EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH2581-WE01

Title:
Algebra II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. (i) Find an element in $\mathbb{Z}[\sqrt{-2}]$ which is neither a unit nor a zero-divisor. (You must prove that the element you give has these properties.)

   (ii) Prove that an element in a commutative ring R cannot be both a unit and a zero-divisor.

2. Let R be the ring $\mathbb{Z}/2 \times \mathbb{Z}/4$.

   (a) Find all the ideals of R. (You may use without proof that any ideal of R is of the form $I \times J$, where I is an ideal of $\mathbb{Z}/2$ and J is an ideal of $\mathbb{Z}/4$.)

   (b) Find all prime ideals of R. (You must prove that your ideals are prime and that the others are not.)

3. Let R be the ring $(\mathbb{Z}/2)[x]/(x^2 + \bar{1})$.

   (a) List the elements of R (no proof required).

   (b) Give the multiplication table for the ring R.

4. (i) Determine the order of the element $(\bar{1}, \bar{2})$ in the group $\mathbb{Z}/2 \times \mathbb{Z}/3$.

   (ii) Write down two *distinct* group isomorphisms from $\mathbb{Z}/3$ to itself. (You must show that the maps you write down are isomorphisms.)

5. (i) Show that a group G is abelian if and only if the map $f : G \to G$ defined by $f(g) = g^2$ is a homomorphism.

   (ii) Suppose that G is a finite group and $\varphi : G \to A_5$ is a surjective homomorphism such that $\operatorname{Ker}\varphi$ has order 3. Determine the order of G.

6. Let $S = \{z \in \mathbb{C} \mid |z| = 1\}$, that is, the set of complex numbers with modulus 1.

   (a) Show that S is a subgroup of $\mathbb{C}^\times$.

   (b) Show that the quotient group $\mathbb{C}^\times/S$ is not finite.

SECTION B

7. Let $f(x) = x^5 + x^4 - x^3 + x^2 + x$ and $g(x) = x^5 - x^4 - x + \bar{1}$, and let

$$R = (\mathbb{Z}/3)[x]/(f(x), g(x)).$$

   (a) Find a monic polynomial $h(x) \in (\mathbb{Z}/3)[x]$ such that $(h(x)) = (f(x), g(x))$ and factorise $h(x)$ into irreducibles.

   (b) Show that R is isomorphic to the product of two non-zero rings (mention the results from the lectures that you use and why they apply to the case at hand).

   (c) Find the number of elements of R.

8. For $\bar{a} \in \mathbb{Z}/5$, let I be the ideal of the ring $(\mathbb{Z}/5)[x]$ such that $I = (x^2 + \bar{a}x + \bar{1})$.

   (a) Find all $\bar{a} \in \mathbb{Z}/5$ such that $(\mathbb{Z}/5)[x]/I$ is a field.

   (b) Find all $\bar{a} \in \mathbb{Z}/5$ such that $x^2 + \bar{a}x + \bar{1}$ has a double root, that is, $x^2 + \bar{a}x + \bar{1} = (x - \bar{\alpha})^2$, for some $\bar{\alpha} \in \mathbb{Z}/5$.

   (c) Prove that if $x^2 + \bar{a}x + \bar{1}$ has a double root, then

$$(\mathbb{Z}/5)[x]/I \cong (\mathbb{Z}/5)[x]/(x^2).$$

9. Let $D_n = \langle r, s \rangle$ for $n \geq 3$, and let $N = \langle r^2 \rangle$ be the subgroup generated by r^2.

   (a) Show that N is normal in D_n.

   (b) For each element $r^{2i} \in N$, find the conjugacy class of r^{2i}.

   (c) Determine the cosets of N in D_n when n is even. (You must prove that you have all the cosets and that your cosets are distinct.)

   (d) Assume that n is even. What is the well known group that D_n/N is isomorphic to? (Justify your answer.)

10. (i) Let σ and τ be the following permutations in S_5:

$$\sigma = (3\,4)(2\,1\,5), \qquad \tau = (1\,2\,3\,4\,5).$$

   (a) Determine σ^3 and $\tau^{-1}\sigma$, writing your answers as products of disjoint cycles.

   (b) Write σ and τ as products of transpositions. Which of them (if any) belongs to A_5?

   (ii) Let G be a finite group acting on a finite set X. Let s be the number of orbits in X of size 1, that is, $s = |\{x \in X \mid gx = x, \text{ for all } g \in G\}|$. Let $X_1, \ldots, X_r$ be the orbits in X of size at least 2.

   (a) Show that

$$|X| = s + \sum_{i=1}^{r} |X_i|.$$

   (b) Suppose that G is a group of order p^n for some prime number p and integer $n \geq 1$. Show that

$$|X| \equiv s \pmod{p}.$$