Sufficient Conditions For A Group Of Homeomorphisms Of The Cantor Set To Be 2-Generated

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The Cantor Set, Cones, Open Sets, Clopen Sets

- By the *Cantor set* we mean the set of rightward infinite words over $\{0,1\}$ which we will denote $\{0,1\}^{\mathbb{N}}$.
- If u is a word over $\{0,1\}$ we will use $\bar{u}$ for the set $\left\{uw \mid w \in \{0,1\}^{\mathbb{N}}\right\}$. We will call such $\bar{u}$ *cones*.
- The *open* subsets of $\{0,1\}^{\mathbb{N}}$ are the arbitrary unions of cones.
- The *clopen* subsets of $\{0,1\}^{\mathbb{N}}$ are the finite unions of cones.
- A *homeomorphism* of $\{0,1\}^{\mathbb{N}}$ is a bijection on $\{0,1\}^{\mathbb{N}}$ such that the image and preimage of any open set is also open.
- We will use $\mathrm{Homeo}(\{0,1\}^{\mathbb{N}})$ for the group of homeomorphisms of $\{0,1\}^{\mathbb{N}}$.

Thompson's Group V

- V is a subgroup of $\mathrm{Homeo}(\{0,1\}^{\mathbb{N}})$ introduced by R. Thompson circa 1965.
- We will define elements of V piecewise with the domains and ranges of pieces being cones.
- For cones $\bar{u}$ and $\bar{v}$ the only bijection $f : \bar{u} \to \bar{v}$ which is allowed to be a piece of an element of V is defined by $(uw)f := (vw)$ for each infinite word w.
- V is simple and 2-generated.

Important Properties

Definition

If X is a set of homeomorphisms of the Cantor set we will say X is *vigorous* if for any proper clopen subset A of $\{0,1\}^{\mathbb{N}}$ and any B, C non-empty proper clopen subsets of A there exists $g \in X$ with $\mathrm{supp}(g) \subseteq A$ and $Bg \subseteq C$.

Definition

If G is a group of homeomorphisms of the Cantor set we will say G is *flawless* if the set

$$\left\{[a,b] \,\middle|\, \mathrm{supp}(a), \mathrm{supp}(b) \subseteq A \text{ for some proper clopen } A \subseteq \{0,1\}^{\mathbb{N}}\right\}$$

generates G.

Lemma

If G is a vigorous subgroup of $\mathrm{Homeo}(\{0,1\}^{\mathbb{N}})$ then G is flawless exactly if G is simple.

Statement Of Theorem

Theorem

If G is a vigorous simple subgroup of $\mathrm{Homeo}(\{0,1\}^{\mathbb{N}})$ *and E is a finitely generated subgroup of G then there exists F a 2-generated subgroup of G containing E.*

Corollary

If G is a finitely generated vigorous simple subgroup of $\mathrm{Homeo}(\{0,1\}^{\mathbb{N}})$ *then G is 2-generated.*

Examples

We will use K for the set of vigorous simple finitely generated subgroups of $\mathrm{Homeo}(\{0,1\}^{\mathbb{N}})$. Note that $\mathrm{Homeo}(\{0,1\}^{\mathbb{N}})$ acts on K by conjugation.

Example

Our first example is V though it has been known to be 2-generated for a while.

Lemma

If G, H are in K then $\langle G \cup H \rangle$ is also in K.

Lemma

If $G \in K$ and $g \in G$ and $h \in \mathrm{Homeo}(\{0,1\}^{\mathbb{N}})$ and A is a non-empty clopen set with $A \cap \mathrm{supp}(g) = \emptyset$ and $\mathrm{supp}(h) \cap \mathrm{supp}(h^g) = \emptyset$ then $\langle G \cup \{[g, h]\} \rangle$ is in K.

Questions

- Are there similar results for other spaces (like manifolds).
- Can we replace vigorous with a weaker condition and still get the theorem?
- Do there exist finitely presented simple groups which are not 2-generated?