How synchronizing are primitive groups?

Wolfram Bentz

CEMAT/CIÊNCIAS, Universidade de Lisboa

Joint work with
João Araújo (CEMAT/CIÊNCIAS, Universidade de Lisboa)
Peter J. Cameron (Mathematical Institute, University of St Andrews)
Gordon Royle (Centre for the Mathematics of Symmetry and Computation, University of Western Australia)
Arthur Schaefer (Mathematical Institute, University of St Andrews)

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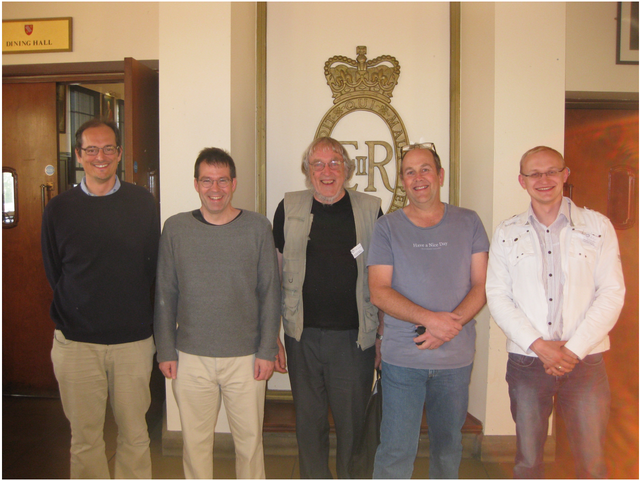
DINING HALL
Have a Nice Day

The prisoner in the dungeon

Imagine the following:

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- You have a complete map of the dungeon, that also shows the favorable yellow door.

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- You do not know which cave you are in.

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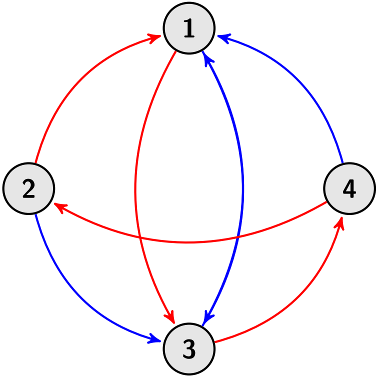

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- So you want to run a route that will always end up in the same spot.
- Once you know your location you can go wherever you want (if everything is connected).

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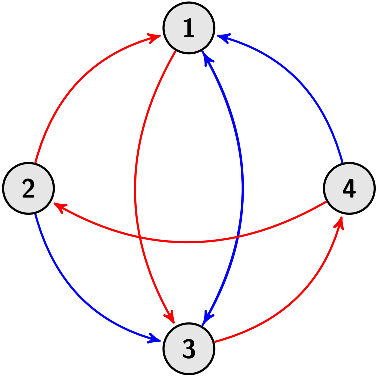

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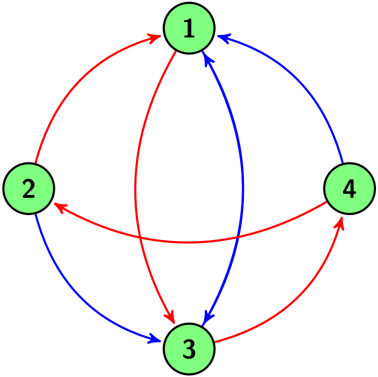

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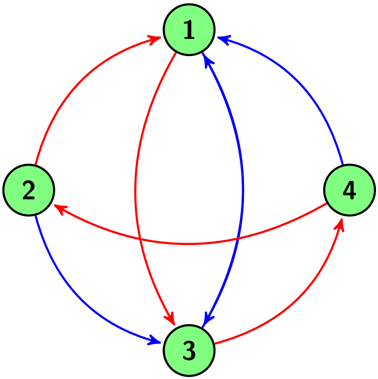

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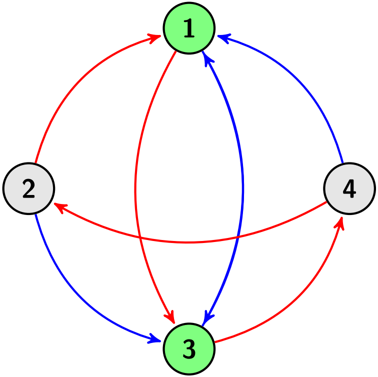

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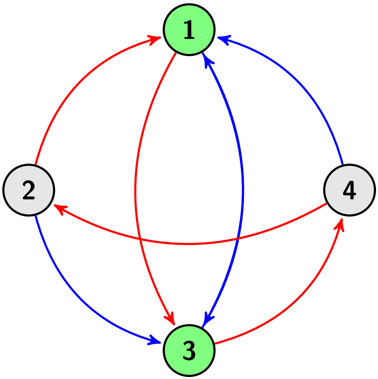

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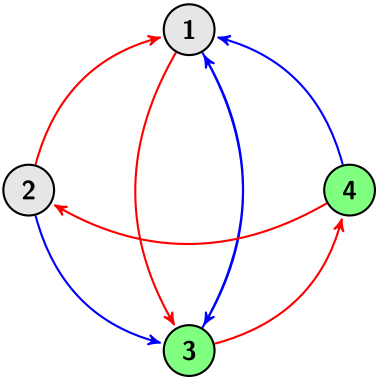

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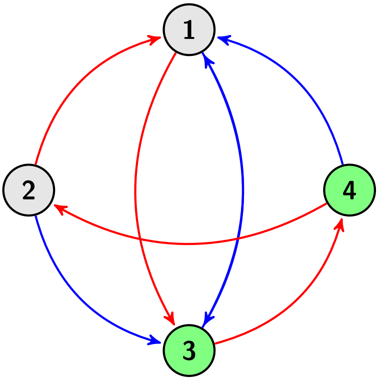

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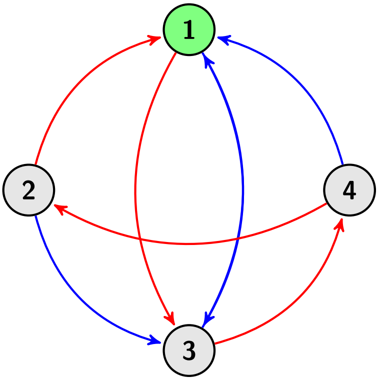

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- Such an option might not always exist or be ideal.

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- Before automatic work can be performed on them, the need to be aligned correctly.

Forward Error Recovery

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- A forward recovery mechanism is a sequence of procedures or instructions that brings the process to a known state *irrespectively of its current state*.

Synchronization for Automata, Semigroups, and Groups

Definition

An automaton is *synchronizing* if there is a sequence of inputs that puts the automaton into a specific state, irrespective of its current state.

Definition

S_n and T_n are, respectively, the symmetric group and the full transformation monoid on the set $X = \{1, \ldots, n\}$.
A set $S \subseteq T_X$ of transpositions is *synchronizing* if the semigroup $\langle S \rangle$ contain a constant map.
We say that a group $G \leq S_n$ *synchronizes* a transformation $t \in T_n \setminus S_n$ if the subsemigroup of T_n generated by $G \cup \{t\}$ contains a constant map.

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- If the group G is primitive, but not synchronizing, let t be a transformation not synchronized by G with minimal rank. Then t has *uniform kernel*, i.e. all kernel classes are of the same size ([Neumann 2009]).
- If $|X|$ is prime, every primitive group over X is synchronizing.

From transformation semigroups to graphs

Let $M \leq T_X$, then the graph $\mathrm{Gr}(M)$ has vertex set X and two vertices $x, y \in X$ are adjacent if there is NO $f \in M$ for which $xf = yf$.

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  - the chromatic number of $\mathrm{Gr}(M)$.

From transformation semigroups to graphs

Theorem

A transformation semigroup M is non-synchronizing if and only if there is a non-null graph Γ such that $M \subseteq \mathrm{End}(\Gamma)$. Moreover, Γ can be assumed to have coinciding clique and chromatic numbers.

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- Suppose that A is a set of vertices, such that the union of their neighborhoods has size $l+d$. Then $|A| \leq \binom{d+1}{2}$.

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It was previously known that a primitive group $G \leq S_n$ synchronizes every map of rank $n-1$ (Rystsov) and $n-2$ (Araújo, Cameron).

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- Rank $n-4$: many cases, tricky cases!

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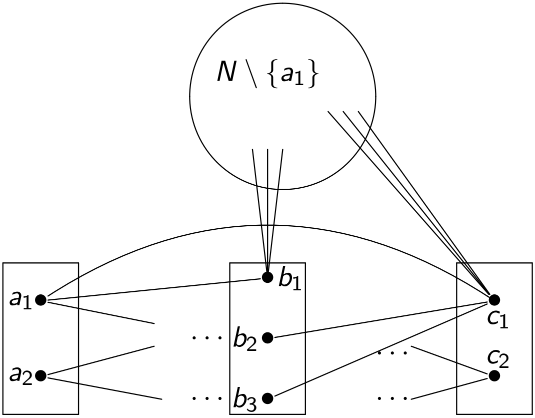

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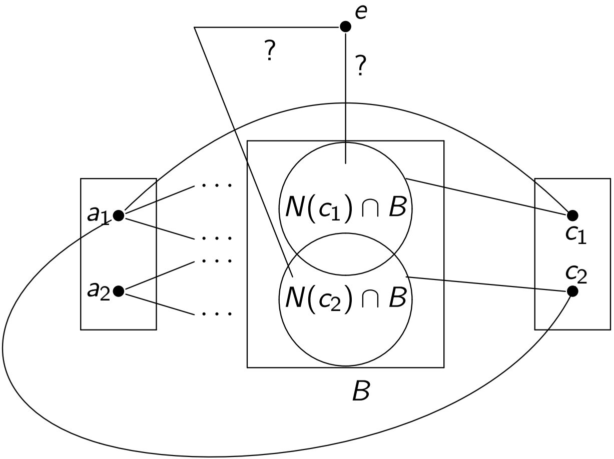

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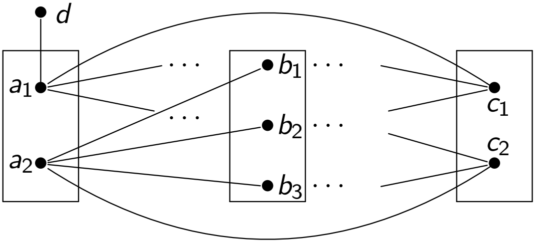

More results for maps of large rank

- Suppose that $j \geq 1$, $d_1, \ldots, d_j \geq 2$, $d = -j + \Sigma d_i$, $l \geq \binom{d+1}{2} + 1$, and G a primitive group. Then G synchronized every map of kernel type $(l, d_1, \ldots, d_j, 1, \ldots, 1)$.

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- Let G act primitively on X with $|X| = n$ and suppose that the induced action of G on X^2 has exactly three orbits. Then G synchronizes every map of rank larger than $n - \left(1 + \sqrt{n-1}/12\right)$.

Maps of small ranks - Flashback

Definition (Araújo, WB, Cameron (ABC))

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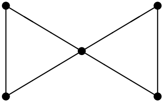
Theorem (Araújo, Cameron)

Every primitive groups synchronizes every non-uniform map of rank at most 4*.*

The tale of the butterfly



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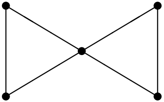

Theorem

There exists a primitive group G acting on a set X of 45 points and a transformation t on X, such that G does not synchronize t and t has kernel type $(15, 10, 10, 5, 5)$.

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- Primitive groups are synchronizing for maps of large rank.
- There are cases of primitive groups with "synchronization deficiencies".
- How deficient can this get in the worst case?

How synchronizing are primitive groups?

- The "typical" primitive group is synchronizing.
- Primitive groups are synchronizing for maps of large rank.
- There are cases of primitive groups with "synchronization deficiencies".
- How deficient can this get in the worst case? See Gordon's talk!

A final conjecture

Conjecture

A primitive group of rank n synchronizes every map of rank r for $n/2 < r < n$.

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Thank you!

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