Metrically Homogeneous Graphs of Generic Type

Gregory Cherlin

THE STATE UNIVERSITY OF NEW JERSEY RUTGERS

July 22, 2015—Durham

1 The classification problem

2 Two dividing lines

3 Generic Type

4 Evidence

Metric Homogeneity

. . . Urysohn has managed to construct a . . . complete metric space with a countable dense subset, which contains any other separable metric space isometrically, and furthermore satisfies a quite strong homogeneity condition; the latter being that one can take the whole space (isometrically) onto itself, so that an arbitrary finite set M is carried over to an equally arbitrary finite set $M1$ congruent to it.

[Alexandrov to Hausdorff (in German), 3.8.24]

Metric Homogeneity

. . . Urysohn has managed to construct a . . . complete metric space with a countable dense subset, which contains any other separable metric space isometrically, and furthermore satisfies a quite strong homogeneity condition; the latter being that one can take the whole space (isometrically) onto itself, so that an arbitrary finite set M is carried over to an equally arbitrary finite set $M1$ congruent to it.
[Alexandrov to Hausdorff (in German), 3.8.24]

$$\mathbb{U} = \bar{\mathbb{U}}_{\mathbb{Q}} \qquad \mathbb{U}_{\mathbb{Q}} = \lim_{\mathcal{F}} \mathbb{Q}\text{-metric}$$

The Urysohn graph

$$\mathbb{U}_{\mathbb{Z}} = \lim_{\mathcal{F}} \mathbb{Z}\text{-metric}$$

$\Gamma_{\mathbb{Z}}$: (a, b) is an edge iff $d(a, b) = 1$.

The Urysohn graph

$$\mathbb{U}_{\mathbb{Z}} = \lim_{\mathcal{F}} \mathbb{Z}\text{-metric}$$

$\Gamma_{\mathbb{Z}}$: (a, b) is an edge iff $d(a, b) = 1$.

Remark

The metric on $\mathbb{U}_{\mathbb{Z}}$ is the path metric in $\Gamma_{\mathbb{Z}}$.

Thus $\mathbb{U}_{\mathbb{Z}}$ is a countable universal graph (allowing isometric embeddings in the connected case).

The Urysohn graph

$$\mathbb{U}_{\mathbb{Z}} = \lim_{\mathcal{F}} \mathbb{Z}\text{-metric}$$

$\Gamma_{\mathbb{Z}}$: (a, b) is an edge iff $d(a, b) = 1$.

Remark

The metric on $\mathbb{U}_{\mathbb{Z}}$ is the path metric in $\Gamma_{\mathbb{Z}}$.

Thus $\mathbb{U}_{\mathbb{Z}}$ is a countable universal graph (allowing isometric embeddings in the connected case).

Local structure: The induced graph Γ_1 on the neighbors of a vertex is the random graph.

The classification problem

Metrically homogeneous graph A connected graph Γ whose associated metric space is homogeneous in Urysohn's sense.

> *. . . In the countable case, an answer to this question might be a step towards a classification of the distance homogeneous graphs.*
> *[Moss,* Distanced Graphs, *1992]*

> *. . . the theory of infinite distance-transitive graphs is open. Not even the countable metrically homogeneous graphs have been determined.*
> *[Cameron,* A census of infinite distance-transitive graphs, *1998]*

What we have

- A natural division between the special and generic types;
- A full classification for the special types;
- A conjectured classification of generic type (nearly uniform);
- A reasonable amount of supporting evidence.

1 The classification problem

2 Two dividing lines

3 Generic Type

4 Evidence

Local type

Γ_i: The induced metric structure at distance i from a basepoint.

Definitely homogeneous.
If there are some edges ($d(x, y) = 1$), and Γ_i is connected, then it is a metrically homogeneous graph, with the graph metric as the induced metric.

Local type

Γ_i: The induced metric structure at distance i from a basepoint.

Definitely homogeneous.
If there are some edges ($d(x,y) = 1$), and Γ_i is connected, then it is a metrically homogeneous graph, with the graph metric as the induced metric.

Γ_1 is a homogeneous graph (possibly edgeless).
We use the local structure as our main dividing line.

Γ_1

Definition

A metrically homogeneous graph has exceptional local type if Γ_1 is imprimitive, or does not contain an infinite independent set.

A metrically homogeneous graph has generic type if Γ_1 is primitive, and two vertices at distance 2 have infinitely many common neighbors.

Γ_1

Definition

A metrically homogeneous graph has exceptional local type if Γ_1 is imprimitive, or does not contain an infinite independent set.

Theorem

The metrically homogeneous graphs of exceptional local type fall into the following classes.

- *Diameter at most 2 (homogeneous graph)*
- *n-cycle*
- *Antipodal of diameter 3, finite*
- *Tree-like $T_{r,s}$, an s-branching tree of r-cliques; excluding the tree $T_{2,\infty}$.*

Γ_1

Definition

A metrically homogeneous graph has exceptional local type if Γ_1 is imprimitive, or does not contain an infinite independent set.
A metrically homogeneous graph has generic type if Γ_1 is primitive, and two vertices at distance 2 have infinitely many common neighbors.

Theorem

The only metrically homogeneous graph which is neither of exceptional local type nor generic type is the regular tree of infinite degree.

$$T_{2,\infty}$$

Major Tool The Lachlan-Woodrow classification

Non-generic type

The metrically homogeneous graphs of non-generic type are classified

1 The classification problem

2 Two dividing lines

3 Generic Type

4 Evidence

Generic type

A metrically homogeneous graph has generic type if Γ_1 is primitive, and two vertices at distance 2 have infinitely many common neighbors.

Generic type

A metrically homogeneous graph has generic type if Γ_1 is primitive, and two vertices at distance 2 have infinitely many common neighbors.

Γ_1: Henson, Random, or an independent set.
And for u, v at distance 2, the common neighbors of u, v give a graph isomorphic to Γ_1.

The Generic Type Conjecture

(Via Fraïssé theory, go to amalgamation classes and forbidden substructures.)

Conjecture (Classification Conjecture Generic Type)

For diameter $\delta \geq 3$, either

$$\mathcal{A} = \mathcal{A}_\Delta \cap \mathcal{A}_H \tag{1}$$

where $\mathcal{A}$ is an amalgamation class determined by triangle constraints, and $\mathcal{A}_H$ is an amalgamation class determined by Henson constraints; or

$$\mathcal{A} = \mathcal{A}_a \cap \mathcal{A}_{H'} \tag{2}$$

where $\mathcal{A}_a$ is an antipodal class and $\mathcal{A}_{H'}$ is determined by antipodal Henson constraints.

Henson Classes

Henson Graph Forbid 1-cliques of order *n*; or, dually, 2-cliques of order *n*.

For $\delta \geq 3$: Forbid some $(1, \delta)$-spaces.

Henson Classes

Henson Graph Forbid 1-cliques of order *n*; or, dually, 2-cliques of order *n*.

For $\delta \geq 3$: Forbid some $(1, \delta)$-spaces.

I'll probably skip the antipodal variation . . .

Triangle constraints

Theorem

Let $\mathcal{A}_\Delta$ be an amalgamation class whose Fraïssé limit is a metrically homogeneous graph of generic type, and whose minimal constraints are of order at most 3. Then

$$\mathcal{A}_\Delta = \mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$$

where K_1, K_2 are parameters controlling the forbidden triangles of small odd perimeter, and C_0, C_1 control the forbidden triangles of large perimeter (even or odd, respectively). Furthermore the parameters are subject to a certain collection of numerical constraints, e.g.

$$\text{If } C = \min(C_0, C_1) \leq 2\delta + K_1, \text{ then } C = 2K_1 + 2K_2 + 1$$

The Generic Conjecture

Minimal constraints should be triangles or Henson constraints
(and then we know everything)

1 The classification problem

2 Two dividing lines

3 Generic Type

4 Evidence

Supporting evidence

- Diameter 3 (with Amato, Macpherson)
- If the conjecture holds in finite diameter then it holds in infinite diameter (hence: induction)
- The bipartite case can be handled under an inductive hypothesis via *halving.*

Still to do: find an inductive treatment of the antipodal case, thereby reducing to the primitive case (via Smith's theorem).

4-triviality

Definition

An amalgamation class is *4-trivial* if any forbidden structure of order 4 either contains a forbidden triangle or is a Henson constraint.

Proposition

In a 4-trivial amalgamation class, the pattern of forbidden triangles is known.

Topological Dynamics

Question

In the primitive generic type case, is the universal minimal flow the space of orders?