Finite Diagram Semigroups: Extending the Computational Horizon

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London Mathematical Society EPSRC Durham Symposium
Permutation groups and transformation semigroups

- What is computable with n states?
- What is the minimal number of states required for a particular computation?
- What is the structure of these computations?

Here we use *computational enumeration*.

Enumerating by Order: Abstract Semigroups

(1954 Gakugei, J., **Notes on finite semigroups and determination of semigroups of order 4**, Tokushima Univ. Math., 5 (1954), 17–27.)

1955 Forsythe, G. E., **SWAC computes 126 distinct semigroups of order 4**, Proc. Amer. Math. Soc., 6 (1955), 443–447.

Tetsuya, K., Hashimoto, T., Akazawa, T., Shibata, R., Inui, T. and Tamura, T., **All semigroups of order at most 5**, *J. Gakugei Tokushima Univ. Nat. Sci. Math.*, 6 (1955), 19–39.

1967 Plemmons, R. J., **There are 15973 semigroups of order 6**, *Math. Algorithms*, 2 (1967), 2–17.

1977 Jürgensen, H. and Wick, P., **Die Halbgruppen der Ordnungen $\leq$ 7**, *Semigroup Forum*, 14 (1) (1977), 69–79.

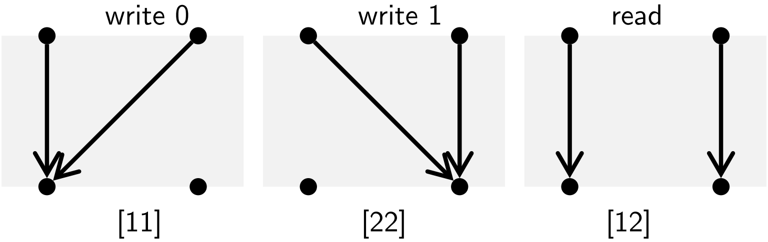
1994 Satoh, S., Yama, K. and Tokizawa, M., **Semigroups of order 8**, *Semigroup Forum*, 49 (1) (1994), 7–29.

Number of semigroups of order n

order	#groups	#semigroups	#3-nilpotent semigroups
1	1	1	0
2	1	4	0
3	1	18	1
4	2	126	8
5	1	1,160	84
6	2	15,973	2,660
7	1	836,021	609,797
8	5	1,843,120,128	1,831,687,022
9	2	52,989,400,714,478	52,966,239,062,973

`http://www-groups.mcs.st-andrews.ac.uk/~jamesm/smallsemi.php`

Flip-flop, the 1-bit memory semigroup

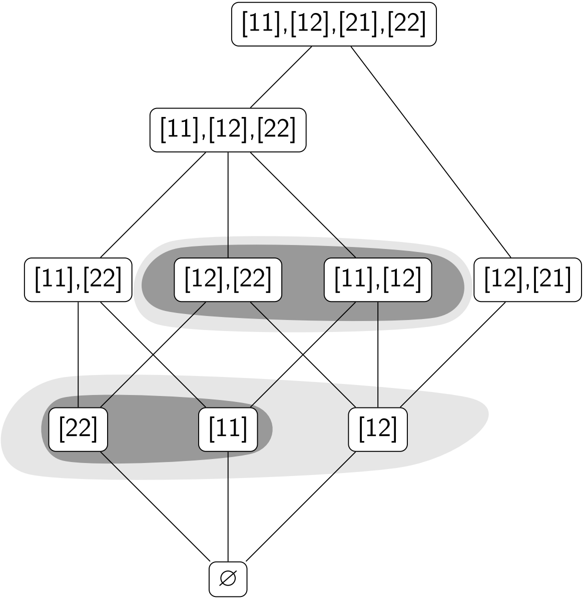


So these are computational devices... $\approx$ automata

With transformation semigroups, we get all semigroups. (Cayley's theorem)

Enumerating by Degree: Transformation Semigroups

Enumeration: subsemigroups of the full transformation semigroup



Data flood

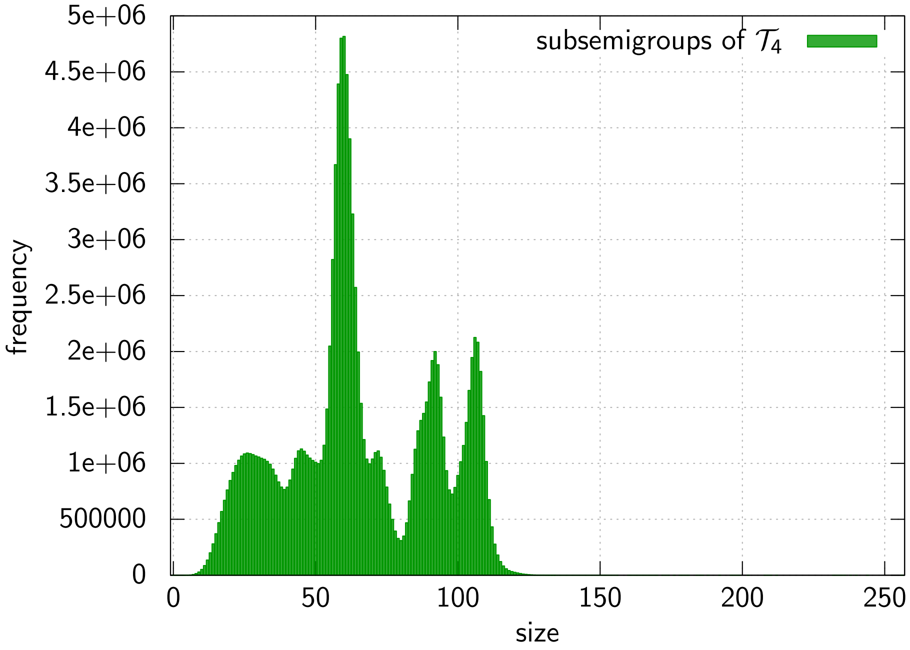
Number of subsemigroups of full transformation semigroups.

	#subsemigroups	#conjugacy classes	#isomorphism classes
$\mathcal{T}_0$	1	1	1
$\mathcal{T}_1$	2	2	2
$\mathcal{T}_2$	10	8	7
$\mathcal{T}_3$	1 299	283	267
$\mathcal{T}_4$	3 161 965 550	132 069 776	131 852 491

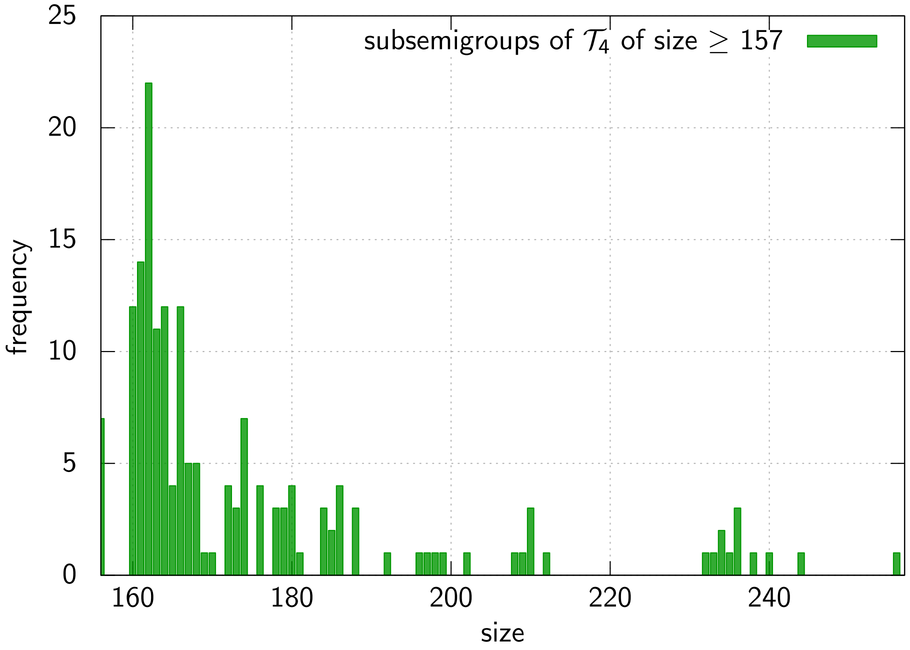
After discounting the state-relabelling symmetries the database of degree 4 transformation semigroups is still around 9GB.

Summary of Results

	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$
#nilpotent	1	2	4	22
#commutative	1	4	18	158
#band	1	5	41	1503
#regular	1	7	116	33285
#subsemigroups	1	7	282	132 069 776



subsemigroups of $\mathcal{T}_4$
frequency
size
5e+06
4.5e+06
4e+06
3.5e+06
3e+06
2.5e+06
2e+06
1.5e+06
1e+06
500000
0
0
50
100
150
200
250



subsemigroups of $\mathcal{T}_4$ of size $\geq$ 157
frequency
size
0
5
10
15
20
25
160
180
200
220
240

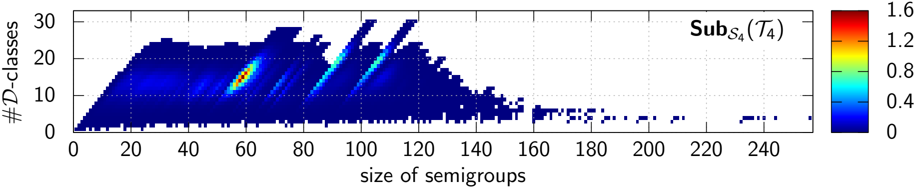
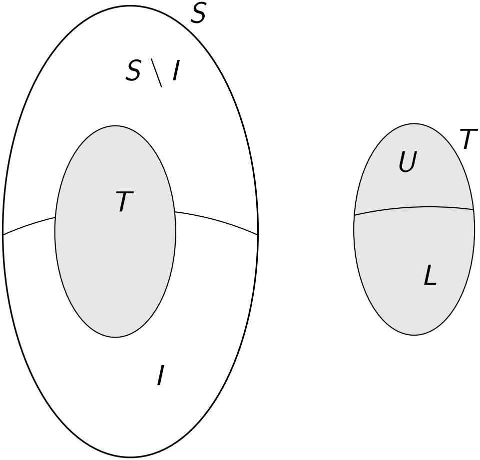


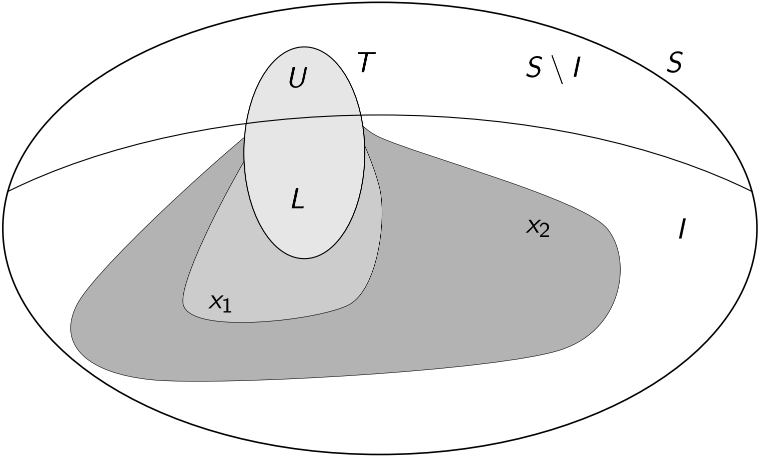
Figure: Size versus the number of $\mathcal{D}$-classes for transformation semigroups up to degree 4. Frequency values in millions.

How is it done?

Upper and lower torsos

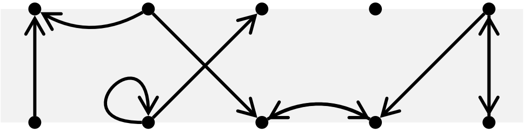


Enumerating lower torsos



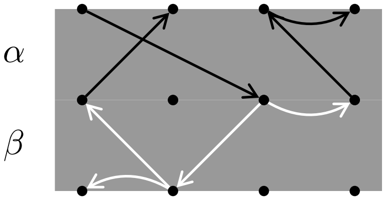
$$|T| << |\langle T \cup \{x_1\}\rangle| << |\langle T \cup \{x_1, x_2\}\rangle|$$

Partitioned Binary Relations

 $\in \mathcal{P}\mathsf{B}_n$,

$$\{(2,1),(2,3'),(5,4'),(5,5'),(1',1),(2',2'),(2',3),(3',4'),(4',3'),(5',5)\}$$

Multiplying Partitioned Binary Relations



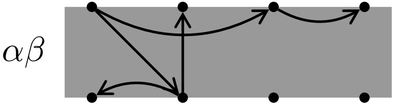
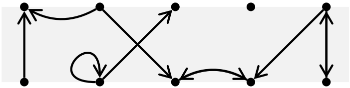
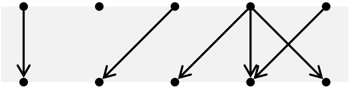


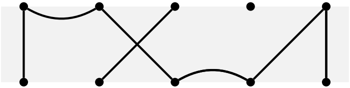
Diagram Semigroups – Typical Elements



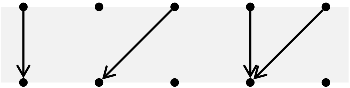
$\in \mathcal{P}\mathsf{B}_n$,



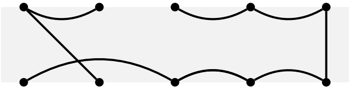
$\in \mathsf{B}_n$,



$\in \mathcal{P}_n$

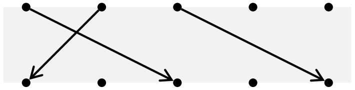


$\in \mathsf{P}\mathcal{T}_n$,

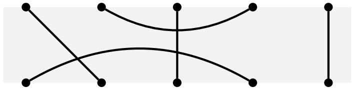


$\in \mathcal{I}_n^*$

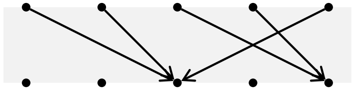
Diagram Semigroups – Typical Elements



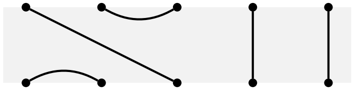
$\in \mathcal{I}_n$,



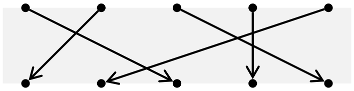
$\in \mathfrak{B}_n$



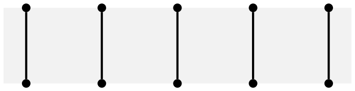
$\in \mathcal{T}_n$,



$\in \mathsf{TL}_n$

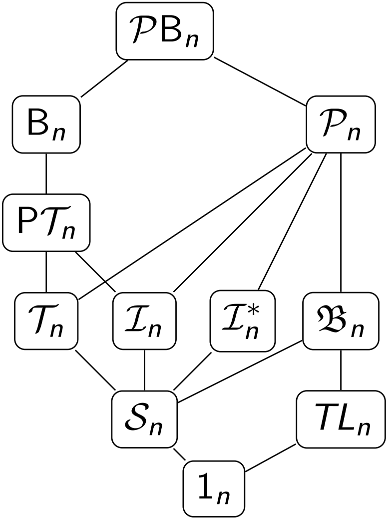


$\in \mathcal{S}_n$,



1_n

Diagram Semigroup Land



	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$\mathcal{P}\mathsf{B}_n$	1262					
B_n	4	385				
$\mathcal{P}_n$	4	272				
$\mathsf{P}\mathcal{T}_n$	4	50	94232			
$\mathcal{I}_n$	4	23	2963			
$\mathcal{I}_n^*$	2	6	795			
$\mathcal{T}_n$	2	8	283	132069776		
$\mathfrak{B}_n$	2	6	42	10411		
TL_n	2	4	12	232	12592	324835618
$\mathcal{S}_n$	1	2	4	11	19	56

Table: Summary table of the numbers of distinct (up to conjugacy) semigroups of given diagram type and degree.

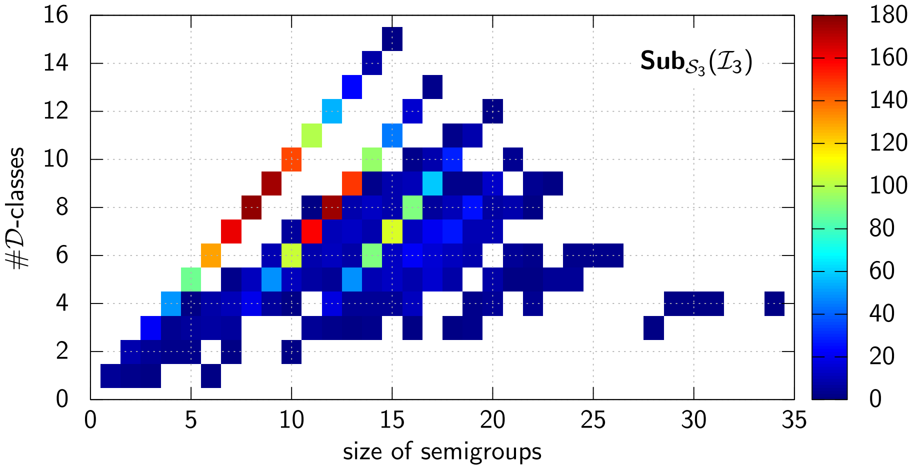


Figure: Size versus the number of $\mathcal{D}$-classes for partial permutation semigroups up to degree 3.

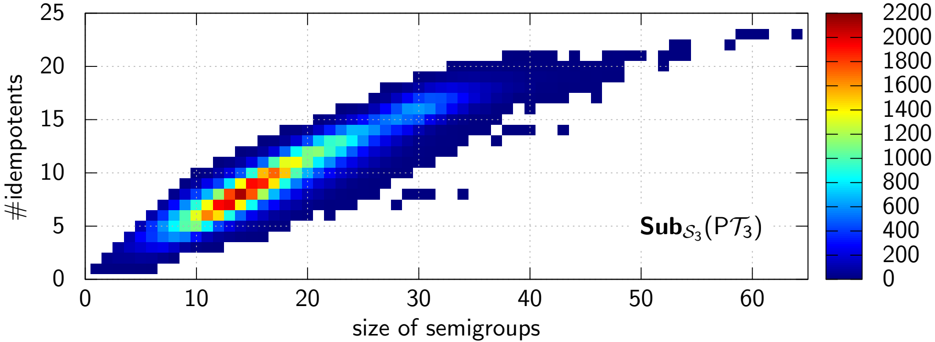


Figure: Size versus the number of idempotents for partial transformation semigroups up to degree 3.

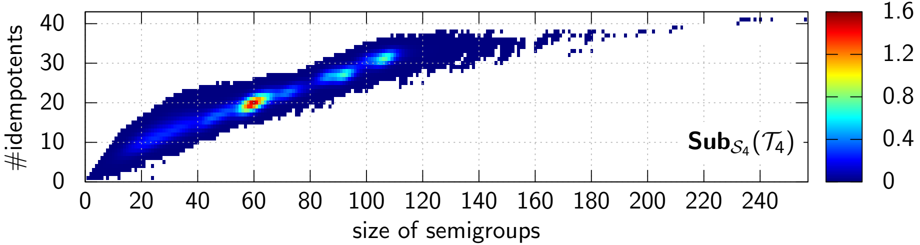


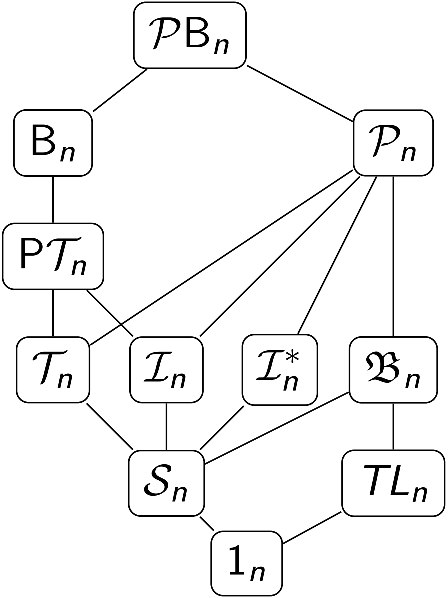
Figure: Size versus the number of idempotents for transformation semigroups up to degree 4. Frequency values in millions.

$\mathcal{P}_1 \hookrightarrow \mathcal{T}_2$
$\mathcal{P}_2 \hookrightarrow \mathcal{T}_5$

$\mathfrak{B}_1 \cong \mathcal{T}_1$
$\mathfrak{B}_2 \hookrightarrow \mathcal{T}_3$

$\mathsf{TL}_1 \cong \mathcal{T}_1$
$\mathsf{TL}_2 \hookrightarrow \mathcal{T}_2$
$\mathsf{TL}_3 \hookrightarrow \mathcal{T}_4$

$\mathcal{P}_1 \hookrightarrow \mathfrak{B}_2$



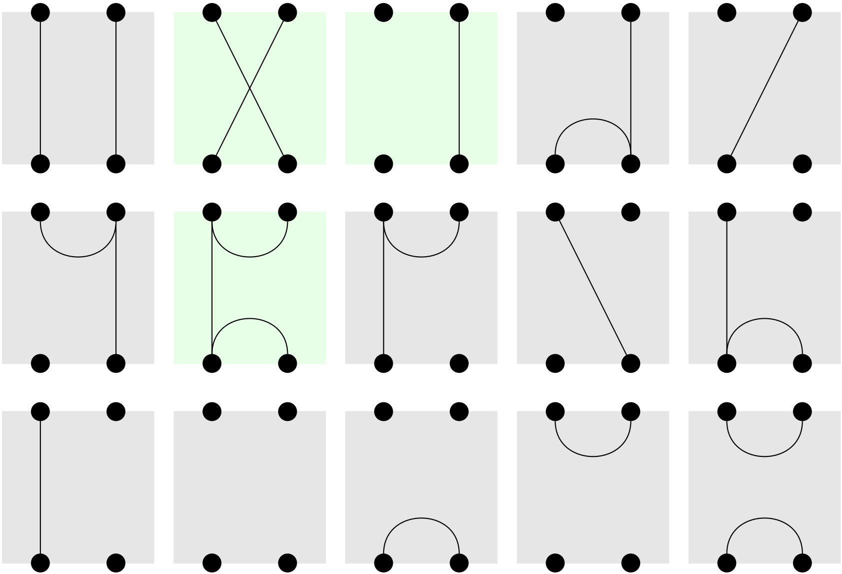
n-generated embeddings

$S \overset{n}{\hookrightarrow} T$ if S embeds into a subsemigroup of T that can be generated by n elements

A simple algorithm:

1. Find all n-generated subsemigroups of T.
2. Filter this set by the property that size is $\geq |S|$.
3. Try constructing embeddings.

$\mathcal{P}_2$, 3-generated



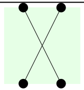
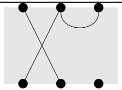
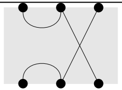
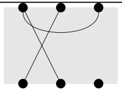
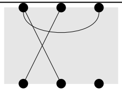
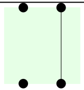
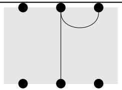
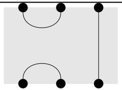
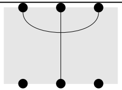
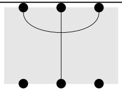
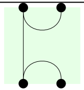
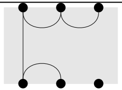
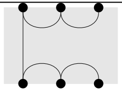
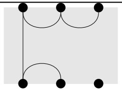
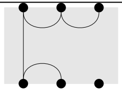
$\mathcal{P}_2 \overset{2}{\hookrightarrow} \mathcal{P}_3$?

The conjecture was: NO.

$\mathcal{P}_2 \overset{2}{\hookrightarrow} \mathcal{P}_3$?

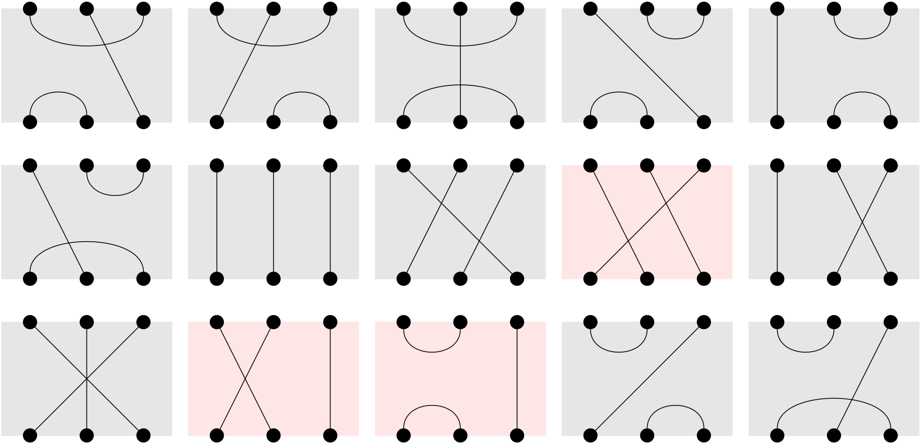
The conjecture was: NO.

But, there are 4, err... 3 distinct ways of this embedding.

Generators of $\mathcal{P}_2$	Solution 1	Solution 2	Solution 3	
				
				
				

Oooooh!!!!

$\mathfrak{B}_3$, the Brauer monoid of degree 3

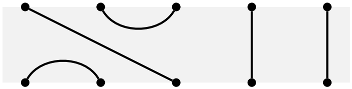


$\mathfrak{B}_3 \overset{2}{\hookrightarrow} \mathfrak{B}_5$? There are at most 21 distinct ways.

$\mathfrak{B}_3 \overset{2}{\hookrightarrow} \mathfrak{B}_4$?, $\mathfrak{B}_4 \overset{2}{\hookrightarrow} \mathfrak{B}_5$? Answers: NO. (as expected)

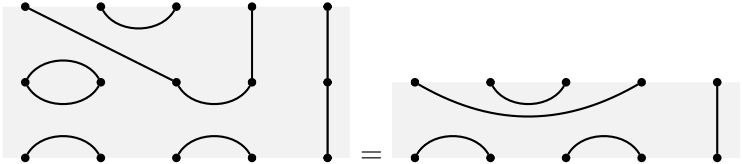
Temperley-Lieb, Jones monoid

Catalan numbers, sequences of well-formed parentheses.

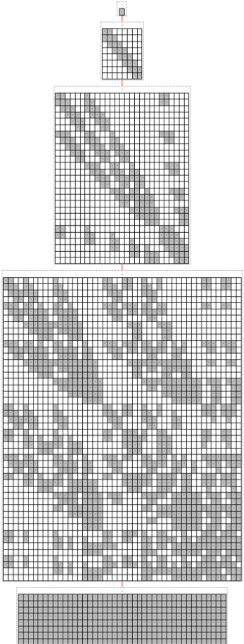


corresponds to (()(()))()

Applications in Physics: statistical mechanics, percolation problem.

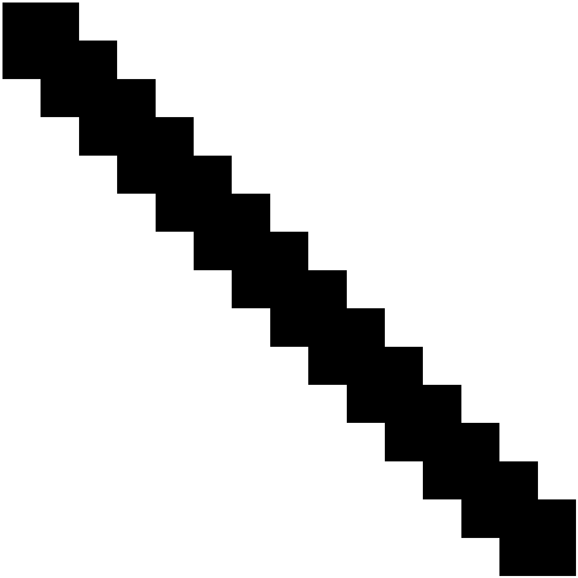


First glimpse through the usual eggbox diagrams, J_9

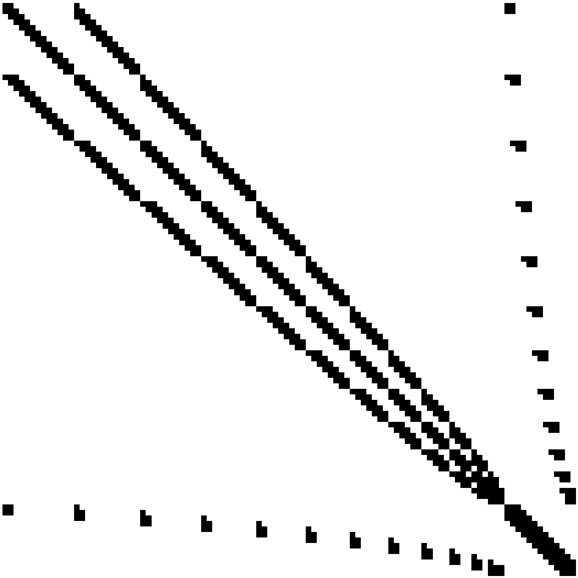


J_{16}, top level

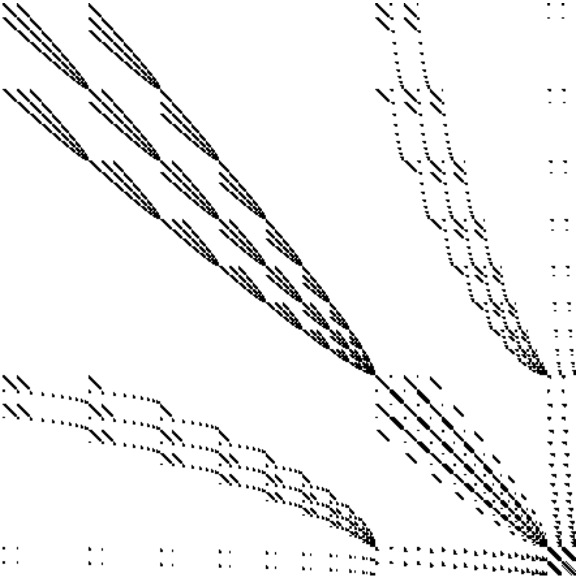
J_{16}, level 2



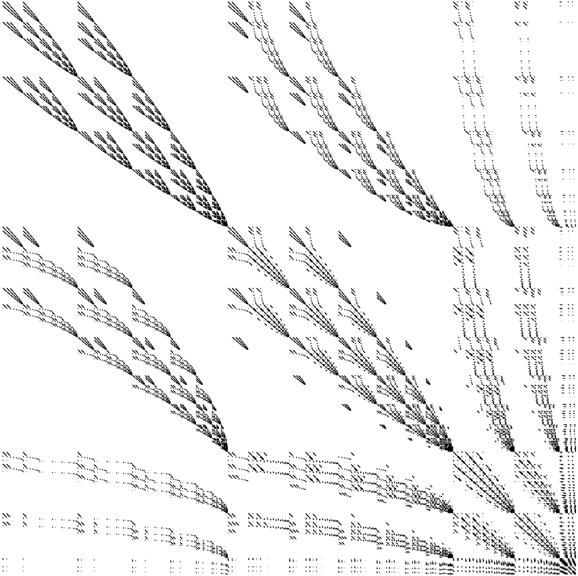
J_{16}, level 3



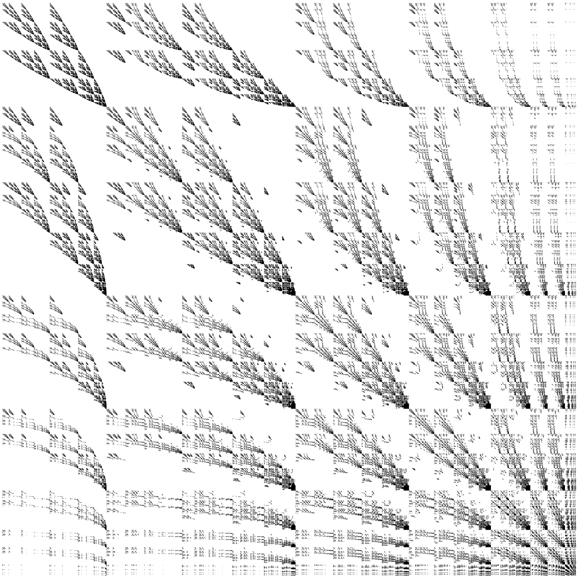
J_{16}, level 4



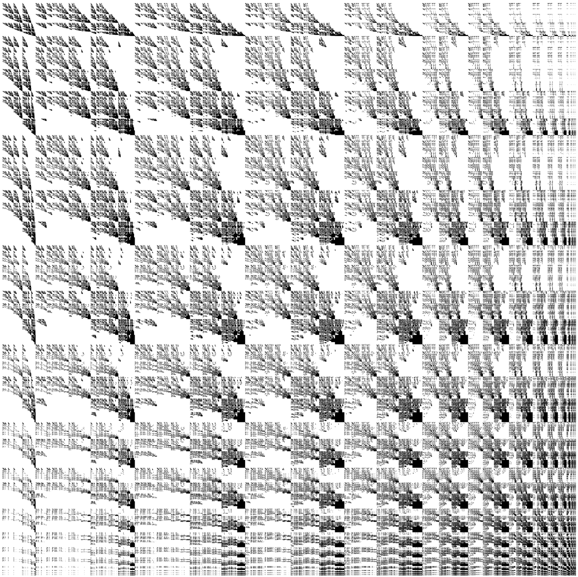
J_{16}, level 5



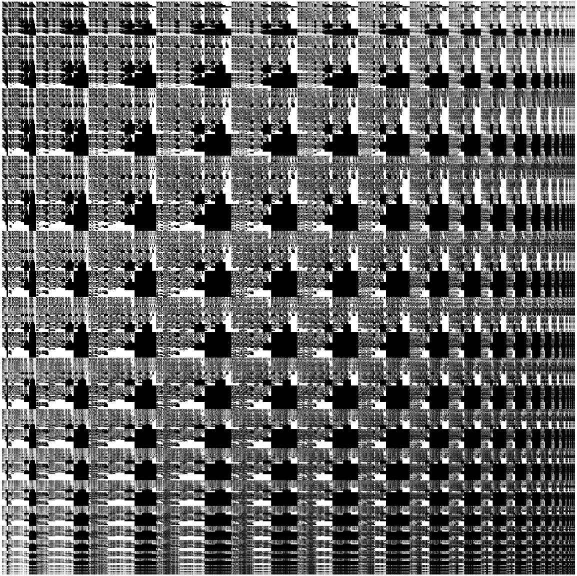
J_{16}, level 6



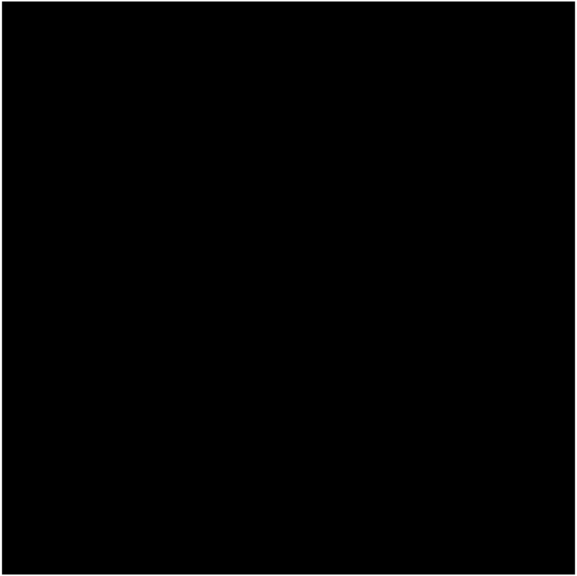
J_{16}, level 7



J_{16}, level 8



J_{16}, level 9



Summary

We produce huge amount of data. $\Longrightarrow$ Open problems pop up everywhere.

In case you can use your data or know how to answer the questions, please talk to us!

Links

GAP	www.gap-system.org
SEMIGROUPS	
www-groups.mcs.st-andrews.ac.uk/~jamesm/semigroups.php	
SUBSEMI	github.com/egri-nagy/subsemi
VIZ	bitbucket.org/james-d-mitchell/viz

Blog on computational semigroup theory:

compsemi.wordpress.com

Other software packages: GraphViz, Gnuplot, TEX

Thank You!