Generating Infinite Random Graphs

U. B. Darji

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THE Random graph
Some Infinite Processes
Our Process
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Basic Examples
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Joint work with Csaba Biró.

Overview

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Objective

The main problem we address in this talk is the following:

Problem

Come up with a natural probabilistic process which generates a large class of "random" infinite graphs.

ER Process Description

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Consider the vertex set $\mathbb{N}$. Let $0 < p < 1$ be fixed. For each pair of distinct integers $n, m \in \mathbb{N}$, put an edge between n and m with probability p. Let G be the resulting graph on $\mathbb{N}$. A classical 1963 Erdős–Rényi theorem states that with probability one, any two such graphs are isomorphic.

ER cont'd 1

In 1964 Rado gave an explicit construction of a graph R which is universal for the collection of all countable graphs. More precisely, he showed that if G and H are any countable graphs and $\phi : G \to H$ a graph homomorphism, then there are embeddings $e_G : G \to R$, $e_H : H \to R$ and a graph homomorphism $\psi : R \to R$ such that $e_H^{-1} \circ \psi \circ e_G = \phi$, i.e., R contains a copy of every countable graph and every graph homomorphism between countable graphs can be lifted to a graph homomorphism of R.

ER cont'd 2

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The constructions of Erdős–Rényi and Rado seem very different but they result in the same graph. The reason for this is that both graphs satisfy the following property: if (A, B) are disjoint, finite sets of vertices, then there are infinitely many vertices v such that there is an edge between v and every element of A and there are no edges between v and any element of B. It can be shown by back and forth method that any two graphs with above property are isomorphic to each other. A graph with this property is often called the Erdős–Rényi graph, the Rado graph or simply the random graph.

Preferential Attachment Models

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In the preferential attachment models, the new vertex is adjacent to earlier vertex or vertices with a probability that is proportional to the current degree of an existing vertex. One of the first examples of empirical study of this model is by Barabási and Albert in 1999, and a rigorous mathematical framework was defined by Bollobás and Riordan in 2004. These, and subsequent works study large finite graphs as opposed to the limiting behavior.

An infinite version of the preferential attachment model was studied by Kleinberg and Kleinberg in 2005. In their paper, the sequence d_i is constant.

Copying models

This model was first introduced by Kumar et al. in 1999 and later, a slightly modified and generalized version was defined by Bonato and Janssen in 2007. In their construction, besides the sequence d_i, an initial finite graph H and a probability $p \in [0, 1]$ is given.

- Let $G_0 = H$.
- To construct G_i, add a new vertex v to G_{i-1} and choose its neighbors as follows.
  - Choose a vertex $u \in V(G_{i-1})$ uniformly at random (called the *copy vertex*). Connect v to each neighbor of u with probability p.
  - Choose a set of d_i vertices from $V(G_{i-1})$ uniformly at random, and connect v to each vertex in this set.
  - Delete multiple edges if necessary.

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The Janson–Severini Process

In 2013, Janson and Severini, independently, introduced a process which is very similar to our. Their construction is the following. For all $i = 1, \ldots,$ let ν_i be a probability distribution on $\{0, 1, \ldots, i\}$. Construct the random graph G_i as follows.

- Let $G_0 = K_1$, the graph on a single vertex.
- Let D_i be a random variable with distribution ν_i, and construct G_i by adding a new vertex to G_{i-1} and connecting it to a uniformly random subset of size D_i of $V(G_{i-1})$.

Description

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Suppose a sequence of integers $\{d_i\}_{i=0}^{\infty}$ is given, with the property $0 \leq d_i \leq i$ for all i. Let $V = \{v_0, v_1, \ldots\}$ be a set of vertices. For $i = 0, 1, \ldots$, in round i, we first choose $A \subseteq \{v_0, \ldots, v_{i-1}\}$ of cardinality d_i with the uniform distribution on the set of all sets of size d_i subsets of $\{v_0, \ldots, v_{i-1}\}$. Then, we add edges $v_i u$ for all $u \in A$. The result is a random graph on countably many vertices. We strive to understand the resulting probability space, in particular, we would like to determine the atoms (graphs with positive probability), and cases when there is only one atom with probability 1. In this latter case, we say that the probability space is *concentrated*.

Examples

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Example

The sequence $0, 1, 1, 1, \ldots$ a.s. generates the ω-tree.

Proposition

Let $\{a_i\}_{i=0}^{\infty}$ be a sequence of real numbers such that $0 < a_i < 1$ and $\{d_i\}$ be a sequence of nonnegative integers. Then,

$$0 < \prod_{i=1}^{\infty} (1 - a_i)^{d_i} \iff \sum_{i=1}^{\infty} d_i a_i < \infty$$

Examples cont'd 2

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Example

A finite sequence and then $1, 1, 1, \ldots$ a.s. generates the following random graph: Generate the random graph of the finite sequence, and attach an ω-tree to each vertex.

Clearly, we can easily choose a sequence $\{d_n\}$ so that the resulting probability space is not concentrated.

Examples cont'd 3

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Example

There exists a sequence $\{d_i\}$ with non-concentrated probability space such that $d_i \in \{1, 2\}$ for $i \geq 1$ and $\{d_i\}$ is not eventually zero.

Proof.

We will construct a sequence consisting mostly of 1's, but infinitely many 2's inserted. The sequence starts with $0, 1, 1, 2$. We set $p_0 = 2/3$, and we note that p_0 is the probability that the first 4 vertices span a triangle. We put $d_k = 2$ in carefully chosen positions so the probability that the resulting graph contains a triangle is positive and less than 1. □

Radocity

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Definition

Let G be a graph. For a nonnegative integer k, we say that G is k-Rado, if every pair of disjoint sets of vertices (A, B) with $|A| \leq k, |B| \leq k$ has infinitely many witnesses.
The number $\text{rado}(G) = \sup\{k : G \text{ is } k\text{-Rado}\}$ is the radocity of G.

Clearly every graph is 0-Rado, and if a graph is k-Rado, it is also k'-Rado for all $k' < k$. Also, G is isomorphic to the Rado graph if and only if $\text{rado}(G) = \infty$.

Radocity Results

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Theorem

As before, let $\{d_i\}$ be such that $0 \le d_i \le i$. Let

$$k_1 = \sup \left\{ t \in \mathbb{N} : \sum_{n=1}^{\infty} \left(\frac{d_n}{n} \right)^t \left(\frac{n - d_n}{n} \right)^t = \infty \right\},$$

$$k_2 = \sup \left\{ t \in \mathbb{N} : \sum_{n=1}^{\infty} \frac{(d_n)_{(t)} (n - d_n)_{(t)}}{(n)_{(2t)}} = \infty \right\}.$$

Then $k_1 = k_2$, and the process a.s. generates a graph of radocity k_1 (and k_2).

Radocity Results cont'd 2

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Corollary

Let $a_n = \min\{\frac{d_n}{n}, \frac{n-d_n}{n}\}$*.*

(i) *If* $\sum_{n=1}^{\infty} a_n^k$ *diverges for all positive integers* k*, then the process almost surely generates the Rado graph.*

(ii) *If there is a positive integer* k *for which* $\sum_{n=1}^{\infty} a_n^k$ *converges, then the process almost surely does not generate the Rado graph.*

Corollary

Let $a_n = \min\{\frac{d_n}{n}, \frac{n-d_n}{n}\}$*. If* $\limsup a_n > 0$*, then the process a.s. generates the Rado graph.*

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Radocity Results cont'd 3

The double random process is when we even chose the sequence in random, choosing d_i with some distribution from the interval $[0, i]$. The following corollary states that in some sense, almost all double random processes will result in the Rado graph.

Corollary (Janson-Severini)

If there exist $\epsilon > 0$, $p_0 > 0$, and M integer such that for $n > M$, $\Pr[\epsilon n \leq d_n \leq (1-\epsilon)n] \geq p_0$, then the double random process a.s. generates the Rado graph.

In particular, if d_n is chosen from $\{0, \ldots, n-1\}$ with uniform distribution, we obtain, a.s., the Rado graph.

Some Lemmas

Lemma

The following statements hold.

i) *If* $\sum d_i/i = \infty$, *then the process a.s. generates a graph in which each vertex is of infinite degree.*

ii) *If* $\sum d_i/i < \infty$, *then the process a.s. generates a graph in which each vertex is of finite degree.*

Some Lemmas cont'd

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Let $\{d_n\}$ be as usual. Define

$$s_n = \sum_{i=0}^{n} d_i.$$

Lemma

Suppose $\sum s_i d_i / i < \infty$. Then the process a.s. generates a graph G which has the property that there exists an $N_1 = N_1(G)$ such that for all $n \geq N_1$ with $d_n > 0$, the vertex v_n will attach back to vertices with current degree 0. More rigorously, the vertex v_n has the property that if $v_j \sim v_n$, and $j < n$, then v_j has no neighbor before v_n.

Sparse Case

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In this section, we assume that $d_i \in \{0, 1\}$.

Lemma

Suppose $\sum d_i/i < \infty$. Then the process a.s. generates a graph with no ray.

Let T be a finite tree. Let F_T be the forest that consists of infinitely many copies of T, as components. Let $F_n = \cup F_T$, where the union is taken over all trees of size n. Note that F_1 is the countably infinite set with no edges, and F_2 is the countably infinite matching.

Main Theorem 0-1 Case

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We will also use the term ω-*tree* for the unique countably infinite tree in which every vertex is of infinite degree.

Theorem

i) *If* $\sum_{i=1}^{\infty} \frac{d_i}{i} = \infty$, *then the space is concentrated, and the atom is a graph whose components are* ω*-trees, and the number of components is equal to the number of zeroes in the sequence.*

ii) *Suppose* $\sum_{i=1}^{\infty} \frac{d_i}{i} < \infty$. *Let* $t_n = \sum_{i=n}^{\infty} \frac{d_i}{i}$, *and* $k = \min\{\kappa \geq 2 : \sum_l d_l t_{l+1}^{\kappa-2} < \infty\}$. *(We set* $k = \infty$ *if the set in question is empty). The space has infinitely many atoms, and all of them are of the form* $F \cup \left[\bigcup_{i<k} F_i\right]$ *where* F *is some finite forest.*

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Questions

Problem

Suppose the degree sequence $\{d_i\}$ is given.

1. *When does the resulting probability space has no atoms?*
2. *When is the resulting space concentrated?*
3. *If the resulting space is concentrated, when might it be a homogeneous structure?*


Thanks for the invite and thanks for listening!!!