EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH1071-WE01

Title: Linear Algebra I

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Use a separate answer book for each Section.

Revision:	

ED01/2017

SECTION A

Use a separate answer book for this Section.

1. (i) Give all solutions to the following system of linear equations, and discuss how the solution sets depend on t:

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & - & x_3 & + & 3x_4 & = & t, \\ -2x_1 & + & 4x_2 & + & 5x_3 & - & 5x_4 & = & 3, \\ 3x_1 & - & 6x_2 & - & 6x_3 & + & 8x_4 & = & -5. \end{array}$$

(ii) Let A be an arbitrary invertible upper triangular (3×3)-matrix.
Let B be the cofactor matrix of A, and C the cofactor matrix of B.
Show that $C = \det(A)A$.

2. (i) Find the inverse of

$$A = \begin{pmatrix} 0 & 1 & 0 & -2 \\ 1 & -1 & 1 & 2 \\ 0 & -2 & 3 & 3 \\ 2 & 2 & -3 & -2 \end{pmatrix}.$$

(ii) Give the orthogonal complement in $\mathbb{R}^4$ of the row space of

$$B = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & -1 & -2 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

[Carefully state any result that you are using.]

3. Let U and U' be vector subspaces of a vector space V.

(i) Define the sum $U + U'$ of U and U'. Show that $U + U'$ is itself a vector space.

(ii) Write down an equation relating $\dim U$, $\dim U'$, $\dim(U \cap U')$ and $\dim(U + U')$.

(iii) Determine the dimensions of U, U', $U \cap U'$ and $U + U'$ where U and U' are the subspaces of $\mathbb{R}^3$ given by

$$U = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} \right\}, \qquad U' = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right\}.$$

4. (i) Let A be a skew-symmetric $(n \times n)$-matrix.
For n odd, show that the determinant of A must be zero.
Show also that, for $n = 4$, the determinant of A is a non-negative real number.

(ii) Sketch the quadrilateral in $\mathbb{R}^2$ with the following vertices, determine if it is a parallelogram or not, and find its area: $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right)$.

5. For $\mathbf{n} = \left(\begin{smallmatrix} -1 \\ 1 \\ 2 \end{smallmatrix}\right)$, define the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n}.$$

(a) Show that T is linear.

(b) Compute the matrix of T with respect to the standard basis of $\mathbb{R}^3$.

(c) Show that $T^2 = T$.

(d) Show that $\ker(T) \cap \operatorname{im}(T) = \{\mathbf{0}\}$. What are the rank and nullity of T?

SECTION B

Use a separate answer book for this Section.

6. Find a solution $x(t), y(t)$ of the system of differential equations

$$\begin{cases} \dot{x}(t) = 7x(t) - 6y(t)\,, \\ \dot{y}(t) = -6x(t) - 2y(t)\,, \end{cases}$$

satisfying the initial condition $x(0) = -1$, $y(0) = 3$.

7. Let V be the vector space $\mathbb{R}[x]_2$ of real polynomials of degree at most two and let $\mathcal{L} : V \mapsto V$ be the linear operator

$$\mathcal{L}\left(p(x)\right) = -p(x+1) + p(x) + \frac{6}{x}\int_0^x p(y)\,dy$$

with $p(x) \in \mathbb{R}[x]_2$. Write down the matrix M, representing $\mathcal{L}$ on V, using the standard basis $\{1, x, x^2\}$, and then find the eigenvalues, the eigenvectors, and the eigenfunctions.

8. (i) Determine whether or not each of the following is an inner product

(a) $(\mathbf{x}, \mathbf{y}) = x_1 y_2 + x_2 y_1\,,$ on $\mathbb{R}^2$;

(b) $\langle \mathbf{z}, \mathbf{w} \rangle = 2z_1\bar{w}_1 - 3z_1\bar{w}_2 - 3z_2\bar{w}_1 + 5z_2\bar{w}_2\,,$ on $\mathbb{C}^2$;

(c)
$$(f, g) = \int_0^1 (1+t)^2 f(t)\, g(t)\, dt\,,$$
on the space of real valued functions $C([0,1])$.

(ii) Let $V = \mathbb{R}^2$, find the orthogonal complement to the vector subspace

$$U = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

in V with respect to the inner product

$$(\mathbf{x}, \mathbf{y}) = 2x_1y_1 + x_1y_2 + x_2y_1 + 5x_2y_2\,.$$

9. (i) Let A be an $n \times n$ anti-symmetric (skew-symmetric) matrix. Show that

$$\mathbf{x}^t A\, \mathbf{x} = 0$$

for all $\mathbf{x} \in \mathbb{R}^n$.

(ii) Let B be an $n \times n$ hermitian matrix. Show that $\mathbf{z}^* B\, \mathbf{z}$ is a real number for all $\mathbf{z} \in \mathbb{C}^n$.

10. (i) Define what a group G is.

(ii) Let

$$G = \left\{ 1, e^{\frac{2\pi i}{5}}, e^{\frac{4\pi i}{5}}, e^{\frac{6\pi i}{5}}, e^{\frac{8\pi i}{5}} \right\}.$$

Show that G is a group under the usual multiplication of complex numbers. Present another group of the same order which is isomorphic to G and give explicitly an isomorphism between the two.