Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH1061-WE02

Title:

Calculus & Probability I paper 2: Probability

Time Allowed:	1 hour 30 minutes	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.

Revision:	

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1. (a) A poker hand consists of 5 cards. A *flush* is a hand for which all cards are the same suit, but not of consecutive denominations (where Ace can be high or low). For example, 2, 3, 4, 5, 7 of Hearts is a flush, but 2, 3, 4, 5, 6 of Hearts is not a flush. Find the probability that a poker hand from a well-shuffled deck is a flush.

   (b) Anne and Barney are playing poker. On each hand, Anne has a 20% chance of bluffing and Barney has a 30% chance of bluffing; the two players bluff independently. What is the probability that Anne is bluffing, given that at least one player is bluffing?

   The probability of being dealt a full house in a poker game is 0.0014. Suppose that, during an evening, you play 100 hands of poker.

   (c) Use Markov's inequality to give an upper bound on the probability that you are dealt more than one full house during the evening.

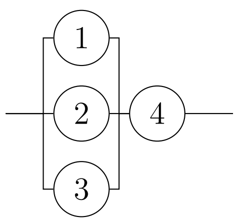
   (d) Explain how a binomial random variable with parameters n and p may be approximated when n is large and p is small. Use this approximation to find an expression for the probability that you are dealt more than one full house during the evening.

2. (i) In my chest of drawers I have three drawers containing socks. One of the drawers contains six black socks and four red socks. Each of the other two drawers contains two black socks and three red socks.

   I pick a drawer at random from the three possibilities (with equal chance of each), and then select two socks at random from those in the drawer (without replacement).

   What is the probability that I end up with a matching pair of socks?

   (ii) Consider the following reliability network:



   Assume that components fail independently, with

$$\mathbb{P}(\text{component } i \text{ works}) = p_i.$$

   Find, as a function of p_1, p_2, p_3, p_4, the (conditional) probability that component 1 works, given that the system works.

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3. Random variable X takes values 1 and -1 with probability $1/2$ of each, and random variable Y also takes values 1 and -1 with probability $1/2$ of each. Suppose that X and Y are independent. Define $Z = XY$, $U = X + Z$ and $V = Y + Z$.

   (a) Show that X and Z are independent. Are the variables X, Y, Z mutually independent?

   (b) Calculate $\mathbb{E}(Z)$ and $\mathbb{C}\mathrm{ov}(X, Z)$, and hence find $\mathbb{E}(U)$ and $\mathbb{V}\mathrm{ar}(U)$.

   (c) Write down the joint probability distribution of U and V.

   (d) Write down the conditional distribution of U given each value of V.

   (e) Confirm in this example that $\mathbb{E}(\mathbb{E}(U \mid V)) = \mathbb{E}(U)$.

4. (a) Define the moment generating function $M_X(t)$ for a random variable X. Show that if $Z \sim \mathcal{N}(0, 1)$ has the standard normal distribution then

   $$M_Z(t) = \mathrm{e}^{t^2/2}, \ t \in \mathbb{R}.$$

   You may use the fact that the probability density function of the standard normal distribution is

   $$f_Z(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}, \ x \in \mathbb{R}.$$

   (b) Deduce from (a) that if $Z \sim \mathcal{N}(0, 1)$, then $\mathbb{E}(Z) = 0$ and $\mathbb{V}\mathrm{ar}(Z) = 1$.

   (c) Deduce from (a) an expression for $M_X(t)$ where $X \sim \mathcal{N}(\mu, \sigma^2)$.

   (d) Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables with $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ for each $i \in \{1, 2, \ldots, n\}$. Let $Y = \sum_{i=1}^n X_i$. What is the distribution of Y? In your explanation, any standard results of moment generating functions that you need should be clearly stated.

5. In a certain game, three fair coins are tossed. If all coins are heads, you gain one point. If all coins are tails, you lose one point. Otherwise, your points total is unchanged. The game is repeated, independently, 400 times. You start with 0 points, and the points total at the end of the 400 games is denoted by S.

   (a) Use Chebyshev's inequality to find a value of $a > 0$ to ensure that

   $$\mathbb{P}(-a < S < a) > 0.95.$$

   (b) Using a suitable approximation, find a value of $a > 0$ such that

   $$\mathbb{P}(-a < S < a) \approx 0.95.$$

   You may use the following values for the inverse of the standard normal distribution function:

x	0.90	0.95	0.975	0.99
$\Phi^{-1}(x)$	1.28	1.64	1.96	2.33

   (c) Suppose that it is found that the games are not independent, but that the covariance between the scores on any two different games is $1/399$. Repeat part (a) under these circumstances.

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