The Brauer Project

Understanding Idempotents in Diagram Semigroups and Algebras

Igor Dolinka[1], James East[2], Athanasios Evangelou[3], Des FitzGerald[3], Nick Ham[3], James Hyde[4], **Nick Loughlin**[5]

[1]Novi Sad [2]Western Sydney [3]Tasmania
[4]St Andrews [5]Newcastle

July 25, 2015

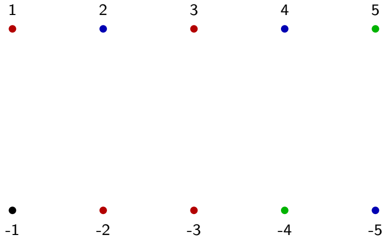
Public Service Announcement

The first paper from this project is available at **arXiv:1408.2021**, and in the Journal of Combinatorial Theory, Series A (JCTA).

The second has appeared in preprint form, at **arXiv:1507.04838**. A third is "in the works."

This is joint work with Igor Dolinka (Novi Sad), James East (Western Sydney), Athanasios Evangelou, Des FitzGerald and Nick Ham (Tasmania) and James Hyde (St Andrews). [**HEHFELD**]

Partitions



Representing a partition by a diagram

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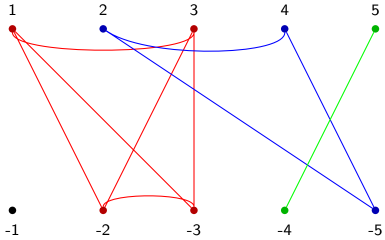

Representing a partition by a diagram

- Join similarly-coloured points;

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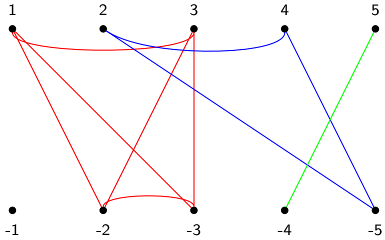

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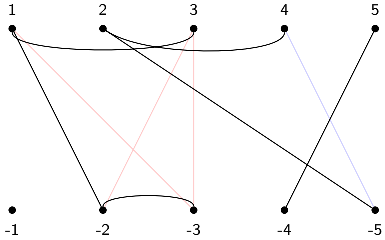

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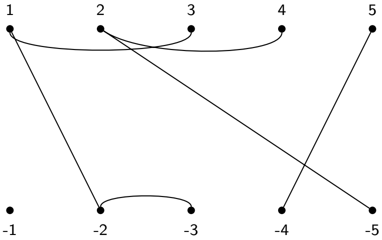

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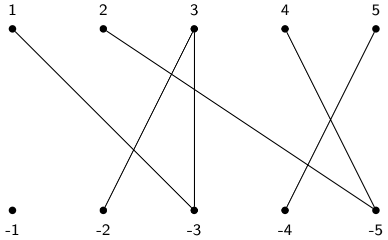

Representing a partition by a diagram

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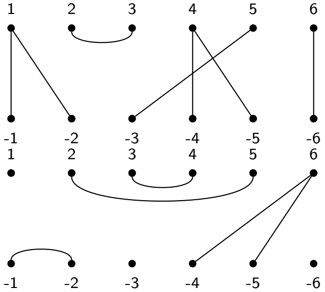
- Choice of spanning forest doesn't matter.

Multiplication of diagrams



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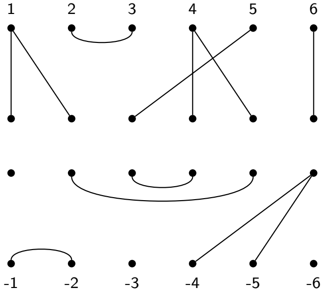

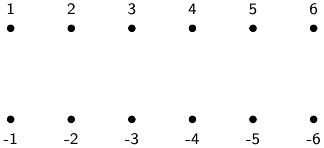

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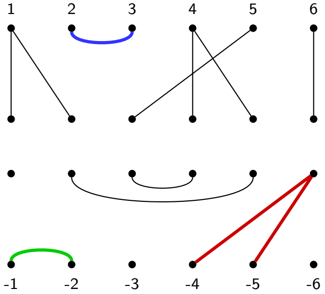

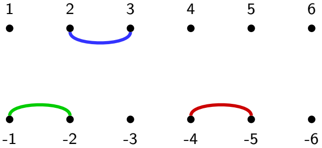

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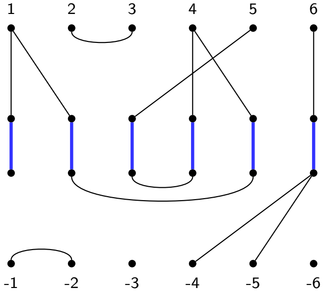

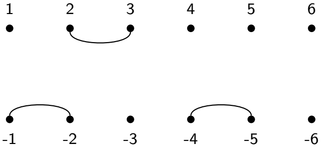

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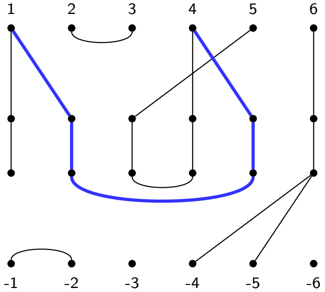

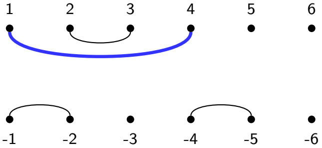

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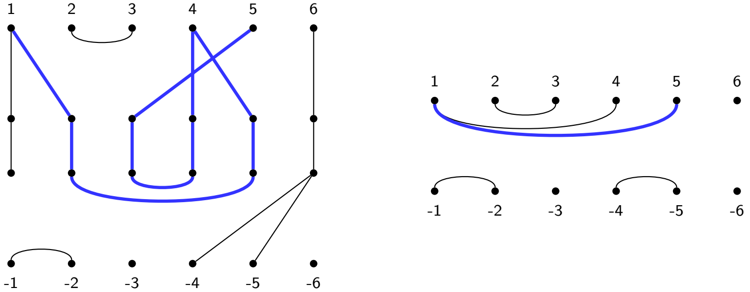

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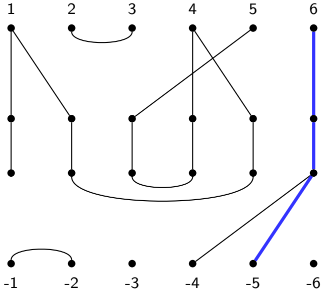

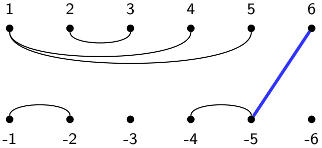

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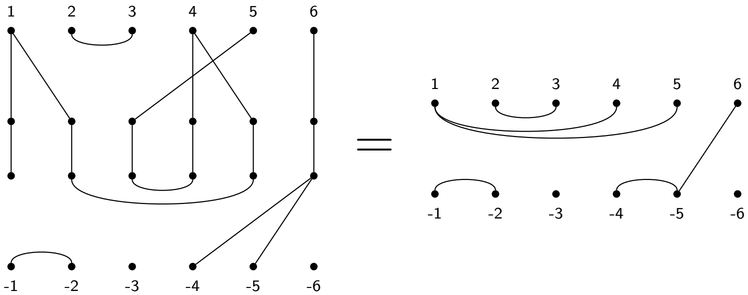

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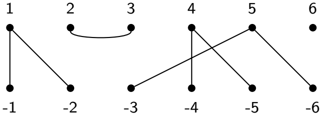
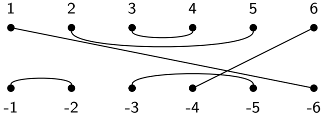
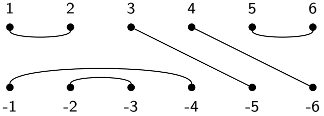
My Favourite Flavours of Partitions



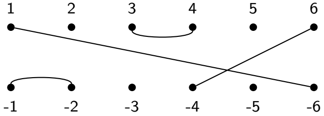
Partition



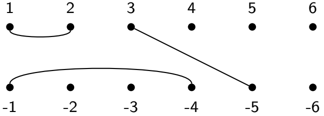
Brauer



Jones



Partial Brauer



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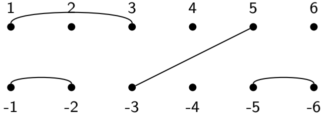
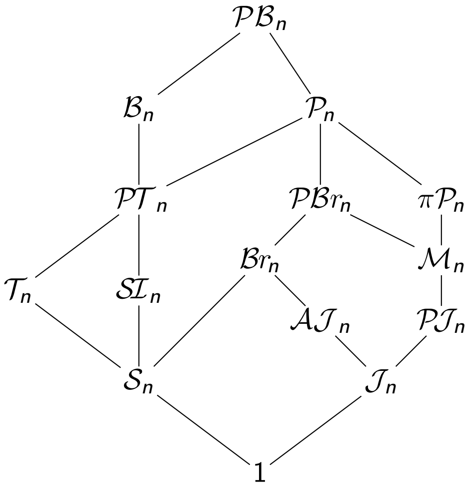


Motzkin

Classes of Diagram Monoids



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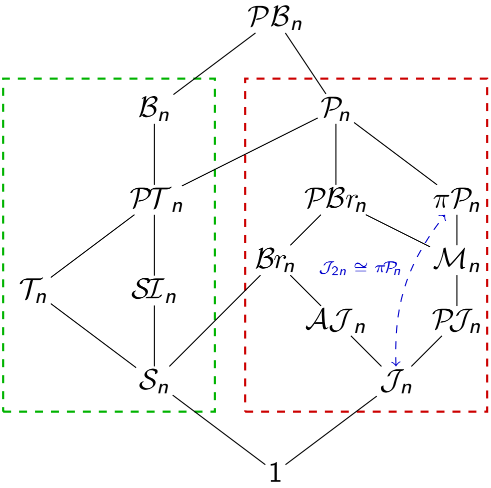

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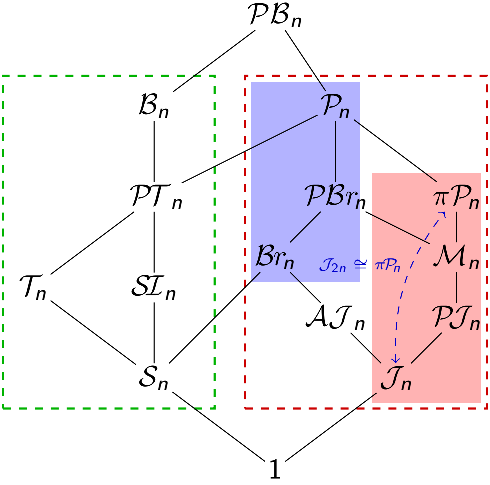

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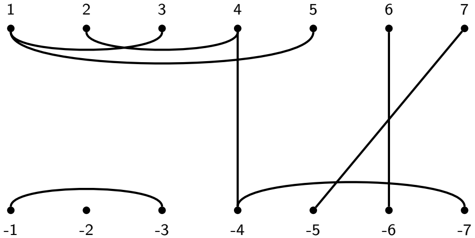
- Planar guys are *aperiodic/combinatorial* (i.e. subgroup-free);
- Usually* $\mathcal{D}$-classes form chain, indexed by number of transversal parts (*not case for partial Jones);
- Nice topological structures on sets of idempotents in "planar" cases.

Algebraic Theory

$\alpha :=$ 

Understand structure by means of *domains* and *kernels*.

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$\text{dom}^{\wedge}(\alpha) = \{2, 4, 6, 7\}$

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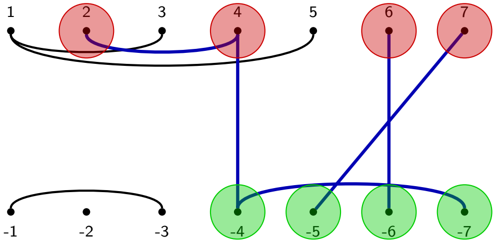

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$$\ker^{\wedge}(\alpha) = \left\{ \begin{array}{c} \{1,3,5\}, \\ \{2,4\}, \\ \{6\}, \{7\} \end{array} \right\}$$

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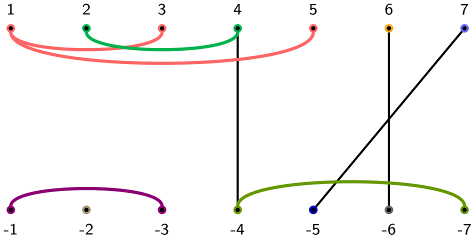

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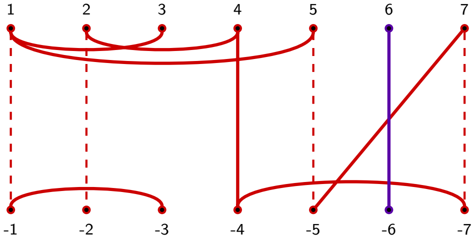

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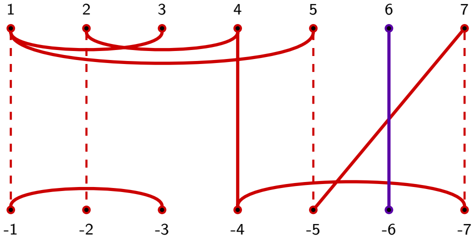

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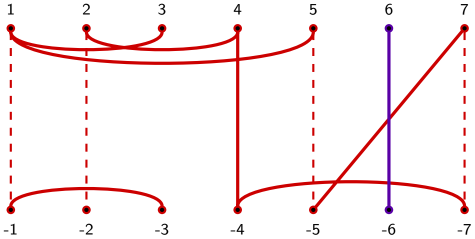

Understand structure by means of *domains* and *kernels*. The *rank* is the size of $\ker(\alpha)$. An element is *irreducible* if it has one kernel class.

Idempotents and Irreducibility

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Corollary (HEHFELD, I)

A partition is idempotent iff each kernel class houses at most one transverse component.

Counting Idempotents using a Partition Trick

Can understand and very quickly enumerate idempotents in $\mathcal{P}_n$, $\mathcal{B}r_n$ and $\mathcal{PB}r_n$.

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Observation

The number $p(n)$ of partitions on n points is equal to

$$p(0) = 1, \qquad p(n) = \sum_{i=1}^{n} i \cdot p(n-i)$$

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Theorem (HEHFELD, I; Theorem 7)

Let $\mathcal{K}_n$ be any of the above. Then the number $e(\mathcal{K}_n)$ of idempotents in $\mathcal{K}_n$ is equal to

$$e(\mathcal{K}_0) = 1, \qquad e(\mathcal{K}_n) = \sum_{i=1}^{n} c(\mathcal{K}_i) \cdot e(\mathcal{K}_{n-i})$$

where $c(\mathcal{K}_n)$ is the number of irreducible idempotents.

The planar case

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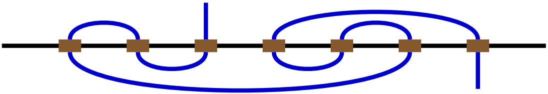

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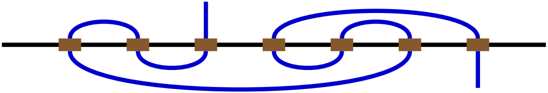

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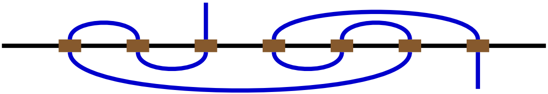

Need new ideas to tackle this problem.

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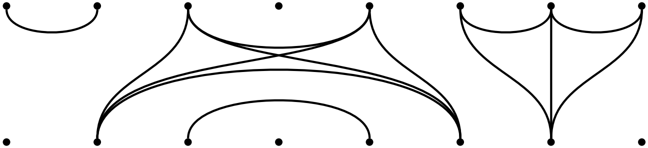

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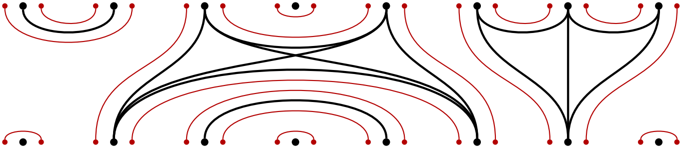

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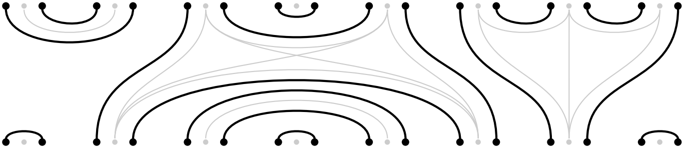

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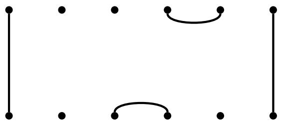

A Motzkin element

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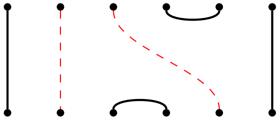

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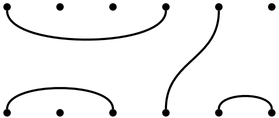

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Interface diagrams

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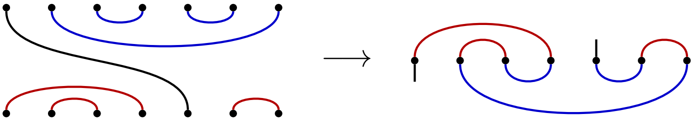

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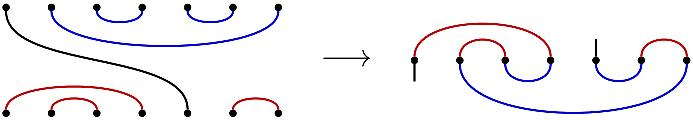

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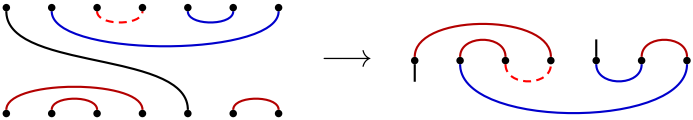

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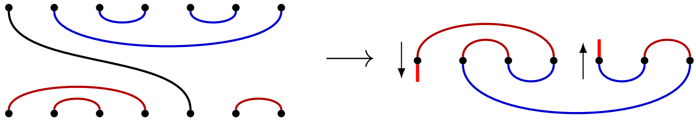

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For idempotents: no cis rays, no half-rays.

A topological structure on idempotents

Maps $\hat{\cdot} : E \longrightarrow D \cap E$ from set E of idempotents to union D of $\mathcal{D}$-classes of rank at most 1.

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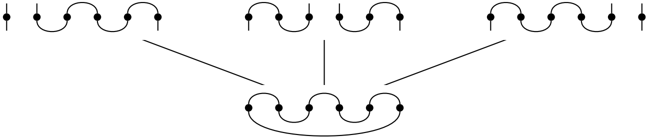

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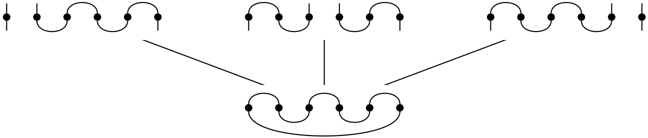

Refines natural order [Higgins, 1994] on idempotents.

Low-rank Idempotents

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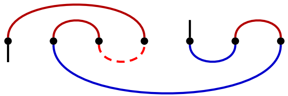
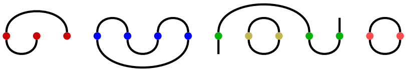
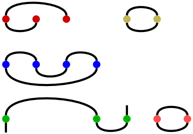
Hat map reduces study to $\mathcal{D}$-classes of rank $\leq$1 and combinatorics on interface diagrams. All rank-0 are idempotent (i.e. basepoints of fibers of hat map); not all rank-1 are:



Connected Idempotents

Can further reduce to studying *connected idempotents* with active return edges marked.





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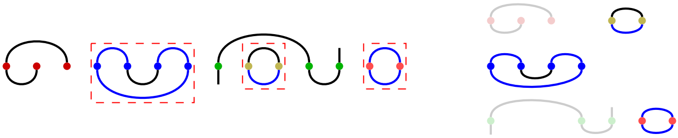

Each connected component contributes $\tau \cdot \beta + 1$, where τ (resp. β) is # top (resp. bottom) return edges.

Shrubs in the space of idempotents

A *shrub* is a rooted tree of height (at most) 1.

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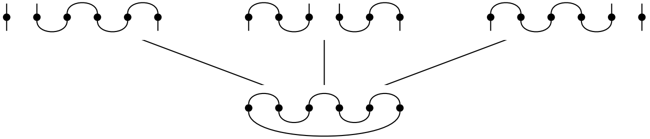

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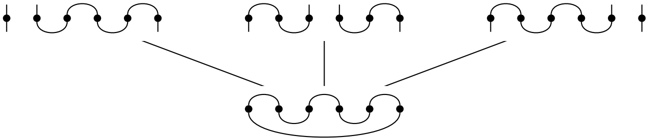
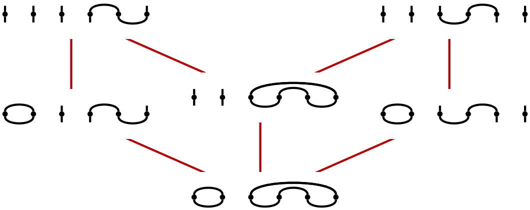


Every shrub is a root with $\tau \cdot \beta$ leaves, τ, β as before.

Calculating number of idempotents

The fiber of a rank ≤ 1 idempotent is a product of shrubs given by connected components with return edges marked.



Calculating idempotents, II

Theorem

The number of idempotents in the Jones monoid of degree n is

$$e(\mathcal{J}_n) = \sum_{e \in D} \delta_e = \sum_{e \in D} \sum_{\substack{c \leq e \\ c \text{ connected}}} (\tau_c \cdot \beta_c + 1) \, .$$

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where D is the set of rank ≤ 1 elements, δ_e is the size of the fibre at e of the hat map, and τ_c and β_c are as above for a connected component with return edges marked.

Calculating idempotents, II

Theorem

The number of idempotents in the Motzkin monoid of degree n is

$$e(\mathcal{J}_n) = \sum_{e \in D \cap E} \delta_e = \sum_{e \in D \cap E} \sum_{\substack{c \leq e \\ c \text{ connected}}} (\tau_c \cdot \beta_c + 1) \, .$$

where D is the set of rank ≤ 1 elements, δ_e is the size of the fibre at e of the hat map, and τ_c and β_c are as above for a connected component with return edges marked.

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- No obvious unique normal forms for elements;
- We have solved this. This will be HEHFELD, III.

References

- I. Dolinka, et al. "Enumeration of idempotents in diagram semigroups and algebras." *J. Comb. Thy A* **131**, 119–152 (2015). **arXiv**:1408.2021. [HEHFELD, I]
- I. Dolinka, et al. "Idempotent Statistics of the Motzkin and Jones Monoids," **arXiv**:1507.04838. [HEHFELD, II]