Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH1571-WE01

Title: Single Mathematics B

Time Allowed:	3 hours	
Additional Material provided:	Tables: Normal distribution.	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.

Revision:	

ED01/2017

1. (i) A particle of constant mass m has position vector

$$\boldsymbol{r}(t) = \cos(2t)\boldsymbol{i} + \sin(3t)\boldsymbol{j}$$

at time t.

Calculate the angular momentum of the particle (about the origin). Is the force acting on the particle a central force?

(ii) (a) Find a normal vector to the plane containing the three points with Cartesian coordinates $(2, 4, 1)$, $(3, 7, 3)$ and $(-2, 6, 2)$.

(b) Find a Cartesian equation for the plane, and the minimum distance from the origin to the plane.

(iii) In cylindrical polar coordinates (ρ, ϕ, z) we can define basis vectors

$$\begin{aligned} \boldsymbol{e}_\rho &= \cos\phi\boldsymbol{i} + \sin\phi\boldsymbol{j} \\ \boldsymbol{e}_\phi &= -\sin\phi\boldsymbol{i} + \cos\phi\boldsymbol{j} \\ \boldsymbol{e}_z &= \boldsymbol{k} \end{aligned}$$

where $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is the Standard Cartesian basis.

Show that the basis $\{\boldsymbol{e}_\rho, \boldsymbol{e}_\phi, \boldsymbol{e}_z\}$ is an orthonormal basis.

2. Find the general solutions of the following ordinary differential equations for functions $y(x)$.

(i)
$$y' + y^2\cos x - \cos x = 0\ .$$

(ii)
$$xy' - y - \frac{3}{x}y^2 = 0\ .$$

(iii)
$$y' - y\sin x - \sin x\cos x = 0\ .$$

3. A function of period 2π is defined by

$$f(x) = \begin{cases} -1 & , \quad -\pi < x \leq 0 \\ 3 & , \quad 0 < x \leq \pi \end{cases} .$$

(a) Sketch this function on the interval $(-2\pi, 2\pi)$.

(b) Find the Fourier series for this function.

(c) Use Parseval's theorem with the above Fourier series to evaluate the sum of $1/N^2$ over all positive odd integers N.

4. (i) Find the Taylor expansion up to and including quadratic terms of the function $f(x, y) = (2 + 3x^2 + y^2)^{3/2}$ about the point $(1, 2)$.

   (ii) Let $x(u) = \sin(2u)$ and $y(u) = 1+u^2$ and find the rate of change of the function $f(x, y) = x^2 e^{3y}$ with respect to u. Calculate the value of $\dfrac{df}{du}$ at $u = \dfrac{\pi}{4}$.

   (iii) Where is the complex valued function $g(x + iy) = |x| + iy$ analytic?

   (iv) Find all complex z where the power series $\sum_{n=0}^{+\infty} 3^n z^n$ absolutely converges.

5. (i) Find $\iint\limits_A \dfrac{3x^2 y}{(1 + y^2)^2} dxdy$, where A is the rectangle defined by $0 \leq x \leq 2$, $0 \leq y \leq 3$.

   (ii) Find the area of the region bounded by the curve $2y = x^2$ and the line $2y + x - 2 = 0$ in the plane.

   (iii) Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1$.

6. (i) Find the moment of inertia of a rectangle defined in the xy-plane by $0 \leq x \leq w$ and $0 \leq y \leq h$ about the y-axis with:

   (a) uniform density ρ,

   (b) density $\rho = 1 + x$.

   (ii) For a function $B(x, y, z) = 5 - xz + y^2$ at the point $(7, -3, 2)$, find both the greatest rate of change of B and the rate of change of B in the direction $\boldsymbol{u} = (2, 1, 0)$.

   (iii) Show that $\nabla \cdot (\phi \boldsymbol{a}) = \phi \nabla \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \nabla \phi$ for a scalar function $\phi(x, y, z)$ and a vector function $\boldsymbol{a} = a_1(x, y, z)\hat{\boldsymbol{i}} + a_2(x, y, z)\hat{\boldsymbol{j}} + a_3(x, y, z)\hat{\boldsymbol{k}}$. Check this formula for the function $\phi = x^2 y^2 z^2$ and $\boldsymbol{a} = (x^2 y, z^2, z^3)$.

7. (i) Fifty independent random variables X_i, $i = 1, 2, \ldots, 50$, have probability distribution $P(X_i = j) = 0.2$ for $j = 1, 2, 3, 4, 5$. Derive the mean and standard deviation of X_i. Specify the approximate probability distribution of the average $\bar{X}$ of these 50 random variables using the Central Limit Theorem.

   (ii) The diameter of shafts made by a certain manufacturing process are known to be Normally distributed with mean 2.500 cm and standard deviation 0.009 cm. A sample of nine such shafts is selected at random. Calculate the probability that the sample mean exceeds 2.506 cm.

   (iii) To check the quality of a very large batch of produced items, a random sample of size 100 is taken. If the number of defective items in the sample does not exceed 2 then the whole batch is accepted, else it is rejected. Let p denote the probability that an item is defective. Use the Binomial distribution to derive an expression for the probability that the batch is accepted. Suppose one is interested in this probability for the values $p = 0.01$ and $p = 0.1$. For both these values of p, provide expressions of suitable approximations to the probability that the batch is accepted. Explain briefly whether or not you expect one of these two approximations, either the one for $p = 0.01$ or the one for $p = 0.1$, to be better than the other.