Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH1041-WE01

Title:
Programming & Dynamics I

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.

Revision:	

ED01/2017

1. (i) The coordinates of a point P are given in two frames by

$$\begin{aligned} \mathbf{r} = \overrightarrow{OP} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ \mathbf{r}' = \overrightarrow{O'P} &= x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}' \end{aligned}$$
$$,$$

where

$$\begin{pmatrix} \mathbf{i}' \\ \mathbf{j}' \\ \mathbf{k}' \end{pmatrix} = R \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix},$$

and the vector $\overrightarrow{OO'} = \cos(t)\mathbf{i} - \cos(t)\mathbf{j} + \sin(t)\mathbf{k}$. Show that the matrix R is orthogonal. Write down the relation between the coordinates (x, y, z) and (x', y', z'). Use this relation to show that a point P with coordinates $(x', y', z') = (0, 1, -\cos(t))$ in the second frame describes a circle in the first frame whose radius you should determine.

(ii) A particle of mass 4 moving in two dimensions has a position given in terms of its polar coordinates (r, θ) in an inertial frame by $r(t) = \cos(\theta(t))$, where $\theta(t) = t^2$. Find the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of the particle expressed in the basis $\mathbf{e}_r$, $\mathbf{e}_\theta$ and calculate its speed and the magnitude of the force acting on it.

(iii) A bead of mass 2 moves horizontally along a straight wire, and is initially at rest at the origin O. It is subjected to an accelerating force

$$F = \frac{1}{(1+x)^2},$$

where x measures its displacement from O along the wire until it reaches the point $x = 3$. At this point the force is turned off, but for $x > 3$ the wire is covered in a sticky substance which produces a braking force on the bead equal in magnitude to the beads velocity v. Calculate the distance X from the origin at which the bead finally stops.

2. (i) A particle of unit mass moves in a potential

$$V(x) = \ln(1 + x^2) + \frac{\alpha}{1 + x^2},$$

where $\alpha > 1$ is a constant.

(a) Find the two stable equilibria $x = x_0$ and $x = x_1$ where $x_0 < x_1$, and sketch the function $V(x)$.

(b) At $t = 0$ the particle is fired from $x = x_0$ with velocity v_0. For what values of v_0 will the particle subsequently pass the point $x = x_1$?

(c) Determine the periods of small oscillations about each stable equilibrium. For what value of α is the period a minimum?

(ii) Write down the relation between a conservative force $\mathbf{F}$ in three dimensions and its corresponding potential $V(x, y, z)$. Find the potential in the case that the force is

$$\mathbf{F} = e^{xy}\left(z\mathbf{i} + \frac{z(xy-1)}{y^2}\mathbf{j} + \frac{1}{y}\mathbf{k}\right).$$

3. An attacking army uses a gun to fire at a city which lies a horizontal distance L from the gun.

(a) If the gun fires its missile accurately at an angle β to the horizontal, express L in terms of the speed s with which the projectile leaves the gun, g the gravitational constant and the angle β.

(b) The defending army has placed an anti-ballistic gun in between the attacking army and the city at a distance l from the city. As the defending gunners are endowed with instantaneous reactions, they fire a defensive projectile towards the attacking army at the same instant that the attackers fire and also at an angle β to the horizontal, so as to hit the incoming missile. Find the speed s' of the defensive projectile as it leaves the anti-ballistic gun.

(c) Unfortunately for the defenders, a small gust of cross-wind at the time the projectiles should have collided results in the two missiles missing one another by a hair's breadth. However in a last ditch attempt to save the city, the defenders at this instant launch a second missile vertically upwards, with the same speed s' as the first shot. Miraculously on this occasion the two missiles collide. Find the ratio l/L in terms of β.

4. A particle of mass m moves in two-dimensions in a potential

$$V(r) = \frac{1}{2}\left(\frac{\alpha}{r^2} + \beta r^2\right)$$

where r is radial distance from the origin in polar coordinates, and α and β are positive constants.

(a) Write down the force corresponding to this potential in terms of two-dimensional polar coordinates. Show that angular momentum L will be conserved for this force. If the particle moves in a circle of radius $r = R$, find R in terms of m, α, β and the particle's angular momentum L.

(b) Show that if $u = 1/r$ then the energy of the particle can be written in the form

$$E = \frac{\tilde{m}}{2}\left(\frac{du}{d\theta}\right)^2 + \tilde{V}$$

where $\tilde{m} = L^2/m$ and $\tilde{V} = \tilde{m}u^2/2 + V(1/u)$.

(c) Find the form for an approximate circular orbit by considering $u = 1/R + \epsilon(\theta)$ where $\epsilon \ll 1$ and R is the solution to part (a) and solving approximately for $\epsilon(\theta)$. Sketch the orbit in the case that $\tilde{m} = 1$, $\alpha = 3$.