EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH2051-WE01

Title:
Numerical Analysis II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. (a) Let $\overline{x}$ and $\overline{y}$ be approximations to x and y, each with relative rounding error not exceeding ϵ_{M} in absolute value. Find an expression for the relative error in estimating $x - y$ by $\overline{x} - \overline{y}$. Is this also bounded by ϵ_{M} or not?

   (b) Calculate $f(0.001)$ correct to six decimal places, where

   $$f(x) = \frac{2\cosh(x) - x\sin(x) - 2}{x^4}.$$

   Justify your belief that your answer has the required accuracy.

2. (a) Show that the backward-difference approximation

   $$D_h = \frac{f(x_0) - f(x_0 - h)}{h}$$

   for approximating $f'(x_0)$ has truncation error $\mathcal{O}(h)$.

   (b) Use Richardson extrapolation to derive a second-order finite-difference formula based on D_h and $D_{h/3}$.

3. Let $f(x) = \mathrm{e}^{-x} - \sin(x)$.

   (a) Prove that the equation $f(x) = 0$ has a unique solution x_* in the interval $(0, 0.8)$.

   (b) Carry out two stages of the interval bisection method to estimate x_*, and give a bound on the magnitude of the error in your estimate.

   (c) How many stages would be necessary to guarantee an error less than 10^{-8}?

4. (a) Find matrices L and U such that $A = LU$, where L is unit lower triangular and U is upper triangular, given that

   $$A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 7 & 9 \\ 5 & 8 & -2 \end{pmatrix}.$$

   (b) Hence solve the linear system

   $$\begin{pmatrix} 1 & 4 & 3 \\ 2 & 7 & 9 \\ 5 & 8 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 53 \\ 53 \\ 53 \end{pmatrix}.$$

5. (a) Calculate the condition number in the 1-norm of the matrix

   $$A = \begin{pmatrix} 10^{-4} & 10^{-4} \\ 1 & 2 \end{pmatrix}.$$

   (b) With reference to an appropriate inequality, explain the practical consequences of a large condition number when trying to solve the linear system $A\boldsymbol{x} = \boldsymbol{b}$.

6. (a) Write down the Lagrange form of the interpolating polynomial $p_2 \in \mathcal{P}_2$ that interpolates a function f at the nodes 0, h and $2h$.

   (b) Hence derive the closed Newton-Cotes approximation $I_2(f)$ for the integral

$$I(f) = \int_0^{2h} f(x)\,\mathrm{d}x.$$

   (c) Starting from the appropriate error formula for polynomial interpolation, and given that $|f'''(x)| \leq M$ for all $x \in [0, 2h]$, show that

$$\big|I(f) - I_2(f)\big| \leq \frac{M}{6} \int_0^{2h} \big|x(x-h)(x-2h)\big|\,\mathrm{d}x.$$

SECTION B

7. (a) The values of a function f and of its first and second derivatives are known at two distinct points a and b. Let $p_5 \in \mathcal{P}_5$ be a polynomial satisfying

$$\begin{aligned} p_5(a) = f(a), \quad p_5'(a) = f'(a), \quad p_5''(a) = f''(a), \\ p_5(b) = f(b), \quad p_5'(b) = f'(b), \quad p_5''(b) = f''(b). \end{aligned}$$

(i) Assuming that such a polynomial exists, prove that it is unique.

(ii) Prove that, when f is sufficiently differentiable,

$$f(x) - p_5(x) = \frac{f^{(6)}(\xi)}{6!}(x-a)^3(x-b)^3 \quad \text{for some } \xi \in (a,b).$$

(b) Suppose that the interval $[0,1]$ is divided into subintervals of length h, and a piecewise-quintic interpolant $q(x)$ is constructed, using the polynomial from part (a) in each subinterval, based on the values of f, f' and f'' at the end-points of that subinterval. Given that $\max_{x\in[0,1]} |f^{(6)}(x)| \leq M$, prove that

$$|f(x) - q(x)| \leq \frac{Mh^6}{46080} \quad \text{for all } x \in [0,1].$$

8. (a) Let f be twice differentiable, with a simple root at $x = x_*$. Stating any result used, prove that Newton's method will converge to x_* for all sufficiently accurate starting values, and that the order of convergence is at least 2.

(b) In what circumstances would the order of convergence of Newton's method be higher than 2?

(c) Prove that the cubic equation

$$6.76x^3 + 11.7x^2 - 12x + 2.5 = 0 \tag{1}$$

has at least one negative root, and carry out two steps of Newton's method starting with $x_0 = -2.6$, giving your answer to six significant figures.

(d) When starting from $x_0 = 0.4$ with the above cubic equation, a Python program for Newton's method gave the following results:

k	x_k	$x_k - x_{k-1}$
1	0.392328	-0.007672
2	0.388477	-0.003851
3	0.386547	-0.001929
4	0.385582	-0.000966
5	0.385099	-0.000483

What does this suggest about a root of the equation near 0.4?

(e) Hence express the roots of the cubic exactly as rational numbers.

9. Consider an overdetermined system of equations $A\boldsymbol{x} = \boldsymbol{b}$ where A is a real $m \times n$ matrix, $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{b} \in \mathbb{R}^m$, with $m > n$.

   (a) Define the residual vector and state whether it belongs to $\mathbb{R}^n$ or to $\mathbb{R}^m$.

   (b) Show that $A\boldsymbol{x}$ is orthogonal to the residual if $\boldsymbol{x}$ satisfies the normal equations

   $$A^\top A\boldsymbol{x} = A^\top \boldsymbol{b}.$$

   (c) If $A = QR$, where Q is a matrix with orthonormal columns and R is an upper triangular matrix (assumed invertible), show that the normal equations reduce to

   $$R\boldsymbol{x} = Q^\top \boldsymbol{b}.$$

   (d) By constructing suitable matrices Q and R, find the least-squares solution when $\alpha = \beta = 0$ to the system

   $$\begin{pmatrix} 1 & 1 \\ \alpha & 2 \\ \beta & 3 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}. \tag{2}$$

   (e) For what values of α and β is there a unique least-squares solution to (2)?

10. (a) Find monic polynomials $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$ of degrees 0, 1 and 2 that are orthogonal on $[0, 1]$ with respect to the weight function $x^{-1/3}$.

   (b) Hence find the nodes and weights (to six significant figures) of a two-point Gaussian quadrature formula for approximating integrals of the form

   $$\int_0^1 f(x)x^{-1/3}\,\mathrm{d}x.$$

   (c) State the degree of exactness of this quadrature formula (without proof).

   (d) Find the error when this quadrature formula is used to approximate

   $$\int_0^1 x^{-1/3}\cos\big(3\arccos(x)\big)\,\mathrm{d}x.$$