Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH2011-WE01

Title:

Complex Analysis II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. (a) Suppose (X, d) is a metric space. What does it mean for (X, d) to be *compact*?
   (b) Show that the open unit interval $(0, 1)$ with its usual metric is not compact.

2. Let S be the region

$$S = \{(x, y) : -1 \leq x \leq 2, -\infty < y < \infty\} \subset \mathbb{R}^2.$$

and let $T \subset \mathbb{R}^2$ be the connected region bounded by the line L_1 passing through $(0, -\sqrt{2})$ and $(\sqrt{2}, 0)$ and by the line L_2 passing through $(0, 2\sqrt{2})$ and $(-2\sqrt{2}, 0)$

   (a) Give an example of a harmonic function $u : S \to \mathbb{R}$ that satisfies $f(-1, y) = 5$ and $f(2, y) = 11$ for all $y \in \mathbb{R}$.
   (b) Hence or otherwise give an example of a harmonic function defined on T which takes the constant value 5 on L_1 and the constant value 11 on L_2.

3. (a) Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function and let $a \in \mathbb{C}$. Define what it means for f to be complex differentiable at a.
   (b) State the Cauchy-Riemann equations.
   (c) Let f be the function defined by

$$f(x + iy) = x \cosh(y) + i \cos(x) \sinh(y)$$

   for $x, y \in \mathbb{R}$.
   Use the Cauchy-Riemann equations to determine where the function f is complex differentiable, stating any results from the course that you use.

4. (a) Let $f(z)$ be holomorphic in a domain D. State the Theorem of Cauchy-Taylor around a point $a \in D$.
   (b) Let $P(z)$ be a polynomial of degree at most d and assume that

$$\int_{|z|=1} \frac{P(z)}{(a\,z - 1)^{n+1}}\ dz = 0,$$

   for some $a > 1$, and all $n = 0, 1, \ldots, d$. Show that $P(z) = 0$.

5. (a) State the Residue Theorem.
   (b) Let α be a non-zero complex number with $|\alpha| < 1$. Show that

$$\int_{|z|=1} \frac{1}{|z - \alpha|^2} \frac{dz}{z} = \frac{2\pi i}{1 - |\alpha|^2}.$$

6. (a) State the local maximum modulus principle.
   (b) Let $f(z) := \left|e^{z^3}\right|$. Explain why $f(z)$ attains a maximum value in the closed disc $|z| \leq 1$, and find this value.

SECTION B

7. (a) Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc. Show that the sequence of functions $f_n : D \to \mathbb{C}$ defined by

$$f_n(z) = z^n$$

converge pointwise but not uniformly.

(b) State the Weierstrass M-test.

(c) Show that

$$\sum_{n=0}^{\infty} \frac{n}{3^n + z^n}$$

converges locally uniformly but not uniformly in the region

$$D = \{z \in \mathbb{C} : |z| < 3\}.$$

8. (a) At which points of $\mathbb{C}$ is the map $f : z \mapsto z^2$ conformal?

(b) Let

$$S_\alpha = \{re^{i\theta} : r > 0, \alpha < \theta < \alpha + \pi/2\}$$

for $\alpha \in \mathbb{R}$.
Determine $f(S_\alpha)$.

(c) Give the Möbius transformation M that maps the ordered triple of points $(-1, i, 1)$ to the ordered triple $(0, 1, \infty)$.

(d) Let R be the region

$$R = \{z \in \mathbb{C} : |z| < 1, \mathrm{Im}(z) > 0\}.$$

Determine $M(R)$.

(e) Give a conformal map that maps R to the open unit disc

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

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9. (a) Show that $2\theta < \pi \sin\theta$, for $0 < \theta < \frac{\pi}{2}$.

   (b) Define the curve $\gamma_{1,r}(t) := re^{it}$, where $0 \le t \le \frac{\pi}{4}$. Using (a) above or otherwise show that
$$\lim_{r\to\infty} \int_{\gamma_{1,r}} e^{iz^2} dz = 0.$$

   (c) Define the curve $\gamma_{2,r}(t) := (r-t)e^{i\pi/4}$, where $0 \le t \le r$. Using the fact that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ or otherwise show that
$$\lim_{r\to\infty} \int_{\gamma_{2,r}} e^{iz^2} dz = \frac{1+i}{\sqrt{2}} \frac{\sqrt{\pi}}{2}.$$

   (d) Using (b) and (c) above or otherwise, show that
$$\int_0^\infty \cos(x^2) dx = \int_0^\infty \sin(x^2) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

10. (a) Let N be a positive integer and set $\alpha_N := \left(N + \frac{1}{2}\right)\pi$. Let γ_N be the square with vertices $(-\alpha_N, -\alpha_N)$, $(\alpha_N, -\alpha_N)$, (α_N, α_N) and $(-\alpha_N, \alpha_N)$, oriented counter-clockwise. Show that
$$\int_{\gamma_N} \frac{dz}{z^2 \sin(z)} = 2\pi i \left(\frac{1}{6} + \frac{2}{\pi^2} \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \right).$$

   (b) Using the identity
$$|\sin(z)|^2 = \sin^2(x) + \sinh^2(y), \quad z = x + iy, x, y \in \mathbb{R},$$
or otherwise show that $|\sin(z)| \ge |\sin(x)|$ and $|\sin(z)| \ge |\sinh(y)|$.

   (c) Show that there exists a $B \in \mathbb{R}$ independent of N such that
$$\left| \int_{\gamma_N} \frac{dz}{z^2 \sin(z)} \right| \le \frac{B}{2N+1}.$$

   (d) Show that
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$