Reconstruction of omega-categorical structures from their endomorphism monoids

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Joint with Manuel Bodirsky, Michael Kompatscher and Michael Pinsker.

Non-reconstructibility

Fact

There exist separable profinite groups G_1, G_2 which are isomorphic as groups, but not as topological groups.

Theorem (DE + P. Hewitt, 1990)

There exist two countable, ω-categorical structures $\mathcal{M}_1, \mathcal{M}_2$ whose automorphism groups are isomorphic as groups, but not as topological groups.

Theorem (M. Bodirsky + DE + M. Kompatscher + M. Pinsker, '14)

There exist two countable, ω-categorical structures $\mathcal{M}_1, \mathcal{M}_2$ whose endomorphism monoids are isomorphic as monoids, but not as topological monoids.

- Can use the same $\mathcal{M}_1, \mathcal{M}_2$.
- Question asked by Lascar ('87); Bodirsky, Pinsker, Pongrácz ('14).

Endomorphisms

Relational structure with domain A: $\mathcal{A} = (A; (R_i : i \in I))$, where

$R_i \subseteq A^{n_i}$, $n_i \in \mathbb{N}$.

Endomorphism of $\mathcal{A}$: $\alpha : A \to A$, $\alpha(R_i) \subseteq R_i$ for all $i \in I$.

$\mathrm{End}(\mathcal{A})$: monoid of endomorphims of $\mathcal{A}$.

CAVEAT: Sensitive to the language (ie. choice of the atomic relations R_i).

$\mathrm{Aut}(\mathcal{A})$: group of units in $\mathrm{End}(\mathcal{A})$.

Topological monoid: $\mathrm{End}(\mathcal{A}) \subseteq A^A$.

Translations

Closed subgroups of $\mathrm{Sym}(A)$ $\leftrightarrow$ $\mathrm{Aut}(\mathcal{A})$, $\mathcal{A}$ relational structure with domain A.

Closed submonoids of A^A $\leftrightarrow$ $\mathrm{End}(\mathcal{A})$, $\mathcal{A}$ relational structure with domain A.

Suppose A is countable:

Closed oligomorphic subgps of $\mathrm{Sym}(A)$ $\leftrightarrow$ $\mathrm{Aut}(\mathcal{A})$, $\mathcal{A}$ ω-categorical.

Oligomorphic: finitely many orbits on A^n, for all $n \in \mathbb{N}$.

Closed submonoids of A^A with oligomorphic unit group $\leftrightarrow$ $\mathrm{End}(\mathcal{A})$, $\mathcal{A}$ ω-categorical.

If $\mathcal{A}$ is ω-categorical the closure of $\mathrm{Aut}(\mathcal{A})$ in $\mathrm{End}(\mathcal{A})$ is the monoid $\mathrm{EEmb}(\mathcal{A})$ of elementary embeddings $\mathcal{A} \to \mathcal{A}$.

Reconstruction questions

Suppose $\mathcal{A}_1, \mathcal{A}_2$ are countable, ω-categorical structures.
Suppose X denotes Aut, End or EEmb.

Suppose $X(\mathcal{A}_1)$ and $X(\mathcal{A}_2)$ are isomorphic as algebraic objects. How are $\mathcal{A}_1$ and $\mathcal{A}_2$ related?

REMARK: If $\mathrm{Aut}(\mathcal{A}_1)$ and $\mathrm{Aut}(\mathcal{A}_2)$ are isomorphic as topological groups, then $\mathcal{A}_1, \mathcal{A}_2$ are biinterpretable.

Failure of automatic continuity

Theorem (Bodirsky, Pinsker, Pongrácz, 2014)

Let $\mathcal{A}$ be countable ω-categorical. Then there is a monoid homomorphism $\xi : \mathrm{EEmb}(\mathcal{A}) \to A^A$ which is not continuous.

Lascar's Theorem

DEFINITION: (1) If S is a topological group, denote by S° the intersection of the closed subgroups of finite index in S.
(2) A countable, ω-categorical structure $\mathcal{A}$ is *G-finite* if for every open subgroup $U \leq \mathrm{Aut}(\mathcal{A})$ the subgroup U° is of finite index in U.

Theorem (Lascar, 1980's)

Suppose $\mathcal{A}_1, \mathcal{A}_2$ are countable, G-finite, ω-categorical structures and $\alpha : \mathrm{EEmb}(\mathcal{A}_1) \to \mathrm{EEmb}(\mathcal{A}_2)$ is an isomorphism of monoids. Then the restriction of α to $\mathrm{Aut}(\mathcal{A}_1)$ is a topological isomorphism between $\mathrm{Aut}(\mathcal{A}_1)$ and $\mathrm{Aut}(\mathcal{A}_2)$. In particular, $\mathcal{A}_1$ and $\mathcal{A}_2$ are biinterpretable.

Start of proof: For $e, f \in \mathrm{EEmb}(\mathcal{A}_1)$, write $e \leq f$ iff there is $k \in \mathrm{EEmb}(\mathcal{A}_1)$ with $e = fk$. Note that this is preserved by α and $e \leq f$ iff $\mathrm{im}(e) \subseteq \mathrm{im}(f)$. So we can recover the poset of elementary submodels of $\mathcal{A}_1$ from the algebraic structure of $\mathrm{EEmb}(\mathcal{A}_1)$. ...

QUESTION: Can we recover $\mathrm{EEmb}(\mathcal{A})$ from the algebraic structure of $\mathrm{End}(\mathcal{A})$ (for ω-categorical $\mathcal{A}$)?

Profinite quotients

Any separable profinite group K embeds as a closed subgroup of $\Pi_{n\in\mathbb{N}}\mathrm{Sym}(n)$.

Fact (Cherlin - Hrushovski)

There is a countable, ω-categorical structure $\mathcal{A}$ and a continuous surjection $\theta : \mathrm{Aut}(\mathcal{A}) \to \Pi_{n\in\mathbb{N}}\mathrm{Sym}(n)$ with kernel $\Phi = (\mathrm{Aut}(\mathcal{A}))^\circ$.

So if $K \leq \Pi_{n\in\mathbb{N}}\mathrm{Sym}(n)$ is closed, then $\Sigma_K = \theta^{-1}(K)$ is a closed, oligomorphic group, $\Sigma_K^\circ = \Phi$, and $\Sigma_K/\Phi \cong K$.

REMARK: If $K_1, K_2 \leq \Pi_{n\in\mathbb{N}}\mathrm{Sym}(n)$ are closed and algebraically isomorphic, there does not seem to be any reason to expect that Σ_{K_1} and Σ_{K_2} should be algebraically isomorphic.

Examples for non-reconstructibility

Fact

There is a separable profinite group G with the following properties:

- *G has a finite, central subgroup $F \neq 1$ such that F has a complement in G and any such complement is dense in G.*
- *G is nilpotent of class 2 and the derived subgroup $G^{(1)}$ is a proper, dense subgroup of the centre $Z(G)$.*

From the first point, there is a subgroup $E \leq G$ with $G = F \times E$, and any such E is dense in G.

If $H = G/F$, then H is algebraically isomorphic to E, but not topologically.

Thus $K = F \times H$ and G are profinite groups which are isomorphic as groups.

Note that $Z(K) = F \times Z(H)$ and $K^{(1)} = 1 \times H^{(1)}$, so the derived group of K is not dense in its centre. So G, K are not topologically isomorphic.

From H to G

Consider $G \stackrel{\pi}{\to} H = G/F$ and $\eta : H \to E$ given by $(\pi|E)^{-1}$ (discontinuous).

G has a base $(G_i : i \leq \omega)$ of open neighbourhoods of 1 where $G_i \trianglelefteq G$ and $\bigcap_{i<\omega} G_i = F$.

Let $H_i = \pi(G_i)$ for $i < \omega$ and $H_\omega = \pi(G_\omega \cap E)$.

Let $X = \coprod_{i<\omega} H/H_i$ and $C = H/H_\omega$.

The action of H on X gives a continuous embedding $H \to \mathrm{Sym}(X)$.

The action of H on $X \cup C$ gives an embedding $H \to \mathrm{Sym}(X \cup C)$ which is not continuous. The closure of the image is topologically isomorphic to G.

PROOF: Identify X with $\coprod_{i<\omega} G/G_i$ and C with G/G_ω via $\alpha : H/H_\omega \to G/G_\omega$ where $\alpha(hH_\omega) = \eta(h)G_\omega$. This is a bijection and $\eta(h)\alpha(kH_\omega) = \alpha(hkH_\omega)$.

From Σ_H to Γ

- Find $\mathcal{A}$ countable, ω-categorical, $\Sigma = \mathrm{Aut}(\mathcal{A})$, with a continuous surjection $\nu : \Sigma \to H$ with kernel $\Phi = \Sigma^\circ$.
- Let $\Psi = \nu^{-1}(H_\omega)$; identify $C = H/H_\omega$ with Σ/Ψ.
- Let $B = A \cup C$ with $i : \Sigma \to \mathrm{Sym}(B)$ the resulting action.
- Let Γ be the closure of $i(\Sigma)$ in $\mathrm{Sym}(B)$.

Lemma

1. Γ *is oligomorphic on B;*
2. $\Gamma = i(\Sigma) \times \Gamma_A$ *and* $\Gamma_A \cong F$;
3. $\Gamma^\circ = i(\Phi)$;
4. Γ/Γ° *is topologically isomorphic to G.*

Conclusion - for automorphism groups

- There is an ω-categorical structure $\mathcal{M}_1$ with domain B and automorphism group Γ.
- There is an ω-categorical structure $\mathcal{M}_2$ with domain B and automorphism group $\Delta = \Sigma \times F$ (topological product).

Theorem

$\mathrm{Aut}(\mathcal{M}_1)$ and $\mathrm{Aut}(\mathcal{M}_2)$ are isomorphic as groups, but not as topological groups.

PROOF: The groups are both isomorphic to $\Sigma \times F$.
Suppose $\beta : \Gamma \to \Delta$ is an isomorphism of topological groups. Then $\beta(\Gamma^\circ) = \Delta^\circ$ and so we have a topological isomorphism $\Gamma/\Gamma^\circ \to \Delta/\Delta^\circ$. But $\Gamma/\Gamma^\circ \cong G$ and $\Delta/\Delta^\circ \cong F \times H$ (topologically). Contradiction. $\square$

Endomorphism monoids

- Canonical language for $\mathcal{A}$: atomic relation for each $\mathrm{Aut}(\mathcal{A})$-invariant subset of A^n (all n).
- Let $\Lambda = \mathrm{End}(\mathcal{A}) = \mathrm{EEmb}(\mathcal{A}) = \bar{\Sigma} \subseteq A^A$.
- $\nu : \Sigma \to H$ extends to a continuous monoid homomorphism $\mu : \Lambda \to H$.
- Λ acts on $C = H/H_\omega = G/G_\omega$ by $f(hH_\omega) = \mu(f)hH_\omega$.
- Obtain embedding $j : \Lambda \to B^B$ (where $B = A \cup C$) extending i.
- Let Ω be the closure of $j(\Lambda)$ in B^B.

Lemma

1. *$\Omega = j(\Lambda) \times \Omega_A$ and $\Omega_A = \Gamma_A$.*
2. *The group of units in Ω is Γ.*

Conclusion - for endomorphism monoids

- Assume $\mathcal{M}_1, \mathcal{M}_2$ have their canonical languages.
- $\Gamma = \mathrm{Aut}(\mathcal{M}_1)$ and $\Omega = \mathrm{End}(\mathcal{M}_1)$.
- $\mathrm{End}(\mathcal{M}_2)$ is isomorphic to the topological product $\Lambda \times F$.
- Both $\mathcal{M}_1, \mathcal{M}_2$ are countable, ω-categorical.

Theorem

$\mathrm{End}(\mathcal{M}_1)$ *and* $\mathrm{End}(\mathcal{M}_2)$ *are isomorphic as monoids, but not as topological monoids.*

PROOF: The monoids are isomorphic to $\Lambda \times F$. A topological isomorphism between them would induce a topological isomorphism between their groups of units, Γ and Δ, which is impossible. $\square$