Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH1561-WE01

Title:
Single Mathematics A

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.

Revision:	

ED01/2017

1. (i) Differentiate $x^{x\sin(x)}$ with respect to x.

   (ii) Compute the following two limits:

$$\lim_{x\to 1}\frac{\sin(\pi x)}{1-x^2}, \qquad \lim_{x\to\infty} 2^{-x}\cos(x^2)\ .$$

   Clearly state which theorem you have used, if any.

   (iii) State the definition of the derivative of a function $f(x)$ as a limit. By working out this limit find the derivative of

$$f(x)=\frac{x+2}{x-1}\ .$$

   Note: do *not* use L'Hôpital's rule here!

2. (i) Compute the following definite integral:

$$\int_0^{\pi/6}\sin(2x)\cos(x)dx\ .$$

   (ii) Compute the following indefinite integral:

$$\int \ln(x)^2 dx\ .$$

   (iii) Compute the following indefinite integral:

$$\int \frac{7x+9}{(x+2)^2(x^2+2x+5)}dx\ .$$

   (iv) Show that, for $z=x+iy$,

$$|\cos(z)|^2 = a\cos(2x)+b\cosh(2y)$$

   where a and b are numbers which you should determine.

3. (i) Find the modulus and argument of

$$\exp\left(\frac{4+3i}{1-i}\right)\ .$$

   (ii) State de Moivre's theorem.

   (iii) Express $\sin(3\theta)$ as a polynomial in $\sin(\theta)$.

   (iv) Find all solutions z to the equation

$$z^4+2z^2+2=0\ .$$

   Give all your answers in polar form.

   (v) Find all solutions z to the equation

$$\cos(z^4)=1\ .$$

   Hint: write $\exp(iz^4)=w$ and solve for w first.

4. (i) (a) Explain the difference between absolute convergence and conditional convergence.
   (b) State the ratio test and quotient tests and state a simple test for determining convergence of an alternating series.
   (c) Explain whether the following series is conditionally convergent, absolutely convergent, or divergent, carefully justifying your answer:

$$\sum_{n=1}^{\infty} \left(\frac{-(n+2)}{3n-2} \right)^n .$$

   (ii) Find the interval of convergence for the power series $\displaystyle\sum_{k=2}^{\infty} (k+1)(-2x)^k$.

5. (a) Write down expressions for the formal Taylor series of a function $f(x)$ about $x = a$.
   (b) Write down the corresponding Taylor polynomials of degree N and the corresponding Lagrange form of the remainder.
   (c) Find the Taylor polynomial of degree 2 about $x = 1$ of the function

$$f(x) = x^{-2/3} .$$

   (d) Write down the Lagrange form of the remainder of this Taylor polynomial evaluated at x and give an upper bound on the absolute value of this remainder when $x = 11/10$ (write your answer as a fraction).

6. (i) Let $A = \begin{pmatrix} 2 & -3 \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & -5 \\ -3 & 1 & -2 \end{pmatrix}$.
   Evaluate the following expressions or give reasons why they are undefined.

$$4B\ , \quad A+B\ , \quad AB\ , \quad BA\ .$$

   (ii) Determine the condition on a, b, c for which the following system of equations

$$\begin{array}{ccccccl} 2x & + & 3y & - & z & = a \\ x & - & y & + & 3z & = b \\ 3x & + & 7y & - & 5z & = c\,. \end{array}$$

   is consistent, and find the general solution when this condition is satisfied.
   (iii) Find the determinant of the following matrix.

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 5 & 2 & 1 \\ 2 & 1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{pmatrix}.$$

7. (i) (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(b) Hence find A^{10}.
[Write the resulting matrix with integers or reduced fractions as entries, using that $3^{10} = 59,049$.]

(ii) Define a group and show that the set of special orthogonal $n \times n$ matrices A form a group under matrix multiplication. [A special orthogonal matrix is one which satisfies $AA^T = I_n$ and $\det A = 1$ where I_n is the $n \times n$ identity matrix.]