Finishing the proof

Conventions

1. I will try to use the colour **magenta** for ideas that deserve a picture.
2. 'Definable':='definable with parameters'.
3. '$\emptyset$-definable':='definable without parameters'.
4. $\mathrm{Def}(\mathcal{M})$ is the set of definable sets in the structure $\mathcal{M}$.
5. $0 \in \mathbb{N}$.
6. Tuples will usually be written in lowercase.
7. $\phi(\mathcal{M})$ is the set defined in a structure $\mathcal{M}$ by the formula $\phi(x)$.

Theorem (Chatzidakis–van den Dries–Macintyre)

Let $\phi(x; y)$ be a formula in the language of rings. Then there exist $C \in \mathbb{R}_{>0}$, $\mu_1, \ldots, \mu_r \in \mathbb{Q}_{>0}$, and $d_1, \ldots, d_r \in \mathbb{N}$ such that for any prime power q and any $b \in \mathbb{F}_q$ there exists $i \in \{1, \ldots, r\}$ such that $\phi(\mathbb{F}_q; b)$ is empty or

$$\left| |\phi(\mathbb{F}_q; b)| - \mu_i q^{d_i} \right| < C q^{d_i - \frac{1}{2}}. \qquad (*_i)$$

Furthermore, for each $i \in \{1, \ldots, r\}$ the set

$$\{b \in \mathbb{F}_q \mid (*_i) \text{ holds}\}$$

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In **red** is the monomial function (of q) that measures the approximate size of the set $\phi(\mathbb{F}_q; b)$:

$$q \longmapsto \mu_i q^{d_i}.$$

Macpherson–Steinhorn:

1. They study classes $\mathcal{C}$ of finite structures in which the C–D–M theorem holds (appropriately restated): *one-dimensional asymptotic classes*.
2. Taking an ultraproduct $\mathcal{M}$ of such classes gives a function

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They axiomatise this as follows:

Definition (Macpherson–Steinhorn, slightly reformulated)

The structure $\mathcal{M}$ is *measurable* if there is

$$h : \mathrm{Def}(\mathcal{M}) \longrightarrow \mathbb{R}e^{\mathbb{N}}$$

such that

1. h is finitely additive, and if X is a singleton then $h(X) = 1e^0$;
2. **'MAC' condition**: For every $\phi(x; y)$,
   1. $\{h(\phi(\mathcal{M}; b)) \mid b \in \mathcal{M}\} = \{r_1 e^{d_1}, \ldots, r_n e^{d_n}\}$ is a finite set **(picture)**, and
   2. $\{b \in \mathcal{M} \mid h(\phi(\mathcal{M}; b)) = r_i e^{d_i}\}$ is $\emptyset$-definable, for each i; and
3. **Fubini**: If $p : X \longrightarrow Y$ is a definable surjection and h takes the constant value re^d on each fibre of p, then

$$h(X) = h(Y) \cdot re^d.$$

Theorem (Macpherson–Steinhorn)

For each pseudofinite field F we have a measuring-function

$$h_F : \mathrm{Def}(F) \longrightarrow \mathbb{R}e^{\mathbb{N}}.$$

Proof.

Pseudofinite fields are elementarily equivalent to non-principal ultraproducts of finite fields. □

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Theorem (Macpherson–Steinhorn)

For the random graph $\mathcal{R}$ we have a measuring-function

$$h_{\mathcal{R}} : \mathrm{Def}(\mathcal{R}) \longrightarrow \mathbb{R}e^{\mathbb{N}}.$$

Proof.

Goes via Paley graphs. These are graphs defined on finite fields $\mathbb{F}_q$ with $q \equiv 1 \pmod 4$: define aEb iff $a - b$ is a square. □

Question

For the generic triangle-free graph $\mathcal{H}_3$, does there exist a measuring-function

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Answer: No. In any measurable structure, 'dimension' is well-founded (model-theoretic language: measurable implies supersimple). However, in $\mathcal{H}_3$ there are infinite descending chains of definable sets with strictly decreasing dimension (model-theoretic language: dividing, tree property of the first kind).

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So it all comes to down to *dimension*.

Definition

$T = (T, +, \cdot, 0, 1, \leq)$ is a *measuring semiring* if

1. $(T, +, 0)$ and $(T, \cdot, 1)$ are commutative monoids,
2. $\cdot$ distributes over $+$,
3. $(T, \leq, 0)$ is a totally ordered set with least element 0,
4. $\forall x, y, z\ (x \leq y \longrightarrow x + z \leq y + z)$,
5. $\forall x, y, z\ (x \leq y \longrightarrow x \cdot z \leq y \cdot z)$,
6. $\forall x\ 0 \cdot x = x \cdot 0 = 0$;

and

7. $\forall x, y, z$ if $x < y$ and (either $y \leq z \leq ny$ or $z \leq y \leq nz$, for some $n \in \mathbb{N}$) then
$$x + z < y + z.$$

Let $T = (T, +, \cdot, 0, 1, \leq)$ be a measuring semiring.

Definition (A.–Macpherson–Steinhorn–Wolf)

The structure $\mathcal{M}$ is *T-measurable* if there is a *measuring-function*

$$h : \mathrm{Def}(\mathcal{M}) \longrightarrow T$$

such that

1. h is finitely additive, and if X is a singleton then $h(X) = 1$;
2. **'MAC' condition**: For every $\phi(x; y)$,
    1. $\{h(\phi(\mathcal{M}; b)) \mid b \in \mathcal{M}\} = \{t_1, ..., t_n\}$ is a finite set, and
    2. $\{b \in \mathcal{M} \mid h(\phi(\mathcal{M}; b)) = t_i\}$ is $\emptyset$-definable, for each i; and
3. **Fubini**: If $p : X \longrightarrow Y$ is a definable surjection and h takes the constant value $t \in T$ on each fibre of p, then

$$h(X) = h(Y) \cdot t.$$

Definition (AMSW)

$\mathcal{M}$ is *generalised measurable* if it is T-measurable for some measuring semiring T.

Free homogeneous structures

Fix a finite relational language $\mathcal{L} := \{R_1, ..., R_r\}$.

Definition (Free amalgamation class)

A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is a *free amalgamation class* if

1. closed under isomorphism and substructure,
2. has the joint embedding property, and
3. has the *free amalgamation property*.

A *free homogeneous* structure is a Fraïssé amalgam of a free amalgamation class.

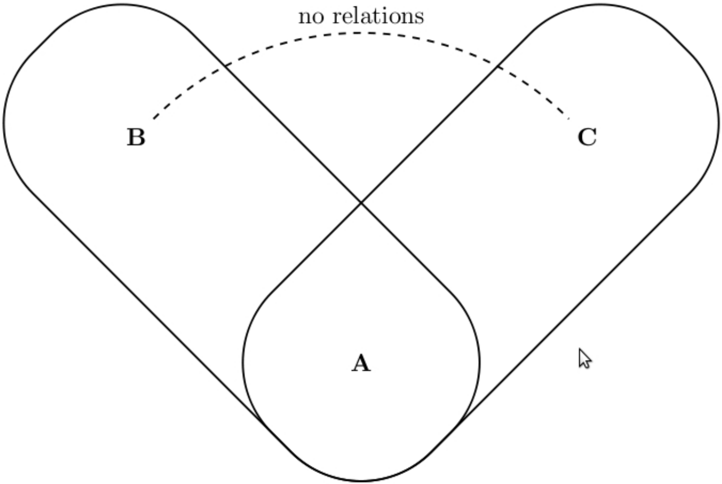

Figure: Free amalgamation

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Facts

If $\mathcal{M}$ is the Fraïssé amalgam of the free amalgamation class $\mathcal{C}$ then $\mathcal{C} = \mathrm{age}(\mathcal{M})$, $\mathcal{M}$ is $\aleph_0$-categorical, (ultra)homogeneous, has quantifier elimination, and algebraic closure is trivial.

Examples

1. $\mathcal{C}_0 := \{\text{finite graphs}\}$, $\mathcal{R} =$ random graph.
2. $\mathcal{C}_n := \{\text{finite } K_n\text{-free graphs}\}$, $\mathcal{H}_n =$ generic K_n-free graph.
3. Hypergraph versions of these.

Non-examples

1. class = finite partial orders, limit = generic partial order.
2. class = finite total orders, limit = $(\mathbb{Q}, <)$.
3. class = finite totally ordered graphs, limit = generic totally ordered graph.

From now on we work with a fixed free homogeneous $\mathcal{L}$-structure $\mathcal{M}$.

We can think of $\mathcal{M} = \mathcal{H}_3$ = generic triangle-free graph.

Definition of Γ_r

Definition

The structure $\Gamma_r = (\Gamma_r, +, 0, -\infty, \leq)$ is defined as follows.

1. Γ_r is the set
$$\Gamma_r^* := \omega^* \oplus_{\text{lex}} (-\omega) \oplus_{\text{lex}} \dots \oplus_{\text{lex}} (-\omega)$$
adjoined by two elements $-\infty$ and 0;
2. the addition $+$ is coordinate-wise on the lexicographic product, 0 is the identity, and $-\infty$ is a zero; and
3. $\leq$ is such that $-\infty < 0 < \Gamma_r^*$.

Example

$$\Gamma_1 = \{-\infty\} \underset{<}{\sqcup} \{0\} \underset{<}{\sqcup} \Big(\omega^* \oplus_{\text{lex}} (-\omega)\Big).$$

Definition

Let $T_r = (T_r, \oplus, \otimes, 0e^{-\infty}, 1e^0, \leq) := \mathbb{N}e^{\Gamma_r}$.

Underlying set:

$$\{me^d \mid m \in \mathbb{N} \setminus \{0\}, d \in \Gamma_r \setminus \{-\infty\}\} \sqcup \{0e^{-\infty}\}$$

1. $m_1 e^{d_1} \oplus m_2 e^{d_2} := \begin{cases} (m_1 + m_2)e^{d_1} & d_1 = d_2 \\ m_1 e^{d_1} & d_1 > d_2 \\ m_2 e^{d_2} & d_1 < d_2; \end{cases}$
2. $m_1 e^{d_1} \otimes m_2 e^{d_2} := (m_1 \cdot m_2)e^{d_1 + d_2}$;
3. $m_1 e^{d_1} \leq m_2 e^{d_2}$ iff (by def.) $d_1 < d_2$ OR ($d_1 = d_2$ AND $m_1 \leq m_2$);
4. $0e^{-\infty}$ is the $\oplus$-identity and a '$\otimes$-zero', and $1e^0$ is the $\otimes$-identity.

Fact

T_r is a measuring semiring.

Theorem (Main Theorem again)

$\mathcal{M}$ is T_r-measurable with measuring-function $h_{\mathcal{M}}$.

Finishing the proof

It remains to argue that

$$h_{\mathcal{M}} : \mathrm{Def}(\mathcal{M}) \longrightarrow T_r$$

really is a measuring-function.

1. Finitely additive
2. Finite sets: a single tuple is a complete type
3. 'mac' condition: follows from $\aleph_0$-categoricity.
4. Fubini condition: