Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH1031-WE01

Title:
Discrete Mathematics

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for the best **FOUR** answers. All questions carry the same marks.

Revision:	

ED01/2017

1. (i) Squares of a three by three grid are filled with three letters A, B or C, with the letters always arranged the correct way up. For example,

$$\begin{matrix} A & B & C \\ A & B & C \\ A & B & C \end{matrix} \quad \text{and} \quad \begin{matrix} A & A & A \\ B & B & B \\ C & C & C \end{matrix}$$

are valid but different configurations.

(a) Find the total number of configurations of the three by three grid.

(b) Find the number of configurations such that the grid contains only 2 A's.

(c) Find the number of configurations with only 2 A's such that the two A's are not vertically or horizontally adjacent to each other. (The A's **are** allowed to be diagonal adjacent to each other.)

(d) Find the number of configurations containing at least one C.

(ii) Solve the recurrence relation $a_0 = 1, a_1 = 2$ and

$$a_n = 8a_{n-1} - 12a_{n-2} \quad \text{for } n \geq 2.$$

2. (i) (a) State the pigeon hole principle.

(b) Show that in any set X of people, there are two members of X who have the same number of friends in X. [It is assumed that $|X|$ is at least 2 and if x is a friend of x', then x' is a friend of x.]

(ii) (a) A pixie can climb stairs one, two or three steps at a time. Find a recurrence relation for the number of distinct ways the pixie can climb a flight of n stairs.

(b) Find a closed form for the generating function for this recurrence relation [please do **not** find the coefficients of the generating function].

3. (i) (a) Write down a formula that expresses the Inclusion-Exclusion principle, being careful to define any notation you use.

(b) Recall that a derangement of a word is an arrangement of the letters in which no letter is in the correct position. Find the number of derangements of `DURHAM`.

(ii) Let d_n denote the number of derangments of a word of length n. Prove that d_n satisfies the following formula

$$d_1 = 0, \quad d_2 = 1, \quad d_n = (n-1)(d_{n-1} + d_{n-2}) \quad (n \geq 3).$$

(*Hint:* **Either** *state and prove a general formula for d_n first and show that it satisfies the recurrence relation* **or** *explain the recurrence relation directly.*)

4. (i) A bank robber holds up a bank and demands n pounds in one pound and two pound coins. He also demands to know the number of ways the cashier can give him the money.

   (a) Let a_n be the number of ways of making n pounds from one pound and two pound coins. Write down a generating function for the sequence a_n.

   (b) Find the number of ways the bank robber can get 50 pounds. (*Hint:* $1 - x^2 = (1 - x)(1 + x)$)

   (ii) Evaluate the sum $\sum_{k=0}^{\infty} \frac{k^2}{2^k}$.

5. (a) Define the complete graph K_n, and the complete bipartite graph $K_{m,n}$. How many vertices does each graph have, and how many edges?

   (b) Define the n-cube graph, Q_n, referring to the usual labels for the vertices. How many vertices does it have, and how many edges?

   The *girth* g of a graph G is the smallest length of a cycle in G. If G has no cycles we say $g = 0$.

   (c) What is the girth of K_n? of $K_{m,n}$? of Q_n? Does this depend on n and m?

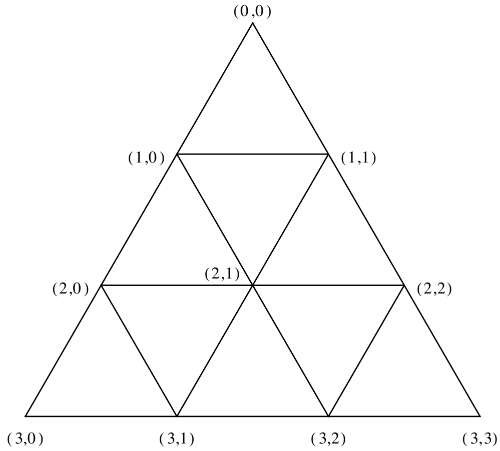
   (d) Write down Euler's formula for planar graphs, and also the two "handshaking" lemmas. Using these, show that if $G = (V, E)$ is a simple, connected, planar graph with $g \neq 0$, then $|E| \leq \frac{g}{g-2}(|V| - 2)$. (You may assume that the boundary of any face contains a cycle.)

   (e) For those values of n and m giving $g \neq 0$, use the inequality you proved in (d) to find out:
   for which values of n could K_n be planar?
   for which values of n and m could $K_{n,m}$ be planar?
   for which values of n could Q_n be planar?

   (f) By drawing planar representations, show that these graphs *are* in fact planar for the values of n and m you found in (e).

   (g) Draw a simple, connected graph, with $g \neq 0$ and $|E| \leq \frac{g}{g-2}(|V| - 2)$, but which is *not* planar.

6. Consider the graph T_3 shown. It is one of a family of graphs $T_n = (V_n, E_n)$, each a pattern of triangles with n layers, and vertices labelled as shown.



(a) Find $|V_n|$ and $|E_n|$. (You may use the fact that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.)

(b) We have a planar representation of each T_n, so we can also let F_n be the set of its faces. Make a conjecture for $|F_n|$, and prove it by induction. (*Hint:* $|F_1| = 2$)

(c) Confirm that your answers for (a) and (b) satisfy Euler's formula for planar graphs.

(d) Draw the following subgraphs of T_3. In each case the vertex set is still V_3.
G_1 is T_3 with edge $(1,0)(1,1)$ removed.
G_2 is T_3 with edges $(2,1)(1,1)$, $(2,1)(2,0)$ and $(2,1)(3,2)$ removed.
G_3 is T_3 with the edges of the cycle $(1,0),(1,1),(2,2),(3,2),(3,1),(2,0),(1,0)$ removed.

Recall the definitions of an Euler circuit and a Hamilton cycle.
Similarly, an *Euler trail* is a trail which uses each edge of the graph just once, and a *Hamilton path* is a path which visits each vertex just once. (Note that an Euler trail, or a Hamilton path, must run from u to v where $u \neq v$.)

(e) Which of the G_i have a Hamilton cycle or a Hamilton path?
Give the cycle if possible, otherwise the path, or if it has neither, try to explain briefly how you know this.

(f) Which of the G_i have an Euler circuit or an Euler trail?
(No need to explain, or to show the circuit or trail.)

(g) Write down Euler's theorem relating Euler circuits to the degrees of vertices in a graph.

(h) Write down a similar result for Euler trails, and prove it. You may use the theorem in (g).