Presentations for symmetric groups encoded by idempotents in the full transformation monoid

Robert D. Gray
University of East Anglia

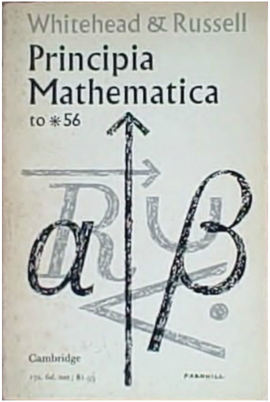
Durham, 29th July 2015

University of East Anglia


FIRE
EXIT
YUM
ESPRESSO
AFTERNOON OFFER
ICED COFFEES

A joke...

Why was the maths book feeling depressed?

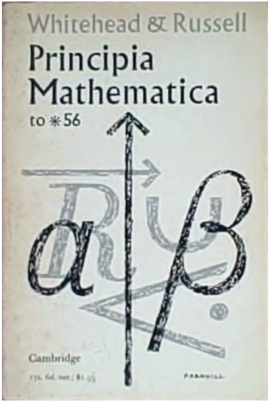


A joke...

Why was the maths book feeling depressed?

Because it had so many problems.

(C. A. Carvalho (2015))



Generators and relations for symmetric groups

S_4 - symmetric group on $\{1, 2, 3, 4\}$.

A generating set

$S_4 = \langle (1\ 2), (2\ 3), (3\ 4) \rangle$

Generators and relations for symmetric groups

S_4 - symmetric group on $\{1, 2, 3, 4\}$.

A generating set

$S_4 = \langle (1\ 2), (2\ 3), (3\ 4) \rangle$

Some relations between these generators

(i) Elements have order 2:

$(1\ 2)(1\ 2) = ()$

(ii) Non-overlapping commute:

$(1\ 2)(3\ 4) = (3\ 4)(1\ 2)$

Generators and relations for symmetric groups

S_4 - symmetric group on $\{1, 2, 3, 4\}$.

A generating set

$S_4 = \langle (1\ 2), (2\ 3), (3\ 4) \rangle$

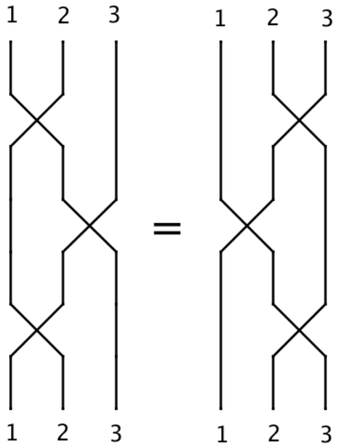
Some relations between these generators

(i) Elements have order 2:

$(1\ 2)(1\ 2) = ()$

(ii) Non-overlapping commute:

$(1\ 2)(3\ 4) = (3\ 4)(1\ 2)$



(iii) Partially overlapping:

$(1\ 2)(2\ 3)(1\ 2) = (2\ 3)(1\ 2)(2\ 3)$

Coxeter presentation

S_r - the symmetric group on $[r] = \{1, 2, \ldots, r\}$.

$S_r = \langle (1\ 2), (2\ 3), \ldots, (r-1\ r) \rangle$

S_r is isomorphic to the group defined by the group presentation:

$$\begin{aligned} \langle\ g_1, \ldots, g_{r-1} \mid\ & g_i^2 = 1 \\ & g_i g_j = g_j g_i \quad |i-j| > 1 \\ & g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \qquad \rangle \end{aligned}$$

- This is called the Coxeter presentation for S_r.
- It defines S_r in terms of the generating set consisting of Coxeter transpositions $(i\ i+1)$ where

$$\text{generating symbol } g_i \quad \longleftrightarrow \quad \text{the generator } (i\ i+1)$$

Aim of my talk

Let $n, r \in \mathbb{N}$ with $1 \leq r \leq n$.

T_n - full transformation monoid, S_r - symmetric group.

- I will give another finite presentation for S_r.
- This presentation will have:
  - Generating symbols $\longleftarrow_{\text{bijection}}\longrightarrow$ rank r idempotents of T_n
  - Relations obtained from certain quadruples of idempotents.

Aim of my talk

Let $n, r \in \mathbb{N}$ with $1 \leq r \leq n$.

T_n - full transformation monoid, S_r - symmetric group.

- I will give another finite presentation for S_r.
- This presentation will have:
  - Generating symbols $\longleftarrow_{\text{bijection}}\longrightarrow$ rank r idempotents of T_n
  - Relations obtained from certain quadruples of idempotents.

I aim to explain:

1. What we proved: The main theorem of the article

   R. Gray and N. Ruškuc, Maximal subgroups of free idempotent generated semigroups over the full transformation monoid.
   Proc. London Math. Soc. 104 (2012) 997–1018.

2. Why we proved it: Motivated by free idempotent generated semigroups.
3. How we proved it: Finding an encoding of the Coxeter presentation in the combinatorics of kernels and images of idempotent transformations.

Idempotent, sets and partitions

Idempotents in T_n

$e \in T_n$ is an idempotent $\Leftrightarrow$ e acts as identity on its image $\mathrm{im}(e)$.

$\epsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 2 & 4 \end{pmatrix}$, $\mathrm{im}(\epsilon) = \{2, 4\}$ with $2\epsilon = 2$, $4\epsilon = 4$, and $\epsilon^2 = \epsilon$.

$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 2 & 2 \end{pmatrix}$, $\mathrm{im}(\beta) = \{2, 4\}$ with $2\beta \neq 2$, and $\beta^2 \neq \beta$.

Idempotent, sets and partitions

Idempotents in T_n

$e \in T_n$ is an idempotent $\Leftrightarrow$ e acts as identity on its image $\text{im}(e)$.

$\epsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 2 & 4 \end{pmatrix}$, $\text{im}(\epsilon) = \{2, 4\}$ with $2\epsilon = 2$, $4\epsilon = 4$, and $\epsilon^2 = \epsilon$.

$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 2 & 2 \end{pmatrix}$, $\text{im}(\beta) = \{2, 4\}$ with $2\beta \neq 2$, and $\beta^2 \neq \beta$.

Images and kernels

Let $\alpha \in T_n$ with $\text{rank}(\alpha) = |\text{im}(\alpha)| = r$.

Associated with α are:
A set $\text{im}(\alpha)$ of size r.
A partition $\ker(\alpha) = \{m\alpha^{-1} : m \in \text{im}(\alpha)\}$ of $[n]$ into r non-empty parts.

Idempotent, sets and partitions

Idempotents in T_n

$e \in T_n$ is an idempotent $\Leftrightarrow$ e acts as identity on its image $\mathrm{im}(e)$.

$\epsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 2 & 4 \end{pmatrix}$, $\mathrm{im}(\epsilon) = \{2, 4\}$ with $2\epsilon = 2$, $4\epsilon = 4$, and $\epsilon^2 = \epsilon$.

$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 2 & 2 \end{pmatrix}$, $\mathrm{im}(\beta) = \{2, 4\}$ with $2\beta \neq 2$, and $\beta^2 \neq \beta$.

Images and kernels

Let $\alpha \in T_n$ with $\mathrm{rank}(\alpha) = |\mathrm{im}(\alpha)| = r$.

Associated with α are:
A set $\mathrm{im}(\alpha)$ of size r.
A partition $\ker(\alpha) = \{m\alpha^{-1} : m \in \mathrm{im}(\alpha)\}$ of $[n]$ into r non-empty parts.

Example: $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 3 & 5 & 2 & 3 \end{pmatrix}$

$\mathrm{im}(\alpha) = \{2, 3, 5\}, \quad \ker(\alpha) = \{\{1, 4\}, \{2, 3, 6\}, \{5\}\}.$

Idempotent, sets and partitions

Let $n, r \in \mathbb{N}$ with $1 \leq r \leq n$.

- $I = \{$partitions of [n] into r non-empty sets$\}$
- $J = \{r$-element subsets of $[n]\}$

For $P \in I$ and $A \in J$ write $A \perp P$ if A is a transversal of P.

Fact: There is a natural bijection

$$\{\text{ idempotents in } T_n \text{ or rank } r\} \quad \longleftarrow_{\text{bijection}}\longrightarrow \quad \{(P, A) \in I \times J : A \perp P\}$$

$$e_{P,A} \text{ with image } A \text{ and kernel } P \quad \longleftrightarrow \quad (P, A) \quad (\text{for } A \perp P)$$

Idempotent, sets and partitions

Let $n, r \in \mathbb{N}$ with $1 \leq r \leq n$.

- $I = \{$partitions of [n] into r non-empty sets$\}$
- $J = \{r$-element subsets of $[n]\}$

For $P \in I$ and $A \in J$ write $A \perp P$ if A is a transversal of P.

Fact: There is a natural bijection

$$\{\text{ idempotents in } T_n \text{ or rank } r\} \quad \longleftarrow_{\text{bijection}} \longrightarrow \quad \{(P, A) \in I \times J : A \perp P\}$$

$$e_{P,A} \text{ with image } A \text{ and kernel } P \quad \longleftrightarrow \quad (P, A) \quad (\text{for } A \perp P)$$

Example

$n = 8$, $r = 3$ with $A \perp P$ being the pair $256 \perp 1247 \mid 35 \mid 68$

	Image $A = 256$
Kernel $\quad P = 1247 \mid 35 \mid 68$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ & 2 & & & 5 & 6 & & \end{pmatrix} = e_{P,A}$

Idempotent, sets and partitions

Let $n, r \in \mathbb{N}$ with $1 \leq r \leq n$.

- $I = \{\text{partitions of [n] into } r \text{ non-empty sets}\}$
- $J = \{r\text{-element subsets of } [n]\}$

For $P \in I$ and $A \in J$ write $A \perp P$ if A is a transversal of P.

Fact: There is a natural bijection

$$\{\text{ idempotents in } T_n \text{ or rank } r\} \quad \longleftarrow_{\text{bijection}}\longrightarrow \quad \{(P, A) \in I \times J : A \perp P\}$$

$$e_{P,A} \text{ with image } A \text{ and kernel } P \quad \longleftrightarrow \quad (P, A) \quad (\text{for } A \perp P)$$

Example

$n = 8$, $r = 3$ with $A \perp P$ being the pair $256 \perp 1247 \mid 35 \mid 68$

		Image $A = 256$
Kernel	$P = 1247 \mid 35 \mid 68$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 2 & & 2 & 5 & 6 & 2 & \end{pmatrix} = e_{P,A}$

Idempotent, sets and partitions

Let $n, r \in \mathbb{N}$ with $1 \leq r \leq n$.

- $I = \{$partitions of [n] into r non-empty sets$\}$
- $J = \{r$-element subsets of $[n]\}$

For $P \in I$ and $A \in J$ write $A \perp P$ if A is a transversal of P.

Fact: There is a natural bijection

$$\{\text{ idempotents in } T_n \text{ or rank } r\} \quad \xleftarrow{\quad}_{\text{bijection}}\xrightarrow{\quad} \quad \{(P, A) \in I \times J : A \perp P\}$$

$$e_{P,A} \text{ with image } A \text{ and kernel } P \quad \longleftrightarrow \quad (P, A) \quad (\text{for } A \perp P)$$

Example

$n = 8$, $r = 3$ with $A \perp P$ being the pair $256 \perp 1247 \mid 35 \mid 68$

	Image $A = 256$
Kernel $\quad P = 1247 \mid 35 \mid 68$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 2 & 5 & 2 & 5 & 6 & 2 & 6 \end{pmatrix} = e_{P,A}$

Graham–Houghton Graph

Let $n, r \in \mathbb{N}$ with $1 \leq r \leq n$.

- $I = \{$partitions of [n] into r non-empty sets$\}$
- $J = \{r$-element subsets of $[n]\}$

For $P \in I$ and $A \in J$ write $A \perp P$ if A is a transversal of P.

The Graham–Houghton Graph Γ_r is the bipartite graph with

Vertices: $I \cup J$, Edges: $P \sim A \Leftrightarrow A \perp P$

Note: $\{$edges of $\Gamma_r\}$ $\longleftarrow_{\text{bijection}}\longrightarrow$ $\{$ idempotents in T_n or rank $r\}$

Graham–Houghton Graph

Let $n, r \in \mathbb{N}$ with $1 \leq r \leq n$.

- $I = \{\text{partitions of [n] into } r \text{ non-empty sets}\}$
- $J = \{r\text{-element subsets of } [n]\}$

For $P \in I$ and $A \in J$ write $A \perp P$ if A is a transversal of P.

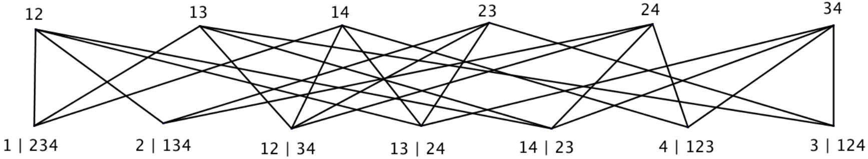
The Graham–Houghton Graph Γ_r is the bipartite graph with

Vertices: $I \cup J$, Edges: $P \sim A \Leftrightarrow A \perp P$

Note: $\{\text{edges of } \Gamma_r\} \longleftarrow_{\text{bijection}} \longrightarrow \{\text{ idempotents in } T_n \text{ or rank } r\}$

Example

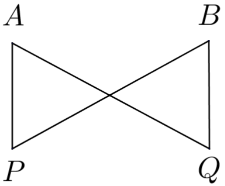
With $n = 4$ the graph Γ_2 is below (note that it is connected).



Singular squares

$(P, Q, A, B) \in I \times I \times J \times J$ is a square if $\{A, B\} \perp \{P, Q\}$.

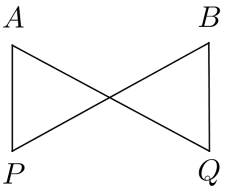
A square (P, Q, A, B) is singular if $\{e_{P,A}, e_{P,B}, e_{Q,A}, e_{Q,B}\}$ is a subsemigroup of T_n.



Singular squares

$(P, Q, A, B) \in I \times I \times J \times J$ is a square if $\{A, B\} \perp \{P, Q\}$.

A square (P, Q, A, B) is singular if $\{e_{P,A}, e_{P,B}, e_{Q,A}, e_{Q,B}\}$ is a subsemigroup of T_n.



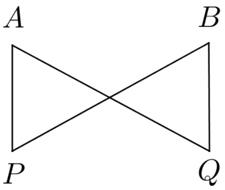
Example: With $n = 4$ and $r = 2$.

	$A = 14$	$B = 23$
$P = 12\|34$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \end{pmatrix}$
$Q = 13\|24$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 2 \end{pmatrix}$

Singular squares

$(P, Q, A, B) \in I \times I \times J \times J$ is a square if $\{A, B\} \perp \{P, Q\}$.

A square (P, Q, A, B) is singular if $\{e_{P,A}, e_{P,B}, e_{Q,A}, e_{Q,B}\}$ is a subsemigroup of T_n.



Example: With $n = 4$ and $r = 2$.

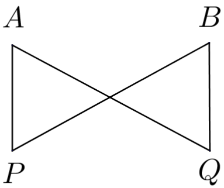
	$A = 14$	$B = 23$
$P = 12\|34$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \end{pmatrix}$
$Q = 13\|24$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 2 \end{pmatrix}$

Not singular since $e_{P,A} e_{Q,B} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 2 & 2 \end{pmatrix} \notin \{e_{P,A}, e_{P,B}, e_{Q,A}, e_{Q,B}\}$.

Singular squares

$(P, Q, A, B) \in I \times I \times J \times J$ is a square if $\{A, B\} \perp \{P, Q\}$.

A square (P, Q, A, B) is singular if $\{e_{P,A}, e_{P,B}, e_{Q,A}, e_{Q,B}\}$ is a subsemigroup of T_n.



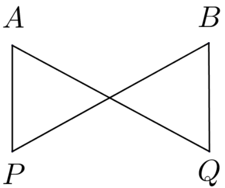
Example: With $n = 4$ and $r = 2$.

	$A = 24$	$B = 34$
$P = 14\|23$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 4 \end{pmatrix}$
$Q = 4\|123$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 \end{pmatrix}$

Singular squares

$(P, Q, A, B) \in I \times I \times J \times J$ is a square if $\{A, B\} \perp \{P, Q\}$.

A square (P, Q, A, B) is singular if $\{e_{P,A}, e_{P,B}, e_{Q,A}, e_{Q,B}\}$ is a subsemigroup of T_n.



Example: With $n = 4$ and $r = 2$.

	$A = 24$	$B = 34$
$P = 14\|23$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 4 \end{pmatrix}$
$Q = 4\|123$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 \end{pmatrix}$

Is a singular square as $\{e_{P,A}, e_{P,B}, e_{Q,A}, e_{Q,B}\}$ is closed.

Graham–Houghton 2-complex $\mathcal{GH}_r$

Let $n, r \in \mathbb{N}$ with $1 \leq r \leq n$.

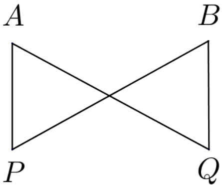
1-skeleton: the Graham–Houghton graph Γ_r

$I = \{$partitions of [n] into r non-empty sets$\}$

$J = \{r$-element subsets of $[n]\}$

Vertices: $I \cup J$, Edges: $P \sim A \Leftrightarrow A \perp P$

2-cells



$\Leftrightarrow$ (P, Q, A, B) is a singular square

Let $H_r = \pi_1(\mathcal{GH}_r)$ denote the fundamental group $\mathcal{GH}_r$.

Roughly speaking: $H_r \cong \langle \underbrace{\text{rank } r \text{ idempotents}}_{\text{generating symbols}} \mid \underbrace{\text{singular squares}}_{\text{defining relations}} \rangle$

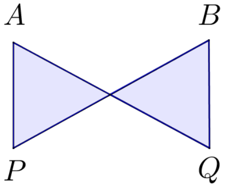
Graham–Houghton 2-complex $\mathcal{GH}_r$

Let $n, r \in \mathbb{N}$ with $1 \leq r \leq n$.

1-skeleton: the Graham–Houghton graph Γ_r
$I = \{$partitions of [n] into r non-empty sets$\}$
$J = \{r$-element subsets of $[n]\}$

Vertices: $I \cup J$, Edges: $P \sim A \Leftrightarrow A \perp P$

2-cells



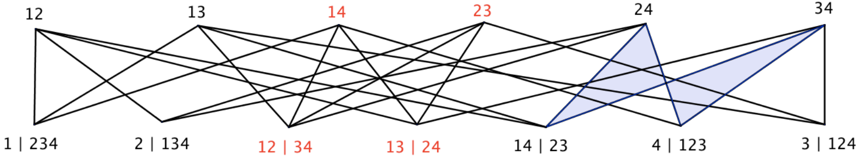
$\Leftrightarrow$ (P, Q, A, B) is a singular square

Let $H_r = \pi_1(\mathcal{GH}_r)$ denote the fundamental group $\mathcal{GH}_r$.

Roughly speaking: $H_r \cong \langle \underbrace{\text{rank } r \text{ idempotents}}_{\text{generating symbols}} \mid \underbrace{\text{singular squares}}_{\text{defining relations}} \rangle$

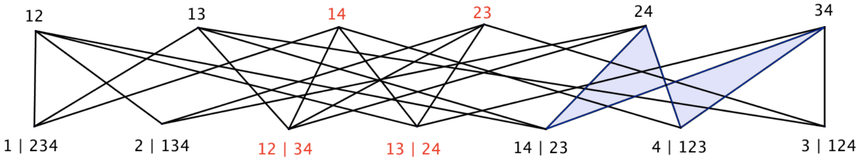
Main theorem

Example: The Graham–Houghton complex $\mathcal{GH}_2$ for T_4:



Main theorem

Example: The Graham–Houghton complex $\mathcal{GH}_2$ for T_4:

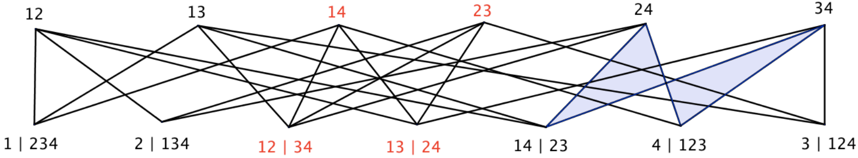


Theorem (RG and Ruškuc (2012))

Let $n, r \in \mathbb{N}$ with $1 \leq r \leq n-2$, and let $\mathcal{GH}_r$ be the Graham–Houghton complex built from the rank r idempotents in T_n. Then the fundamental group $H_r = \pi_1(\mathcal{GH}_r)$ is isomorphic to the symmetric group S_r.

Main theorem

Example: The Graham–Houghton complex $\mathcal{GH}_2$ for T_4:



Theorem (RG and Ruškuc (2012))

Let $n, r \in \mathbb{N}$ with $1 \leq r \leq n - 2$, and let $\mathcal{GH}_r$ be the Graham–Houghton complex built from the rank r idempotents in T_n. Then the fundamental group $H_r = \pi_1(\mathcal{GH}_r)$ is isomorphic to the symmetric group S_r.

[**Note:** When $r = n - 1 \Rightarrow \Gamma_r$ has no squares $\Rightarrow H_{n-1}$ is the fundamental group of a graph, and hence is a free group.]

- Why did we prove this?
- How did we prove this?

Idempotent generated semigroups

S - semigroup, $E = E(S)$ - idempotents $e = e^2$ of S

Definition. S is idempotent generated if $\langle E(S) \rangle = S$

Idempotent generated semigroups

S - semigroup, $E = E(S)$ - idempotents $e = e^2$ of S

Definition. S is idempotent generated if $\langle E(S) \rangle = S$

- Many natural examples
  - Howie (1966) - $T_n \setminus S_n$, the non-invertible transformations;
  - Erdös (1967) - singular part of $M_n(\mathbb{F})$, semigroup of all $n \times n$ matrices over a field $\mathbb{F}$;
  - Putcha (2006) - conditions for a reductive linear algebraic monoid to have the same property;
  - Fountain and Lewin (1992) - endomorphism monoids of finite dimensional independence algebras;
  - East (2011) - $\mathcal{P}_n \setminus S_n$, the non-invertible elements of the partition monoid.

Idempotent generated semigroups

S - semigroup, $E = E(S)$ - idempotents $e = e^2$ of S

Definition. S is idempotent generated if $\langle E(S) \rangle = S$

- Many natural examples
  - Howie (1966) - $T_n \setminus S_n$, the non-invertible transformations;
  - Erdös (1967) - singular part of $M_n(\mathbb{F})$, semigroup of all $n \times n$ matrices over a field $\mathbb{F}$;
  - Putcha (2006) - conditions for a reductive linear algebraic monoid to have the same property;
  - Fountain and Lewin (1992) - endomorphism monoids of finite dimensional independence algebras;
  - East (2011) - $\mathcal{P}_n \setminus S_n$, the non-invertible elements of the partition monoid.
- Idempotent generated semigroups are "general"
  - Every semigroup S embeds into an idempotent generated semigroup.

Free idempotent generated semigroups

S - semigroup, $E = E(S)$ - idempotents of S

Nambooripad (1979): The set of idempotents E carries a certain abstract structure, that of a biordered set.

Big idea: Fix a biorder E and investigate those semigroups whose idempotents carry this fixed biorder structure.

Free idempotent generated semigroups

S - semigroup, $E = E(S)$ - idempotents of S

Nambooripad (1979): The set of idempotents E carries a certain abstract structure, that of a biordered set.

Big idea: Fix a biorder E and investigate those semigroups whose idempotents carry this fixed biorder structure.

Within this family there is a unique free object $IG(E)$ which is the semigroup defined by presentation:

$$IG(E) = \langle E \mid e \cdot f = ef \ \ (e,f \in E,\ \{e,f\} \cap \{ef,fe\} \neq \emptyset) \ \rangle$$

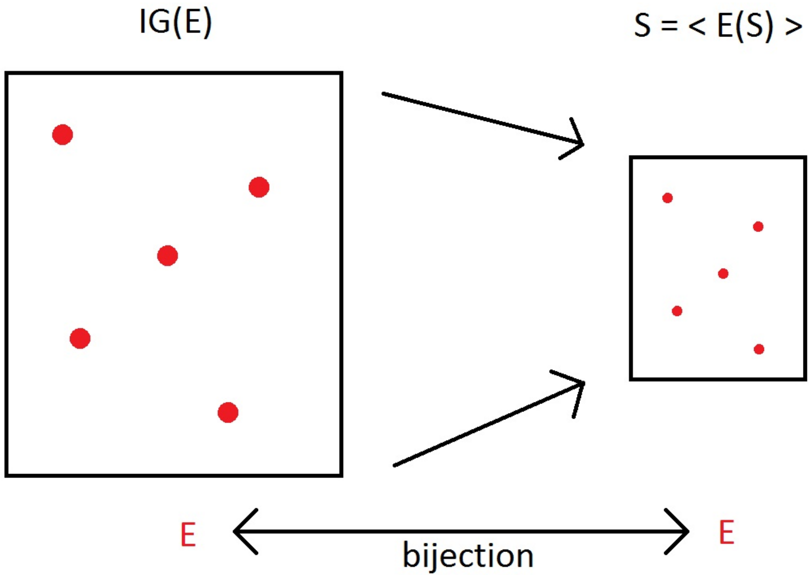
$IG(E)$ is called the free idempotent generated semigroup on E.

First steps towards understanding $IG(E)$

Theorem (Easdown (1985))

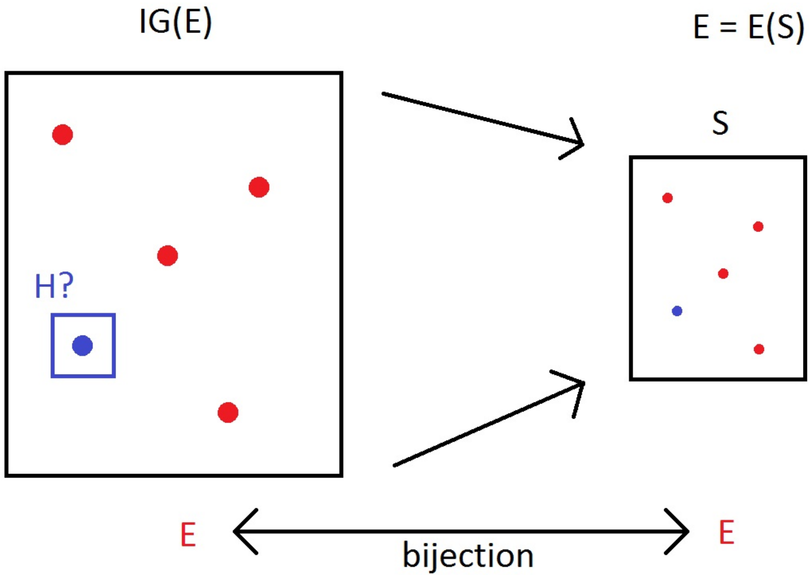
Let S be an idempotent generated semigroup with $E = E(S)$. Then $IG(E)$ is an idempotent generated semigroup and there is a surjective homomorphism $\phi : IG(E) \to S$ which is bijective on idempotents.

Conclusion. It is important to understand $IG(E)$ if one is interested in understanding an arbitrary idempotent generated semigroups.


IG(E)
S = < E(S) >
E
bijection
E

Maximal subgroups of $IG(E)$

Question. Which groups can arise as maximal subgroups of a free idempotent generated semigroups?


IG(E)
E = E(S)
S
H?
E
bijection
E

Maximal subgroups of $IG(E)$

Question. Which groups can arise as maximal subgroups of a free idempotent generated semigroups?

- Work of Pastijn (1977, 1980), Nambooripad and Pastijn (1980), McElwee (2002) led to a conjecture that all these groups must be free groups.

Maximal subgroups of $IG(E)$

Question. Which groups can arise as maximal subgroups of a free idempotent generated semigroups?

- Work of Pastijn (1977, 1980), Nambooripad and Pastijn (1980), McElwee (2002) led to a conjecture that all these groups must be free groups.
- Brittenham, Margolis & Meakin (2009) - gave the first counterexample to this conjecture by showing $\mathbb{Z} \oplus \mathbb{Z}$ can arise.

Maximal subgroups of $IG(E)$

Question. Which groups can arise as maximal subgroups of a free idempotent generated semigroups?

- Work of Pastijn (1977, 1980), Nambooripad and Pastijn (1980), McElwee (2002) led to a conjecture that all these groups must be free groups.
- Brittenham, Margolis & Meakin (2009) - gave the first counterexample to this conjecture by showing $\mathbb{Z} \oplus \mathbb{Z}$ can arise.
- RG & Ruskuc (2012) proved that *every group* is a maximal subgroup of some free idempotent generated semigroup.

New question

What can be said about maximal subgroups of $IG(E)$ where $E = E(S)$ for semigroups S that arise in nature?

$IG(E)$ for $E = E(T_n)$

Let $E = E(T_n)$ where T_n is the full transformation monoid.

Howie (1966): $\langle E(T_n) \rangle = (T_n \setminus S_n) \cup \{\mathrm{id}\}$.

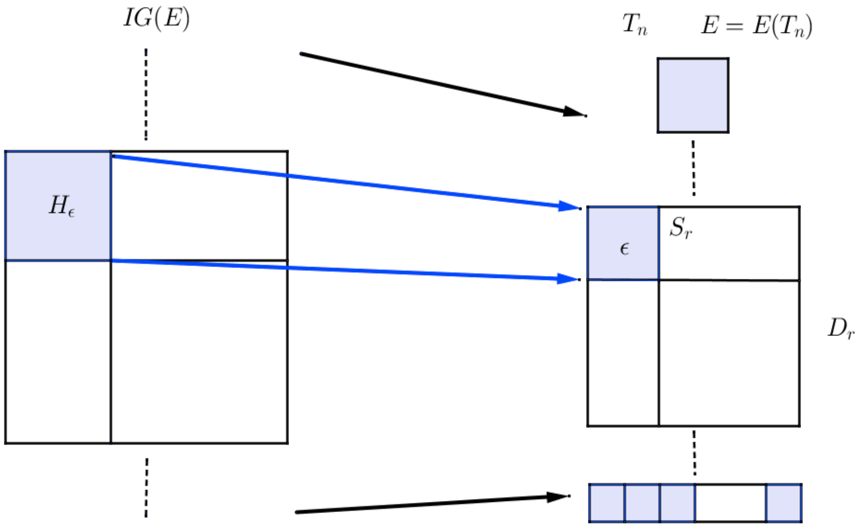
Easdown (1985): We may identify $E = E(T_n) = E(IG(E))$.
Fix an idempotent transformation $\epsilon \in T_n$ of rank r.

Problem: Identify the maximal subgroup H_ϵ of

$$IG(E) = \langle E \mid e \cdot f = ef \ \ (e,f \in E,\ \{e,f\} \cap \{ef,fe\} \neq \emptyset) \ \rangle$$

containing ϵ.

General fact: H_ϵ is a homomorphic preimage of the corresponding maximal subgroup of T_n, namely the symmetric group S_r.



$IG(E)$
T_n
$E = E(T_n)$
H_ϵ
ϵ
S_r
D_r

Reinterpreting our result

Theorem (Brittenham, Margolis & Meakin (2009))

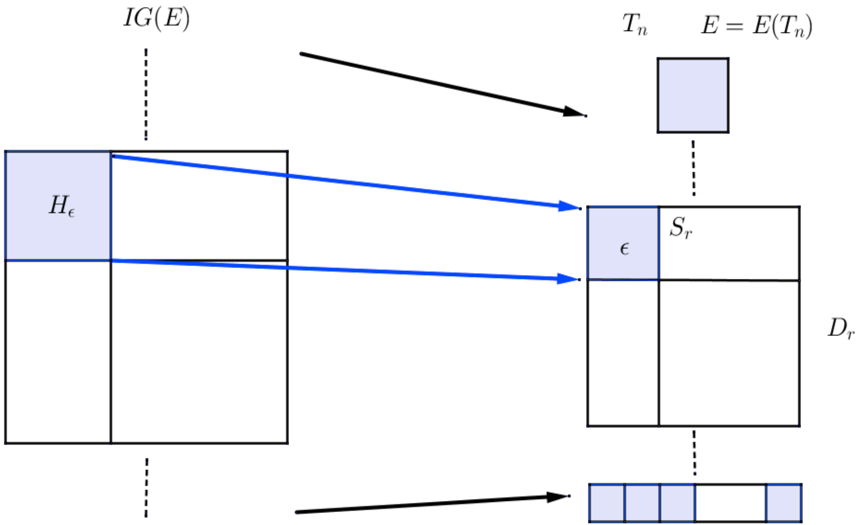
Let S be a regular semigroup and set $E = E(S)$. Then

$$\{ \text{ maximal subgroups of } IG(E) \} \quad = \quad \{ \text{ fundamental groups of Graham–Houghton complexes of } S \}$$

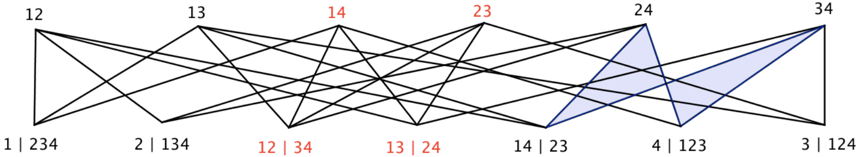
So, our result on fundamental groups of GH-complexes of T_n says:

Theorem (RG and Ruškuc (2012))

Let T_n be the full transformation semigroup, let E be its set of idempotents, and let $\epsilon \in E$ be an arbitrary idempotent with image size r ($1 \leq r \leq n-2$). Then the maximal subgroup H_ϵ of the free idempotent generated semigroup $IG(E)$ containing ϵ is isomorphic to the symmetric group S_r.


$IG(E)$
T_n
$E = E(T_n)$
H_ϵ
ϵ
S_r
D_r

Main theorem



Theorem (RG and Ruškuc (2012))

Let $n, r \in \mathbb{N}$ with $1 \leq r \leq n-2$, and let $\mathcal{GH}_r$ be the Graham–Houghton complex built from the rank r idempotents in T_n. Then the fundamental group $H_r = \pi_1(\mathcal{GH}_r)$ is isomorphic to the symmetric group S_r.

- Why did we prove this?
- How did we prove this?

Computing the group H_r

The group $H_r = \pi_1(\mathcal{GH}_r)$ is then defined by the presentation with generators

$$F = \{f_{P,A} \;:\; P \in I,\ A \in J,\ A \perp P\},$$

and the defining relations

$$\begin{array}{ll} f_{P,A} = 1 & ((P,A) \in \mathcal{T} \text{ a spanning tree of } \Gamma_r) \\ f_{P,A}^{-1} f_{P,B} = f_{Q,A}^{-1} f_{Q,B} & ((P,Q,A,B) \text{ a singular square}). \end{array}$$

Observation: This presentation has lots of generators so if this is a presentation for S_r then it must have a lot of redundancy.

Idea: Our hope is to show this is a presentation for S_r. So, ultimately each generator $f_{P,A}$ will need to be equal (in the group defined by the presentation) to some element of S_r.

So, for each $P \in I,\ A \in J,\ A \perp P$ we want to define an element $\lambda(P,A) \in S_r$ which we aim to prove is the element represented by the generator $f_{P,A}$.

The label function

For each set A and partition P with $A \perp P$ write:

$$A = \{a_1, \dots, a_r\},\ a_1 < \cdots < a_r,$$
$$P = \{P_1, \dots, P_r\},\ \min P_1 < \cdots < \min P_r.$$

Then write

$$\begin{pmatrix} P_1 & P_2 & \dots & P_r \\ a_{l_1} & a_{l_2} & \dots & a_{l_r} \end{pmatrix}, \quad \lambda(P, A) = \begin{pmatrix} 1 & 2 & \dots & r \\ l_1 & l_2 & \dots & l_r \end{pmatrix} \in S_r.$$

The label function

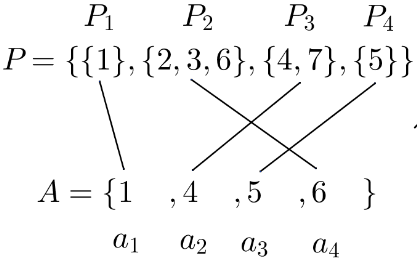
For each set A and partition P with $A \perp P$ write:

$$A = \{a_1, \ldots, a_r\},\ a_1 < \cdots < a_r,$$
$$P = \{P_1, \ldots, P_r\},\ \min P_1 < \cdots < \min P_r.$$

Then write

$$\begin{pmatrix} P_1 & P_2 & \ldots & P_r \\ a_{l_1} & a_{l_2} & \ldots & a_{l_r} \end{pmatrix}, \quad \lambda(P, A) = \begin{pmatrix} 1 & 2 & \ldots & r \\ l_1 & l_2 & \ldots & l_r \end{pmatrix} \in S_r.$$

Example: $n = 7, r = 4$



$$\lambda(P, A) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

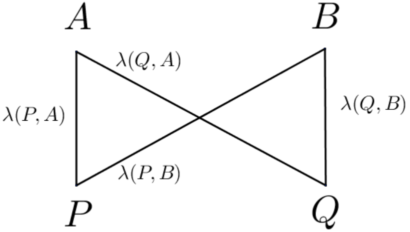
Singular squares and labels

Fact: We can read off the singular squares using λ. A square

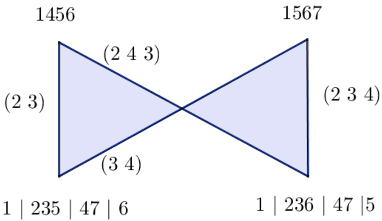
$$(P, Q, A, B) \text{ is singular} \Leftrightarrow \lambda(P, A)^{-1}\lambda(P, B) = \lambda(Q, A)^{-1}\lambda(Q, B).$$

We can think of λ as labelling each edge of the Graham–Houghton graph.

Example



$$\lambda(P, A)^{-1}\lambda(P, B) = \lambda(Q, A)^{-1}\lambda(Q, B).$$



$$(2\ 3)^{-1}(3\ 4) = (2\ 4\ 3) = (2\ 4\ 3)^{-1}(2\ 3\ 4)$$

First we prove the group H_r is defined by the presentation

Generators: $F = \{f_{P,A} \;:\; P \in I,\ A \in J,\ A \perp P\}$

Relations:

(I) $f_{P,A} = 1$ whenever $\lambda(P,A) = 1$

(II) $f_{P,A}^{-1} f_{P,B} = f_{Q,A}^{-1} f_{Q,B}$ where (P,Q,A,B) is a square: $\lambda(P,A)^{-1}\lambda(P,B) = \lambda(Q,A)^{-1}\lambda(Q,B)$.

$\lambda(P,A)$ $\lambda(P,B)$

$\lambda(Q,A)$ $\lambda(Q,B)$

First we prove the group H_r is defined by the presentation

Generators: $F = \{f_{P,A} \ : \ P \in I, \ A \in J, \ A \perp P\}$

Relations:

(I) $f_{P,A} = 1$ whenever $\lambda(P,A) = 1$

(II) $f_{P,A}^{-1} f_{P,B} = f_{Q,A}^{-1} f_{Q,B}$ where (P,Q,A,B) is a square: $\lambda(P,A)^{-1}\lambda(P,B) = \lambda(Q,A)^{-1}\lambda(Q,B)$.

$\lambda(P,A)$	$\lambda(P,B)$
$\lambda(Q,A)$	$\lambda(Q,B)$

Basic idea of the proof that $H_r \cong S_r$

Use Tietze transformations to transform the presentation above into the classical Coxeter presentation for S_r.

First we prove the group H_r is defined by the presentation

Generators: $F = \{f_{P,A} \ : \ P \in I, \ A \in J, \ A \perp P\}$

Relations:

(I) $f_{P,A} = 1$ whenever $\lambda(P,A) = 1$

(II) $f_{P,A}^{-1} f_{P,B} = f_{Q,A}^{-1} f_{Q,B}$ where (P,Q,A,B) is a square: $\lambda(P,A)^{-1}\lambda(P,B) = \lambda(Q,A)^{-1}\lambda(Q,B)$.

$\lambda(P,A)$ $\lambda(P,B)$

$\lambda(Q,A)$ $\lambda(Q,B)$

Basic idea of the proof that $H_r \cong S_r$

Use Tietze transformations to transform the presentation above into the classical Coxeter presentation for S_r.

1. Eliminate all generators $f_{P,A}$ with $\lambda(P,A) \neq (i\; i+1)$ expressing them as products of generators with labels $(i\; i+1)$.

First we prove the group H_r is defined by the presentation

Generators: $F = \{f_{P,A} \ : \ P \in I, \ A \in J, \ A \perp P\}$

Relations:

(I) $f_{P,A} = 1$ whenever $\lambda(P,A) = 1$

(II) $f_{P,A}^{-1} f_{P,B} = f_{Q,A}^{-1} f_{Q,B}$ where (P,Q,A,B) is a square: $\lambda(P,A)^{-1}\lambda(P,B) = \lambda(Q,A)^{-1}\lambda(Q,B)$.

$\lambda(P,A)$ $\lambda(P,B)$

$\lambda(Q,A)$ $\lambda(Q,B)$

Basic idea of the proof that $H_r \cong S_r$

Use Tietze transformations to transform the presentation above into the classical Coxeter presentation for S_r.

1. Eliminate all generators $f_{P,A}$ with $\lambda(P,A) \neq (i\, i+1)$ expressing them as products of generators with labels $(i\, i+1)$.
2. Show that if $\lambda(P,A) = (i\, i+1) = \lambda(Q,B)$ then $f_{P,A} = f_{Q,B}$ is a consequence of (I) & (II).

First we prove the group H_r is defined by the presentation

Generators: $F = \{f_{P,A} \ : \ P \in I,\ A \in J,\ A \perp P\}$

Relations:

(I) $f_{P,A} = 1$ whenever $\lambda(P,A) = 1$

(II) $f_{P,A}^{-1} f_{P,B} = f_{Q,A}^{-1} f_{Q,B}$ where (P,Q,A,B) is a square: $\lambda(P,A)^{-1}\lambda(P,B) = \lambda(Q,A)^{-1}\lambda(Q,B)$.

$\lambda(P,A)$	$\lambda(P,B)$
$\lambda(Q,A)$	$\lambda(Q,B)$

Basic idea of the proof that $H_r \cong S_r$

Use Tietze transformations to transform the presentation above into the classical Coxeter presentation for S_r.

1. Eliminate all generators $f_{P,A}$ with $\lambda(P,A) \neq (i\ i+1)$ expressing them as products of generators with labels $(i\ i+1)$.
2. Show that if $\lambda(P,A) = (i\ i+1) = \lambda(Q,B)$ then $f_{P,A} = f_{Q,B}$ is a consequence of (I) & (II).
3. We are left with a presentation with generators in one-one correspondence with the Coxeter generators of S_r. To finish the proof we show that the Coxeter relations are consequences.

Table of labels

$I \times J$	A	B
P	$(\ )$	$(2\ 3)$
Q	$(1\ 2)$	$(1\ 3\ 2)$

Spot singular squares

$$(\ )^{-1}(2\ 3) = (1\ 2)^{-1}(1\ 3\ 2)$$

Table of generating symbols

$I \times J$	A	B
P	$f_{P,A}$	$f_{P,B}$
Q	$f_{Q,A}$	$f_{Q,B}$

Deduce relations

$$f_{P,A} = 1, \text{ and } f_{Q,A} f_{P,B} = f_{Q,B}$$

Spotting relations from Coxeter presentation for S_r

$$\langle g_1, \ldots, g_{r-1} \mid g_i^2 = 1, \quad g_i g_j = g_j g_i \quad (|i-j| > 1), \quad g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \rangle$$

Example: Case $n = 7$, $r = 4$ finding a relation $g_i g_j = g_j g_i$.

	A 1 2 5 6	B 2 3 5 6	C 2 3 4 5
P = 1 3 4 7 | 2 | 5 | 6	( )	(1 2)	
Q = 1 3 7 | 2 | 4 6 | 5	(3 4)	(1 2)(3 4)	(1 2)
R = 1 2 7 | 3 | 4 6 | 5		(3 4)	( )

Spotting relations from Coxeter presentation for S_r

$$\langle g_1, \ldots, g_{r-1} \mid g_i^2 = 1, \quad g_i g_j = g_j g_i \quad (|i-j| > 1), \quad g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \rangle$$

Example: Case $n = 7$, $r = 4$ finding a relation $g_i g_j = g_j g_i$.

	A 1 2 5 6	B 2 3 5 6	C 2 3 4 5
$\mathsf{P} = 1\ 3\ 4\ 7 \mid 2 \mid 5 \mid 6$	( )	(1 2)	
$\mathsf{Q} = 1\ 3\ 7 \mid 2 \mid 4\ 6 \mid 5$	(3 4)	(1 2)(3 4)	(1 2)
$\mathsf{R} = 1\ 2\ 7 \mid 3 \mid 4\ 6 \mid 5$		(3 4)	( )

Deductions: $f_{Q,A}\, f_{P,B} = f_{Q,B} = f_{Q,C}\, f_{R,B}, \quad f_{Q,A} = f_{R,B}, \quad f_{P,B} = f_{Q,C}$

$\therefore \quad f_{Q,A}\, f_{P,B} = f_{P,B}\, f_{Q,A}$ where $\lambda(P,B) = (1\ 2)$ and $\lambda(Q,A) = (3\ 4)$.

Related results and future work

Analogous results have since been proved for:

- Endomorphism monoids of free G-acts

  I. Dolinka, V. Gould and D. Yang,
  Free idempotent generated semigroups and endomorphism monoids of free G-acts.
  Journal of Algebra. 429 (2015), 133–176.

- The full linear monoid $M_n(\mathbb{F})$

  I. Dolinka and R. D. Gray,
  Maximal subgroups of free idempotent generated semigroups over the full linear monoid.
  Trans. Amer. Math. Soc. 366(1) (2014), 419–455.

Note: For the full linear monoid we currently only know the groups for $r < \frac{n}{3}$ (we get the general linear group $GL_r(\mathbb{F})$) but we do not know what the groups are for higher values of r.