EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH2071-WE01

Title:
Mathematical Physics II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. A point particle of mass m moves under gravity on the surface of a smooth bowl described by the paraboloid $z = x^2 + y^2$ where (x, y, z) are Cartesian co-ordinates with the z-axis pointing vertically upwards. Take as generalised co-ordinates q_1, q_2 where

$$\begin{aligned} x &= q_1 \cos(q_2) \\ y &= q_1 \sin(q_2) \\ z &= q_1^2 \,. \end{aligned}$$

   (a) Find $\dot{x}$, $\dot{y}$ and $\dot{z}$ in terms of the generalised co-ordinates and their time derivatives.

   (b) Find the kinetic energy of the particle in terms of its generalised co-ordinates and their time derivatives.

   (c) Find the Lagrangian of the particle by writing the gravitational potential energy in terms of generalised co-ordinates and using your result for the kinetic energy.

   (d) Which co-ordinate is ignorable, and what is the corresponding conserved quantity?

2. This question concerns Noether's theorem. An infinitesimal transformation of generalised co-ordinates

$$q_i \to q_i' = q_i + \epsilon a_i(q_1, .., q_n), \quad \dot{q}_i \to \dot{q}_i' = \dot{q}_i + \epsilon \dot{a}_i(q_1, .., q_n)\,,$$

is a symmetry of a Lagrangian L if L changes in the following way

$$L(q_1, .., q_n, \dot{q}_1, .., \dot{q}_n) \to L(q_1', .., q_n', \dot{q}_1', .., \dot{q}_n') = L(q_1, .., q_n, \dot{q}_1, .., \dot{q}_n) + \epsilon \dot{G}$$

up to terms quadratic in the infinitesimal parameter ϵ.

   (a) Let δL denote the difference $\delta L = L(q_1', .., q_n', \dot{q}_1', .., \dot{q}_n') - L(q_1, .., q_n, \dot{q}_1, .., \dot{q}_n)$. Write δL in terms of $\dot{G}$. Using the chain-rule obtain another expression for δL, this time in terms of a_i, $\dot{a}_i$ and partial derivatives of L.

   (b) Use these two expressions and the Euler-Lagrange equations to show that $\dot{Q} = 0$ where

$$Q = \sum_{i=1}^{n} a_i \frac{\partial L}{\partial \dot{q}_i} - G\,.$$

   (c) Show that

$$q_1 \to q_1' = q_1 + \epsilon q_2, \quad q_2 \to q_2' = q_2 - \epsilon q_1$$

   is a symmetry of

$$L = (q_1^2 + q_2^2)(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)\,.$$

   (d) Find the corresponding Noether charge Q.

3. The displacement $u(x,t)$ of a particular string satisfies the wave-equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

and the energy of the portion of the string between $x = a$ and $x = b$ is given by

$$E(a,b) = \frac{1}{2}\int_a^b \left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 \right) dx.$$

(a) Use the wave-equation to show that

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 \right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)$$

subject to the second partial derivatives of u being continuous.

(b) Hence show that

$$\frac{d}{dt}E(a,b) = \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_a^b.$$

(c) If the string is semi-infinite, so that $u(x,t)$ is only defined for $-\infty < x \leq 0$ and an incident wave is reflected at the origin so that $u(x,t)$ is given by the real part of

$$e^{ik(x-t)} + Re^{-ik(x+t)}$$

find R in the following cases

i. $u(0,t) = 0$,

ii. $\frac{\partial u(0,t)}{\partial x} = 0$.

4. Consider a particle which is confined inside the region $-L < x < L$. At a given time t the wavefunction of the particle's state is $\psi(x,t) = A\,(x^2 - L^2)$.

(a) Fix the constant A for the state to be physical.

(b) Calculate the expectation values $\langle\, \hat{x}\,\rangle$ and $\langle\, \hat{p}\,\rangle$ with $\hat{x}$ and $\hat{p}$ being the position and momentum operators corrspondingly.

(c) How would your answer for A and $\langle\, \hat{x}\,\rangle$ be modified for the wavefunction $\psi(x,t) = A\, e^{i\,f(x)}\,(x^2 - L^2)$ with $f(x)$ a real function of x?

5. (a) Write the expression of the expectation value of an observable $\hat{\Omega}$ making use of the inner product and the state vector ψ.

(b) Write the expression which defines the uncertainty of the observable $\hat{\Omega}$.

(c) Calculate the uncertainty of $\hat{\Omega}$ for one of its eigenstates with corresponding eigenvalue ω.

(d) Discuss the time dependence of the expectation value of $\hat{\Omega}$ assuming that $\hat{\Omega}$ commutes with the Hamiltonian $\hat{H}$ of the system.

6. (a) Write down the one dimensional Hamiltonian operator for a particle of mass m subject to a potential which is classically given by the function $V(x)$.

(b) Suppose the potential has a global minimum V_{min} such that $V(x) > V_{min}$ for $x \in \mathbb{R}$. Find a real number B such that $\langle\, \hat{H}\,\rangle \geq B$ for any physical state.

(c) Can there be a physical state which gives $\langle\, \hat{H}\,\rangle = V_{min}$? Briefly justify your answer.

SECTION B

7. (a) Write down Hamilton's equations of motion for a system described by co-ordinates $q_1, .., q_n$ and and their conjugate momenta $p_1, .., p_n$. Define the Poisson bracket of two functions A and B of these co-ordinates and momenta, and use it and Hamilton's equations to show that

$$\dot{A} = \{A, H\} \, ,$$

where A has no explicit dependence on t.

(b) Find the Hamiltonian for the Lagrangian

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 - q_1^2 - q_2^2\right) \, .$$

(c) By computing their Poisson brackets with the Hamiltonian show that $M = p_1 p_2 + q_1 q_2$ and $L = p_1 q_2 - p_2 q_1$ are constants of the motion.

(d) Find $\{L, M\}$.

(e) Use the Jacobi identity $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$ to show that $\{L, M\}$ is conserved.

8. Two coupled pendula are described by the Lagrangian

$$L = \frac{1}{2}\left(\frac{1}{2}\dot{\theta}_1^2 + \dot{\theta}_2^2 + \cos\theta_1 + 2\cos\theta_2 - (\theta_1 - \theta_2)^2\right)$$

(a) Show that $\theta_1 = \theta_2 = 0$ is one solution of the Euler-Lagrange equations.

(b) Consider motion close to this solution. Expand L in powers of θ_1, θ_2, $\dot{\theta}_1$ and $\dot{\theta}_2$ up to and including quadratic terms to obtain an approximation which we will call L_2.

(c) Write the Euler-Lagrange equations for the approximate Lagrangian L_2 in the matrix form

$$\frac{d^2}{dt^2}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} = \mathbf{A}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix}$$

and find the two by two matrix $\mathbf{A}$.

(d) Find the general solution for $\theta_1(t)$ and $\theta_2(t)$ to this matrix equation.

(e) If initially $\dot{\theta}_1 = \dot{\theta}_2 = 0$ and $\theta_1 = \theta_2 = 1$ find $\theta_1(t)$ and $\theta_2(t)$ for subsequent values of t.

9. Consider a quantum mechanical particle of mass m inside an infinite potential well whose left hand side wall is at $x = 0$ and the right hand side one is at $x = L$. The potential in the region $0 \leq x \leq L$ is equal to zero.

   (a) You are given that the wavefunctions $\psi_n(x)$ of the Hamiltonian eigenstates are either $\psi_n(x) = A_n \sin\left(\frac{n\pi}{L}x\right)$ or $\psi_n(x) = A_n \cos\left(\frac{n\pi}{L}x\right)$ for some constants A_n and positive integers n which label the different eigenfunctions. Decide which ones are the correct wavefunctions for the problem at hand. Justify your answer. Fix the constants A_n for the eigenstates to form an orthonormal basis.

   (b) Write down the Hamiltonian. Derive its spectrum using the eigenfunctions you chose to be the correct ones in the previous question.

   (c) At $t = 0$ the wavefunction of the particle's state is given by the wavefunction $\psi(x, t = 0) = C\ (\psi_1(x) + i\,\psi_2(x))$. Choose an appropriate C so that the initial state is physical and write down the time evolved wavefunction $\psi(x, t)$ at time $t > 0$.

   (d) Compute the expectation value $\langle\, \hat{x}\, \rangle$ of the position operator $\hat{x}$ as a function of time for the time dependent state you constructed in the previous question.
   Hint: You are given that $\int_0^1 y \sin(\pi y)\sin(2\pi y)dy = -8/(9\pi^2)$

10. Consider the one dimensional quantum mechanical simple harmonic oscillator of mass m and frequency ω. The annihilation operator is defined through

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}\ (m\omega\,\hat{x} + i\,\hat{p})\ ,$$

   where $\hat{x}$ is the position operator and $\hat{p}$ is the momentum operator. The wavefunction $\psi_0(x)$ of the ground state satisfies $\hat{a}\psi_0(x) = 0$. You can assume that $\psi_0(x)$ is correctly normalised when answering the following questions.

   (a) Compute the commutator $[\hat{a}, \hat{a}^\dagger]$ and use induction to show that $[\hat{a}, (\hat{a}^\dagger)^n] = n\,(\hat{a}^\dagger)^{n-1}$ for any positive integer n.

   (b) Express $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$.

   (c) Consider the wavefunctions $\psi_n(x) = c_n\,(\hat{a}^\dagger)^n\psi_0(x)$ with n positive integers. Compute the norm of ψ_n and choose the constants c_n so that the wavefunctions $\psi_n(x)$ correspond to physical states.

   (d) Consider the operator $\hat{N} = \hat{a}^\dagger\hat{a}$ and check whether this is Hermitian or not. Show that $\psi_n(x)$ are eigenfunctions of $\hat{N}$ and find the corresponding eigenvalues. What is the implication of these facts for the inner product between ψ_n and ψ_m for $n \neq m$?

   (e) Compute the expectation values $\langle\, \hat{x}\, \rangle$ for each of the state wavefunctions $\psi_n(x)$.