Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH1051-WE01

Title:

Analysis I

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.

Revision:	

ED01/2017

1. Find the limits of the following sequences as $n \to \infty$, or show that the limit does not exist. State explicitly every result that you use.

   (i)
   $$x_n = \frac{n^2 + e^{-n}}{5n^3},$$

   (ii)
   $$x_n = \frac{(n!)^2}{(n-1)!(n+1)!},$$

   (iii)
   $$x_n = n^{1+\frac{1}{n}}(n - \sqrt{n^2+5})$$

2. (i) Find all real values of x for which $|2x+1| \leq |3x-6|$.

   (ii) Using the standard triangle inequality $|x+y| \leq |x| + |y|$, derive the following inequality for any pair of real numbers $a, b \in \mathbb{R}$:
   $$|\,|a| - |b|\,| \leq |a-b|.$$

3. (a) Give the (ϵ, δ)-definition of continuity of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$.

   (b) Using quantifiers, give the precise logical (ϵ, δ)-formulation that $f : \mathbb{R} \to \mathbb{R}$ is not continuous at a point $x \in \mathbb{R}$.

   (c) Using the (ϵ, δ)-definition, show for any pair of functions $f, g : \mathbb{R} \to \mathbb{R}$: If f is continuous at x and g is continuous at $f(x)$, then $g \circ f$ is also continuous at x.

4. (i) Let $g : [0, \sqrt{2\pi}] \to \mathbb{R}$, $g(x) = \sin(x^2)$. Calculate explicitly $g^{-1}([0,1])$.

   (ii) Let $f : [1, \infty) \to \mathbb{R}$ be defined as
   $$f(x) = \frac{3x+1}{x(2+\cos(x))}.$$

   Find $\inf_{x \in [1,\infty)} f(x)$.

5. (a) State the Completeness Axiom of the real numbers.

   (b) Let (x_n) be a bounded, monotone increasing sequence of real numbers. Using the Completeness Axiom of the real numbers, show that (x_n) is convergent. *Hint:* Show first that the set $\{x_n \mid n \in \mathbb{N}\}$ has a supremum and then show that (x_n) must converge to this supremum.

   (c) Use (b) to show that the sequence (a_n), defined by

$$a_1 = 0, \quad a_{n+1} = a_n + \frac{\sin^2(n)}{2^n} \text{ for all } n \in \mathbb{N},$$

   is convergent. *Hint:* Use the geometric series and its limit

$$\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$$

   for $|c| < 1$.

6. Determine whether or not the following series converge:

   (a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{n^5+2}\right)^{1/3},$

   (b) $\displaystyle\sum_{n=2}^{\infty} \left(n\sqrt{\log n}\right)^{-1}.$

7. (i) Discuss the convergence of the series $\Sigma_{n=1}^{\infty} [\pi n \cos(\pi n)]^{-1}$.

   (ii) Compute the radius of convergence of the power series $\Sigma_{n=1}^{\infty} a_n z^n$, where $a_n = (4n)!\,(4n)^{-2n}/(n!)^2$.

8. (a) Determine whether or not the following integral converges:

$$\int_0^1 \sqrt{\frac{x^2+1}{x}}\,dx.$$

   (b) Let $f(x) = x^p/\sqrt{x^5+1}$ for $x > 0$. Determine all the values of the real constant p for which the integral $\int_1^{\infty} f(x)\,dx$ converges.

9. (a) Let (x_n) be a real sequence. Prove that if the series $\sum_{n=1}^{\infty} |x_n|$ converges, then $\sum_{n=1}^{\infty} x_n$ converges as well.

   (b) Define a sequence of functions (f_n) on the interval $[0,1]$ by

$$f_n(x) = x^{2n}/(1 + x^{2n}).$$

   Find the pointwise limit $f(x)$ as $n \to \infty$, and determine whether or not this limit is uniform.

10. (a) Let (f_n) be a sequence of continuous real-valued functions on the interval $[0,1]$, such that $f_n \to f$ as $n \to \infty$, uniformly on $[0,1]$. Prove that $\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx$ as $n \to \infty$.

   (b) Determine whether the series

$$\sum_{n=1}^{\infty} n^{-1} x^n (1 - x)$$

   is uniformly convergent on the interval $[0,1]$. (Hint: first compute the maximum of the function $f_n(x) = x^n(1-x)$ on $[0,1]$.)