

EXAMINATION PAPER

Examination Session: May

2017

Year:

Exam Code:

MATH2071-WE01

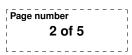
Title:

Mathematical Physics II

Time Allowed:	3 hours			
Additional Material provided:	None			
Materials Permitted:	None			
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.		
Visiting Students may use dictionaries: No				

Instructions to Candidates:	Credit will be given for: the best FOUR answers from Section and the best THREE answers from S Questions in Section B carry TWICE in Section A.	Section B.	arks as those
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Revision:





SECTION A

1. A point particle of mass m moves under gravity on the surface of a smooth bowl described by the paraboloid $z = x^2 + y^2$ where (x, y, z) are Cartesian co-ordinates with the z-axis pointing vertically upwards. Take as generalised co-ordinates q_1, q_2 where

$$\begin{aligned} x &= q_1 \cos(q_2) \\ y &= q_1 \sin(q_2) \\ z &= q_1^2 . \end{aligned}$$

- (a) Find \dot{x} , \dot{y} and \dot{z} in terms of the generalised co-ordinates and their time derivatives.
- (b) Find the kinetic energy of the particle in terms of its generalised co-ordinates and their time derivatives.
- (c) Find the Lagrangian of the particle by writing the gravitational potential energy in terms of generalised co-ordinates and using your result for the kinetic energy.
- (d) Which co-ordinate is ignorable, and what is the corresponding conserved quantity?
- 2. This question concerns Noether's theorem. An infinitesimal transformation of generalised co-ordinates

$$q_i \to q'_i = q_i + \epsilon a_i(q_1, ..., q_n), \quad \dot{q}_i \to \dot{q}'_i = \dot{q}_i + \epsilon \dot{a}_i(q_1, ..., q_n),$$

is a symmetry of a Lagrangian L if L changes in the following way

$$L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n) \to L(q'_1, ..., q'_n, \dot{q}'_1, ..., \dot{q}'_n) = L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n) + \epsilon \dot{G}$$

up to terms quadratic in the infinitesimal parameter ϵ .

- (a) Let δL denote the difference $\delta L = L(q'_1, ..., q'_n, \dot{q}'_1, ..., \dot{q}'_n) L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n)$. Write δL in terms of \dot{G} . Using the chain-rule obtain another expression for δL , this time in terms of a_i , \dot{a}_i and partial derivatives of L.
- (b) Use these two expressions and the Euler-Lagrange equations to show that $\dot{Q} = 0$ where

$$Q = \sum_{i=1}^{n} a_i \frac{\partial L}{\partial \dot{q}_i} - G \,.$$

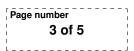
(c) Show that

$$q_1 \to q_1' = q_1 + \epsilon q_2, \quad q_2 \to q_2' = q_2 - \epsilon q_1$$

is a symmetry of

$$L = (q_1^2 + q_2^2)(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2).$$

(d) Find the corresponding Noether charge Q.



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- 3. The displacement u(x,t) of a particular string satisfies the wave-equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

and the energy of the portion of the string between x = a and x = b is given by

$$E(a,b) = \frac{1}{2} \int_{a}^{b} \left(\left(\frac{\partial u}{\partial t} \right)^{2} + \left(\frac{\partial u}{\partial x} \right)^{2} \right) dx.$$

(a) Use the wave-equation to show that

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)$$

subject to the second partial derivatives of u being continuous.

(b) Hence show that

$$\frac{d}{dt}E(a,b) = \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_a^b.$$

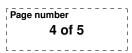
(c) If the string is semi-infinite, so that u(x,t) is only defined for $-\infty < x \leq 0$ and an incident wave is reflected at the origin so that u(x,t) is given by the real part of

$$e^{ik(x-t)} + Re^{-ik(x+t)}$$

find R in the following cases

i.
$$u(0,t) = 0$$
,
ii. $\frac{\partial u(0,t)}{\partial x} = 0$.

- 4. Consider a particle which is confined inside the region -L < x < L. At a given time t the wavefunction of the particle's state is $\psi(x,t) = A(x^2 L^2)$.
 - (a) Fix the constant A for the state to be physical.
 - (b) Calculate the expectation values $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$ with \hat{x} and \hat{p} being the position and momentum operators correspondingly.
 - (c) How would your answer for A and $\langle \hat{x} \rangle$ be modified for the wavefunction $\psi(x,t) = A e^{i f(x)} (x^2 L^2)$ with f(x) a real function of x?
- 5. (a) Write the expression of the expectation value of an observable $\hat{\Omega}$ making use of the inner product and the state vector ψ .
 - (b) Write the expression which defines the uncertainty of the observable $\hat{\Omega}$.
 - (c) Calculate the uncertainty of $\hat{\Omega}$ for one of its eigenstates with corresponding eigenvalue ω .
 - (d) Discuss the time dependence of the expectation value of $\hat{\Omega}$ assuming that $\hat{\Omega}$ commutes with the Hamiltonian \hat{H} of the system.
- 6. (a) Write down the one dimensional Hamiltonian operator for a particle of mass m subject to a potential which is classically given by the function V(x).
 - (b) Suppose the potential has a global minimum V_{min} such that $V(x) > V_{min}$ for $x \in \mathbb{R}$. Find a real number B such that $\langle \hat{H} \rangle \geq B$ for any physical state.
 - (c) Can there be a physical state which gives $\langle \hat{H} \rangle = V_{min}$? Briefly justify your answer.



SECTION B

7. (a) Write down Hamilton's equations of motion for a system described by coordinates $q_1, ..., q_n$ and and their conjugate momenta $p_1, ..., p_n$. Define the Poisson bracket of two functions A and B of these co-ordinates and momenta, and use it and Hamilton's equations to show that

$$\dot{A} = \{A, H\},\$$

where A has no explicit dependence on t.

(b) Find the Hamiltonian for the Lagrangian

$$L = \frac{1}{2} \left(\dot{q}_1^2 + \dot{q}_2^2 - q_1^2 - q_2^2 \right) \,.$$

- (c) By computing their Poisson brackets with the Hamiltonian show that $M = p_1p_2 + q_1q_2$ and $L = p_1q_2 p_2q_1$ are constants of the motion.
- (d) Find $\{L, M\}$.
- (e) Use the Jacobi identity $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$ to show that $\{L, M\}$ is conserved.
- 8. Two coupled pendula are described by the Lagrangian

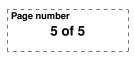
$$L = \frac{1}{2} \left(\frac{1}{2} \dot{\theta}_1^2 + \dot{\theta}_2^2 + \cos \theta_1 + 2\cos \theta_2 - (\theta_1 - \theta_2)^2 \right)$$

- (a) Show that $\theta_1 = \theta_2 = 0$ is one solution of the Euler-Lagrange equations.
- (b) Consider motion close to this solution. Expand L in powers of θ_1 , θ_2 , $\dot{\theta}_1$ and $\dot{\theta}_2$ up to and including quadratic terms to obtain an approximation which we will call L_2 .
- (c) Write the Euler-Lagrange equations for the approximate Lagrangian L_2 in the matrix form

$$\frac{d^2}{dt^2} \left(\begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right) = \mathbf{A} \left(\begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right)$$

and find the two by two matrix **A**.

- (d) Find the general solution for $\theta_1(t)$ and $\theta_2(t)$ to this matrix equation.
- (e) If initially $\dot{\theta}_1 = \dot{\theta}_2 = 0$ and $\theta_1 = \theta_2 = 1$ find $\theta_1(t)$ and $\theta_2(t)$ for subsequent values of t.



- 9. Consider a quantum mechanical particle of mass m inside an infinite potential well whose left hand side wall is at x = 0 and the right hand side one is at x = L. The potential in the region $0 \le x \le L$ is equal to zero.
 - (a) You are given that the wavefunctions $\psi_n(x)$ of the Hamiltonian eigenstates are either $\psi_n(x) = A_n \sin\left(\frac{n\pi}{L}x\right)$ or $\psi_n(x) = A_n \cos\left(\frac{n\pi}{L}x\right)$ for some constants A_n and positive integers n which label the different eigenfunctions. Decide which ones are the correct wavefunctions for the problem at hand. Justify your answer. Fix the constants A_n for the eigenstates to form an orthonormal basis.
 - (b) Write down the Hamiltonian. Derive its spectrum using the eigenfunctions you chose to be the correct ones in the previous question.
 - (c) At t = 0 the wavefunction of the particle's state is given by the wavefunction $\psi(x, t = 0) = C \ (\psi_1(x) + i \ \psi_2(x))$. Choose an appropriate C so that the initial state is physical and write down the time evolved wavefunction $\psi(x, t)$ at time t > 0.
 - (d) Compute the expectation value ⟨ x̂ ⟩ of the position operator x̂ as a function of time for the time dependent state you constructed in the previous question.
 Hint: You are given that ∫₀¹ y sin(πy) sin(2πy)dy = -8/(9π²)
- 10. Consider the one dimensional quantum mechanical simple harmonic oscillator of mass m and frequency ω . The annihilation operator is defined through

$$\hat{a} = \frac{1}{\sqrt{2\hbar m \omega}} \left(m \omega \, \hat{x} + i \, \hat{p} \right) \,,$$

where \hat{x} is the position operator and \hat{p} is the momentum operator. The wavefunction $\psi_0(x)$ of the ground state satisfies $\hat{a}\psi_0(x) = 0$. You can assume that $\psi_0(x)$ is correctly normalised when answering the following questions.

- (a) Compute the commutator $[\hat{a}, \hat{a}^{\dagger}]$ and use induction to show that $[\hat{a}, (\hat{a}^{\dagger})^n] = n (\hat{a}^{\dagger})^{n-1}$ for any positive integer n.
- (b) Express \hat{x} and \hat{p} in terms of \hat{a} and \hat{a}^{\dagger} .
- (c) Consider the wavefunctions $\psi_n(x) = c_n (\hat{a}^{\dagger})^n \psi_0(x)$ with *n* positive integers. Compute the norm of ψ_n and choose the constants c_n so that the wavefunctions $\psi_n(x)$ correspond to physical states.
- (d) Consider the operator $\hat{N} = \hat{a}^{\dagger}\hat{a}$ and check whether this is Hermitian or not. Show that $\psi_n(x)$ are eigenfunctions of \hat{N} and find the corresponding eigenvalues. What is the implication of these facts for the inner product between ψ_n and ψ_m for $n \neq m$?
- (e) Compute the expectation values $\langle \hat{x} \rangle$ for each of the state wavefunctions $\psi_n(x)$.