Invariant random subgroups of locally finite groups

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The space of subgroups

- Let G be a countably infinite group and let

$$\mathrm{Sub}_G \subset \mathcal{P}(G) = \{0, 1\}^G = 2^G$$

be the set of subgroups $H \leqslant G$.

Observation

Sub_G is a closed subset of 2^G.

Proof.

If $S \in 2^G$ isn't a subgroup, then either

- $S \in \{ T \in 2^G \mid 1 \notin T \}$,

or there exist $a, b \in G$ such that

- $S \in \{ T \in 2^G \mid a, b \in T \text{ and } ab^{-1} \notin T \}$.

□

Invariant random subgroups

- Note that $G \curvearrowright \mathrm{Sub}_G$ via conjugation: $H \overset{g}{\mapsto} g\,H\,g^{-1}$.

Definition (Miklós Abért)

A G-invariant probability measure ν on Sub_G is called an invariant random subgroup or IRS.

A Boring Example

If $N \trianglelefteq G$, then the Dirac measure δ_N is an IRS of G.

Stabilizer distributions

Observation

- Suppose that $G \curvearrowright (Z, \mu)$ is a measure-preserving action on a probability space.
- Let $f : Z \to \mathrm{Sub}_G$ be the G-equivariant map defined by $z \mapsto G_z = \{ g \in G \mid g \cdot z = z \}$.
- Then the stabilizer distribution $\nu = f_* \mu$ is an IRS of G.
- If $B \subseteq \mathrm{Sub}_G$, then $\nu(B) = \mu(\{ z \in Z \mid G_z \in B \})$.

Theorem (Abért-Glasner-Virag 2012)

If ν is an IRS of G, then ν is the stabilizer distribution of a measure-preserving action $G \curvearrowright (Z, \mu)$.

Ergodicity

Definition

A measure-preserving action $G \curvearrowright (Z, \mu)$ *is ergodic if* $\mu(A) = 0, 1$ *for every G-invariant* μ*-measurable subset* $A \subseteq Z$*.*

Observation

If $G \curvearrowright (Z, \mu)$ is ergodic, then the corresponding stabilizer distribution ν is an ergodic IRS of G.

Theorem (Creutz-Peterson 2013)

If ν *is an ergodic IRS of G, then* ν *is the stabilizer distribution of an ergodic action* $G \curvearrowright (Z, \mu)$*.*

Strongly simple locally finite groups

Definition

A countable group G is strongly simple if the only ergodic IRS of G are δ_1 and δ_G.

Theorem (Kirillov 1965 & Peterson-Thom 2013)

If K is a countably infinite field and $n \geq 2$, then $PSL(n, K)$ is strongly simple.

Open Problem

Classify the strongly simple locally finite groups.

Definition

A countably infinite group G is locally finite if we can express $G = \bigcup_{i \in \mathbb{N}} G_i$ as the union of an increasing chain of finite groups.

Inductive limits of finite alternating groups

Definition

- *G is an $L(\mathrm{Alt})$-group if we can express $G = \bigcup_{i \in \mathbb{N}} G_i$ as the union of an increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$, where $|\Delta_0| \geq 5$.*
- *Here we allow arbitrary embeddings $G_i \hookrightarrow G_{i+1}$.*

Theorem (Thomas-Tucker-Drob 2015)

It is possible to classify the strongly simple $L(\mathrm{Alt})$-groups ... and to classify the ergodic IRS's of the non-strongly simple $L(\mathrm{Alt})$-groups.

Inductive limits of finite alternating groups

Definition

Suppose that $\Sigma \subseteq \Delta_{i+1}$ is a G_i-orbit.

- *Σ is **trivial** if $|\Sigma| = 1$.*
- *Σ is **natural** if $G_i = \mathrm{Alt}(\Delta_i) \curvearrowright \Sigma$ is isomorphic to $\mathrm{Alt}(\Delta_i) \curvearrowright \Delta_i$.*
- *Otherwise, Σ is **exceptional**.*

Notation/Theorem

- *$n_i = |\Delta_i|$.*
- *e_{i+1} is the number of $x \in \Delta_{i+1}$ which lie in an exceptional G_i-orbit.*
- *s_{i+1} is the number of natural G_i-orbits on Δ_{i+1}.*
- *If $i < j$, then $s_{ij} = s_{i+1}s_{i+2}\cdots s_j$ is the number of natural G_i-orbits on Δ_j.*

The classification theorem

Definition (Zalesskii)

$G = \bigcup_{i \in \mathbb{N}} G_i$ *is a diagonal limit if* $e_{i+1} = 0$ *for all* $i \in \mathbb{N}$.

Theorem (Thomas-Tucker-Drob 2015)

If G is an $L(\mathrm{Alt})$*-group, then G has a nontrivial ergodic IRS if and only if G can be expressed as an almost diagonal limit of finite alternating groups.*

Definition

$G = \bigcup_{i \in \mathbb{N}} G_i$ *is an almost diagonal limit if* $s_{i+1} > 0$ *for all* $i \in \mathbb{N}$ *and* $\sum_{i=1}^{\infty} e_i / s_{0i} < \infty$.

Classifying of the ergodic IRS's of diagonal limits

From now on, we suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is a diagonal limit.

The analysis initially splits into two cases:

- G has linear natural orbit growth.
- G has sublinear natural orbit growth.

The sublinear case then splits into two cases:

- $G \not\cong \mathrm{Alt}(\mathbb{N})$.
- $G \cong \mathrm{Alt}(\mathbb{N})$.

Linear vs sublinear natural orbit growth

Proposition (Leinen-Puglisi 2003)

For each $i \in \mathbb{N}$, the limit $a_i = \lim_{j \to \infty} s_{ij}/n_j$ exists.

Proof.

If $i < j < k$, then $s_{ik} = s_{ij} s_{jk}$ and $s_{jk} n_j \leq n_k$. Hence we obtain that

$$\frac{s_{ik}}{n_k} = \frac{s_{ij}}{n_j} \cdot \frac{s_{jk} n_j}{n_k} \leq \frac{s_{ij}}{n_j}.$$

□

Definition (Leinen-Puglisi 2003)

G has linear natural orbit growth if $a_i > 0$ for some (equivalently every) $i \in \mathbb{N}$. Otherwise, G has sublinear natural orbit growth.

A natural candidate for a nontrivial ergodic IRS

Clearly we can suppose that

- $\Delta_0 = \{ \alpha^0_\ell \mid \ell < t_0 = n_0 \}$.
- $\Delta_{i+1} = \{ \sigma ^\frown k \mid \sigma \in \Delta_i, 0 \leq k < s_{i+1} \} \cup \{ \alpha^{i+1}_\ell \mid 0 \leq \ell < t_{i+1} \}$

and that the embedding $\varphi_i : \mathrm{Alt}(\Delta_i) \to \mathrm{Alt}(\Delta_{i+1})$ satisfies

- $\varphi_i(g)(\sigma ^\frown k) = g(\sigma) ^\frown k$
- $\varphi_i(g)(\alpha^{i+1}_\ell) = \alpha^{i+1}_\ell$.

Let Δ consist of the infinite sequences of the form

$$(\alpha^i_\ell, k_{i+1}, k_{i+2}, k_{i+3}, \cdots)$$

where $0 \leq k_j < s_j$. Then $G \curvearrowright \Delta$ via

$$g \cdot (\alpha^i_\ell, k_{i+1}, \cdots, k_j, k_{j+1} \cdots) = (g(\alpha^i_\ell, k_{i+1}, \cdots, k_j), k_{j+1} \cdots), \quad g \in G_j.$$

A natural candidate for a nontrivial ergodic IRS

For each $\sigma \in \Delta_i$, let $\Delta(\sigma) \subseteq \Delta$ be the set of sequences of the form

$$\sigma^\frown(\, k_{i+1}, k_{i+2}, k_{i+3}, \cdots\,).$$

Then the $\Delta(\sigma)$ form a clopen basis for a locally compact topology on Δ; and $G \curvearrowright \Delta$ via homeomorphisms.

Question

When is there a G-invariant ergodic probability measure on Δ?

The Pointwise Ergodic Theorem

Theorem (Vershik 1974 & Lindenstrauss 1999)

Suppose that $G = \bigcup G_i$ is locally finite and that $G \curvearrowright (Z, \mu)$ is ergodic. If $B \subseteq Z$ is μ-measurable, then for μ-a.e $z \in Z$,

$$\mu(B) = \lim_{i \to \infty} \frac{1}{|G_i|} |\{ g \in G_i \mid g \cdot z \in B \}|.$$

The Pointwise Ergodic Theorem

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Linear vs sublinear natural orbit growth

Proposition

If μ is a G-invariant ergodic probability measure on Δ and $\sigma \in \Delta_i$, then

$$\mu(\Delta(\sigma)) = \lim_{j \to \infty} s_{ij}/n_j = a_i.$$

Corollary

If G has sublinear natural orbit growth, then no such μ exists.

Proof.

Supposing that μ exists, we have that

$$1 = \mu(\Delta) = \sum_{i \in \mathbb{N}} \sum_{0 \leq \ell < t_i} \mu(\Delta(\alpha^i_\ell)) = 0.$$

□

The proof of the proposition

- Choose $z \in \Delta$ such that
$$\mu(\Delta(\sigma)) = \lim_{j\to\infty} \frac{1}{|G_j|} |\{ g \in G_j \mid g \cdot z \in \Delta(\sigma) \}|.$$
- Let $z = (\alpha^r_\ell, k_{r+1}, k_{r+2}, k_{r+3}, \cdots)$ and for each $j > r$, let
$$z_j = (\alpha^r_\ell, k_{r+1}, k_{r+2}, k_{r+3}, \cdots , k_j) \in \Delta_j.$$
- For each $j > \max\{ i, r \}$, let $S_j \subseteq \Delta_j$ be the set of sequences of the form $\sigma \,\widehat{}\, (d_{i+1}, d_{i+2}, \cdots , d_j)$.
- Then $|S_j| = s_{ij}$ and we have that
$$\{ g \in G_j \mid g \cdot z \in \Delta(\sigma) \} = \{ g \in G_j \mid g \cdot z_j \in S_j \}.$$
- It now follows that
$$\mu(\Delta(\sigma)) = \lim_{j\to\infty} \frac{1}{|G_j|} |\{ g \in G_j \mid g \cdot z_j \in S_j \}| = \lim_{j\to\infty} |S_j|/|\Delta_j| = a_i.$$

The ergodic IRS's for linear natural orbit growth

Theorem

If G has linear natural orbit growth, then there exists a unique G-invariant ergodic probability measure μ on Δ.

Non-obvious Corollary

If G has linear natural orbit growth, then the diagonal action $G \curvearrowright (\Delta^r, \mu^{\otimes r})$ is ergodic for all $r \geq 1$.

Theorem (Thomas-Tucker-Drob 2015)

If G has linear natural orbit growth and $\nu \neq \delta_1, \delta_G$ is an ergodic IRS, then there exists $r \geq 1$ such that ν is the stabilizer distribution of $G \curvearrowright (\Delta^r, \mu^{\otimes r})$.

The basic strategy for groups of linear orbit growth

Observation

- Suppose that G has linear natural orbit growth and that ν_r is the stabilizer distribution of $G \curvearrowright (\Delta^r, \mu^{\otimes r})$.
- Then for ν_r-a.e. $H \in \mathrm{Sub}_G$, for all but finitely many $i \in \mathbb{N}$, there exists $\Sigma_i \subset \Delta_i$ with $|\Delta_i \smallsetminus \Sigma_i| = r$ such that $H \cap \mathrm{Alt}(\Delta_i) = \mathrm{Alt}(\Sigma_i)$.

Target

- *Suppose that G has linear natural orbit growth and that $\nu \neq \delta_1, \delta_G$ is the stabilizer distribution of the ergodic action $G \curvearrowright (Z, \mu)$.*
- *Then there exists $r \geq 1$ such that for ν-a.e. $H \in \mathrm{Sub}_G$, for all but finitely many $i \in \mathbb{N}$, there exists $\Sigma_i \subset \Delta_i$ with $|\Delta_i \smallsetminus \Sigma_i| = r$ such that $H \cap \mathrm{Alt}(\Delta_i) = \mathrm{Alt}(\Sigma_i)$.*

Another application of the pointwise ergodic theorem

- Let $G = \bigcup G_i$ be locally finite and let $G \curvearrowright (Z, \mu)$ be ergodic.
- For each $z \in Z$ and $i \in \mathbb{N}$, let $\Omega_i(z) = \{ g \cdot z \mid g \in G_i \}$.

Theorem

With the above hypotheses, for μ-a.e. $z \in Z$, for all $g \in G$,

$$\mu(\operatorname{Fix}_Z(g)) = \lim_{i \to \infty} |\operatorname{Fix}_{\Omega_i(z)}(g)| / |\Omega_i(z)|.$$

Remark

Note that the $|\operatorname{Fix}_{\Omega_i(z)}(g)| / |\Omega_i(z)|$ is the probability that an element of $(\Omega_i(z), \mu_i)$ is fixed by $g \in G_i$, where μ_i is the uniform probability measure on $\Omega_i(z)$

Computing the normalized permutation character

Definition

The *normalized permutation character* *of the action* $G_i \curvearrowright \Omega_i(z)$ *is*

$$\chi_i(g) = |\operatorname{Fix}_{\Omega_i(z)}(g)|/|\Omega_i(z)|.$$

- Note that $G_i \curvearrowright \Omega_i(z)$ is isomorphic to $G_i \curvearrowright G_i/H_i$, where $H_i = \{ h \in G_i \mid h \cdot z = z \}$ is the stabilizer of z.

Proposition

If χ_i *is the normalized permutation character corresponding to the action* $G_i \curvearrowright G_i/H_i$*, then*

$$\chi_i(g) = \frac{|g^{G_i} \cap H_i|}{|g^{G_i}|} = \frac{|\{s \in G_i \mid sgs^{-1} \in H_i\}|}{|G_i|}.$$

The basic strategy for groups of linear orbit growth

- Let $G = \bigcup G_i$ have linear natural orbit growth, where $G_i = \mathrm{Alt}(\Delta_i)$.
- Let ν be the stabilizer distribution of the ergodic $G \curvearrowright (\, Z, \mu\,)$.
- Then we can suppose that there exists $1 \neq g \in G$ such that $\mu(\, \mathrm{Fix}_Z(g)\,) \neq 0$. Otherwise, $\nu = \delta_1$.
- Choose a μ-random point $z \in Z$ such that for all $g \in G$,

$$\mu(\, \mathrm{Fix}_Z(g)\,) = \lim_{i \to \infty} |\, \mathrm{Fix}_{\Omega_i(z)}(g)\,| / |\, \Omega_i(z)\,|;$$

  and let $H = \{\, h \in G \mid h \cdot z = z\,\}$ be the ν-random subgroup.
- Then we can suppose that $\mu(\, \mathrm{Fix}_Z(h)\,) > 0$ for all $h \in H$.

The basic strategy for groups of linear orbit growth

- We must analyse the action of $H_i = H \cap G_i$ on Δ_i.
- Let $h \in H$ be an element of prime order p.
- Regarded as an element of $\mathrm{Alt}(\Delta_i)$, let h be a product of c_i p-cycles.
- Then there exists a constant b such that $c_i \geq b\, n_i$.
- By Stirling's Formula, there exist constants $r, s > 0$ such that

$$|h^{\mathrm{Alt}(\Delta_i)}| > r\, s^{n_i}\, n_i^{n_i(p-1)b}$$

The basic strategy for groups of linear orbit growth

- Suppose that $H_i \curvearrowright \Delta_i$ is primitive for infinitely many $i \in \mathbb{N}$.

Theorem (Praeger-Saxl 1979)

If $H_i < \mathrm{Alt}(n_i)$ is a proper primitive subgroup, then $|H_i| < 4^{n_i}$.

- But this means that

$$\begin{aligned}\mu(\,\mathrm{Fix}_Z(g)\,) &= \lim_{i\to\infty} \frac{|\,h^{\mathrm{Alt}(\Delta_i)} \cap H_i\,|}{|\,h^{\mathrm{Alt}(\Delta_i)}|} \\ &\leq \lim_{i\to\infty} \frac{|\,H_i\,|}{|\,h^{\mathrm{Alt}(\Delta_i)}|} = 0,\end{aligned}$$

which is a contradiction!

Another natural candidate for a nontrivial ergodic IRS

Observation

- Suppose that G has linear natural orbit growth and that ν_r is the stabilizer distribution of $G \curvearrowright (\, \Delta^r, \mu^{\otimes r}\,)$.
- Then for ν_r-a.e. $H \in \mathrm{Sub}_G$, for all but finitely many $i \in \mathbb{N}$, there exists $\Sigma_i \subset \Delta_i$ with $|\Delta_i \smallsetminus \Sigma_i| = r$ such that $H \cap \mathrm{Alt}(\Delta_i) = \mathrm{Alt}(\Sigma_i)$.

Basic Idea

Construct an IRS ν which concentrates on subgroups

$$H = \bigcup \mathrm{Alt}(\Sigma_i), \qquad \Sigma_i \subset \Delta_i,$$

such that $|\Delta_i \smallsetminus \Sigma_i| \to \infty$.

Another natural candidate for a nontrivial ergodic IRS

Let Σ consist of the sequences $(\Sigma_i)_{i \in \mathbb{N}}$ such that:

- $\Sigma_i \subseteq \Delta_i$
- $\mathrm{Alt}(\Sigma_{i+1}) \cap G_i = \mathrm{Alt}(\Sigma_i)$.

For each $X \subseteq \Delta_i$, let $\Sigma(X) \subseteq \Sigma$ be the sequences such that $\Sigma_i = X$. Then the $\Sigma(X)$ form a basis for a locally compact topology on Σ; and $G \curvearrowright \Sigma$ via homeomorphisms.

Fix some $\beta_0 = \beta \in \mathbb{R}^+$ and let $\beta_{i+1} = \beta_i / s_{i+1}$. Then we can define a G-invariant probability measure μ_β on Σ by

$$\mu_\beta(\Sigma(X)) = \left(1/e^{\beta_i}\right)^{|X|} \left(1 - 1/e^{\beta_i}\right)^{n_i - |X|}$$

for each $X \subseteq \Delta_i$. Note that $1/e^{\beta_i} = (1/e^{\beta_{i+1}})^{s_{i+1}}$.

Another natural candidate for a nontrivial ergodic IRS

Question

When is μ_β ergodic?

Proposition

If G has linear natural orbit growth, then μ_β is not ergodic.

Proof.

If $\sigma = (\,\Delta_i\,)_{i \in \mathbb{N}}$, then $\{\,\sigma\,\}$ is G-invariant. Furthermore,

$$\mu_\beta(\,\{\,\sigma\,\}\,) = \lim_{i \to \infty} \mu_\beta(\,\Sigma(\Delta_i)\,) = \lim_{i \to \infty} \frac{1}{e^{\beta n_i / s_{0i}}} = \frac{1}{e^{\beta / a_0}}$$

□

The ergodic decomposition of μ_β

Remark

- Suppose that G has linear natural orbit growth and let $\lambda = \beta / a_0$.
- For $r \geq 1$, let ν_r be the stabilizer distribution of $G \curvearrowright (\Delta^r, \mu^{\otimes r})$.
- Write $\nu_0 = \delta_G$.
- Then the ergodic decomposition of μ_β is given by

$$\mu_\beta = \frac{1}{e^\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \nu_r.$$

The ergodic IRS's for sublinear natural orbit growth

Theorem

If G has sublinear natural orbit growth, then μ_β is ergodic.

Theorem (Thomas-Tucker-Drob 2015)

If $G \neq \mathrm{Alt}(\mathbb{N})$ has sublinear natural orbit growth and $\nu \neq \delta_1, \delta_G$ is an ergodic IRS, then there exists $\beta \in \mathbb{R}^+$ such that ν is the stabilizer distribution of $G \curvearrowright (\Sigma, \mu_\beta)$.

Remark (Vershik)

$\mathrm{Alt}(\mathbb{N})$ has a much richer collection of ergodic IRS's.

An open problem

Question

If G is a countably infinite simple locally finite group and ν is an ergodic IRS of G, does ν necessarily concentrate on the subgroups $H \leqslant G$ of a fixed isomorphism type?

Remark

- Clearly ν concentrates on the subgroups $H \leqslant G$ with a fixed skeleton; i.e. with a fixed set of isomorphism types of finite subgroups.
- However, since the skeleton is usually the set of all finite groups, this observation is not very useful.

The End