Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH2031-WE01

Title:
Analysis in Many Variables II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. (a) Compute the gradient, ∇f, of $f(x, y, z) = x^3 + xy + z^4$. In which direction is this function decreasing the fastest at the point $(1, 1, 1)$, and what is the equation for the tangent plane to the surface $f(x, y, z) = 3$ at this point?

   (b) Let $F(t)$ be the value of $f(x, y, z)$ restricted to the curve $x = t$, $y = t^2$, $z = 1/t$. Use the chain rule to calculate $\frac{dF}{dt}$.

2. (i) (a) Sketch the three-dimensional vector field $\mathbf{A}(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ in the $z = 0$ plane.

   (b) Compute the curl of $\mathbf{A}$ and comment on its value in relation to your sketch.

   (ii) Using index notation, calculate the curl of $\mathbf{v} = (\mathbf{a} \cdot \mathbf{x})\mathbf{x}$, where $\mathbf{a}$ is a constant.

3. (a) Give the definition of the limit of a function:

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = l$$

   where $\mathbf{x}$ and $\mathbf{a}$ are n dimensional vectors.

   (b) Define what is meant by $f(\mathbf{x})$ being continuous at $\mathbf{a}$.

   (c) For the function

$$f(\mathbf{x}) \ = \ \frac{-2xy}{x^2 + y^2}$$

   test whether $\lim_{\mathbf{x} \to \mathbf{0}} f(\mathbf{x})$ exists.

4. A solid cylinder C of radius 1 and height 1 is defined by

$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1,\ 0 \leq z \leq 1\}.$$

   Show that the paraboloid $P = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$ cuts C into two pieces of equal volume.

5. (a) Use Green's theorem in the plane to show that

$$\oint_C \big((x + y)\, dy + (x - y)\, dx\big) = K\, A\,,$$

   where C is a curve in the x,y plane, A is the area enclosed by C, and K is a constant which you should determine.

   (b) Use the divergence theorem to show that $\int_S \mathbf{x} \cdot d\mathbf{A} = L\, V$, where the surface S encloses a volume V in $\mathbb{R}^3$, and L is a constant which you should determine.

6. Find the coefficients a, b, c and d in the following generalised function identities:

   (a) $x\delta(x - 2) = a\, \delta(x - 2)$,

   (b) $x\delta'(x - 2) = b\, \delta(x - 2) + c\, \delta'(x - 2)$,

   (c) $x\delta(x^3 - 2x^2 + x - 2) = d\, \delta(x - 2)$.

SECTION B

7. (a) Suppose the level curve defined by $f(x, y) = c$ can be written as the function $y = g(x)$, where $f(x, y)$ is differentiable. Derive an expression for dg/dx in terms of the partial derivatives of $f(x, y)$.

(b) State the implicit function theorem as applied to the function $f(x, y)$ from question 7.(a).

(c) Consider the function

$$f(x, y) \;=\; (3x + y)e^{3xy}.$$

Determine whether or not the curve $f(x, y) = c$ can be written in the form $y = g(x)$, and if not, state clearly the points (x_0, y_0) and corresponding values of c where problems occur.

(d) Using $f(x, y)$ as given in question 7.(c), determine whether or not the curve $f(x, y) = c$ can be written in the form $x = h(y)$, and if not, state clearly the points (x_0, y_0) where problems occur.

(e) Using $f(x, y)$ as given in question 7.(c), are there any points where the curve $f(x, y) = c$ can neither be written as $y = g(x)$ nor as $x = h(y)$?

8. (a) Given a scalar function f defined on $\mathbb{R}^n$, explain what is meant by a critical point of f.

(b) Now suppose $f(x, y)$ is a scalar function on $\mathbb{R}^2$. State sufficient conditions for a critical point of f to be either (i) a local maximum, (ii) a local minimum, or (iii) a saddle point of f, in terms of the 2×2 Hessian matrix with components $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$:

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

(c) Find and classify all critical points of the scalar function

$$f(x, y) \;=\; x^4 + 4xy + y^4.$$

(d) Using $f(x, y)$ as given in question 8.(c), use the method of Lagrange multipliers to find the critical points of $f(x, y)$ subject to the constraint $g(x, y) = 0$ where $g(x, y) = x + y$. Comment on your results as they relate to question 8.(c).

9. (a) State Stokes' theorem and use it to show that if $\nabla \times \mathbf{v} = 0$ in a simply connected region D of $\mathbb{R}^3$ and C is a closed curve in D, then $\oint_C \mathbf{v} \cdot d\mathbf{x} = 0$.

(b) Show further that if C_1 and C_2 are two curves in the region D with the same starting and ending points, then $\int_{C_1} \mathbf{v} \cdot d\mathbf{x} = \int_{C_2} \mathbf{v} \cdot d\mathbf{x}$.

(c) Now consider the vector field $\mathbf{V}$ defined for $x^2 + y^2 \neq 0$ by

$$\mathbf{V}(x, y, z) = \left(\frac{-y}{x^2 + y^2}\,, \frac{x}{x^2 + y^2}\,, z \right).$$

Compute $\nabla \times \mathbf{V}$ at all points where it is defined, and evaluate the line integrals $\int_{C_1} \mathbf{V} \cdot d\mathbf{x}$ and $\int_{C_2} \mathbf{V} \cdot d\mathbf{x}$ where C_1 is the semi-circle $\mathbf{x}(t) = (2\cos(t), 2\sin(t), 0)$, $0 \leq t \leq \pi$, and C_2 is the semi-circle $\mathbf{x}(t) = (2\cos(t), -2\sin(t), 0)$, $0 \leq t \leq \pi$. Explain why your answer does not contradict the result from part (b).

(d) Now let $\mathbf{V}$ be as in part (c), C_3 be the semi-ellipse $\mathbf{x}(t) = (2\cos(t), \sin(t), 0)$, $0 \leq t \leq \pi$, and C_4 be the semi-ellipse $\mathbf{x}(t) = (2\cos(t), -\sin(t), 0)$, $0 \leq t \leq \pi$. Using parts (b) and (c), find the values of $\int_{C_3} \mathbf{V} \cdot d\mathbf{x}$ and $\int_{C_4} \mathbf{V} \cdot d\mathbf{x}$.

10. (a) Show that the series

$$\varphi(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \sinh(n\pi y)$$

satisfies Laplace's equation $\nabla^2 \varphi = 0$ inside the unit square $0 < x < 1$, $0 < y < 1$ (you can assume that you can interchange the orders of integration and differentiation). Show also that $\varphi(0, y) = \varphi(1, y) = 0$ for $0 \leq y \leq 1$, and that $\varphi(x, 0) = 0$ for $0 \leq x \leq 1$.

(b) In addition to the boundary conditions imposed in part (a), it is now imposed that $\varphi(x, 1) = f(x)$ for $0 \leq x \leq 1$, for some given function $f(x)$. Obtain a formula for the coefficients A_n in the expansion of $\varphi(x, y)$. The formula $\sin(A)\sin(B) = \frac{1}{2}(\cos(A - B) - \cos(A + B))$ can be used without proof.

(c) Using your result from part (b), find the series solution to Laplace's equation on the unit square with boundary conditions

$$\varphi(0, y) = \varphi(1, y) = 0, \quad 0 \leq y \leq 1\,; \quad \varphi(x, 0) = 0\,,\ \varphi(x, 1) = x, \quad 0 \leq x \leq 1\,.$$

(d) Find an expression for the solution $\psi(x, y)$ to Laplace's equation on the unit square with boundary conditions

$$\psi(0, y) = 0\,,\ \psi(1, y) = y, \quad 0 \leq y \leq 1\,; \quad \psi(x, 0) = 0\,,\ \psi(x, 1) = x, \quad 0 \leq x \leq 1\,.$$

(Hint: use your result from part (c), and the principle of superposition.)